

Bismut type formulae for differential forms

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Abstract. Formulae are given $dP_t\phi$, $d^*P_t\phi$, and $\Delta P_t\phi$ for P_t the heat semigroup acting on a q -form ϕ . The formulae are Brownian motion expectations of ϕ composed with random translations determined by Weitzenböck curvature terms. Derivatives of the curvature are not involved. © Académie des Sciences/Elsevier, Paris

Formules de Bismut pour les formes différentielles

Résumé. Nous donnons des expressions pour $dP_t\phi$, $d^*P_t\phi$ et $\Delta P_t\phi$, où P_t désigne le semi-groupe de la chaleur agissant sur les formes différentielles d'ordre q . Elles s'écrivent comme l'espérance de la fonctionnelle ϕ composée avec les transports parallèles aléatoires associés aux courbures de Weitzenböck. Les dérivées de la courbure n'interviennent pas dans ces formules. © Académie des Sciences/Elsevier, Paris

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1. Soient M une variété riemannienne compacte, Δ le laplacien avec la convention de signe $\Delta = -(d\delta + \delta d)$ et $\mathcal{R} : \wedge^*TM \rightarrow \wedge^*TM$ est la courbure de Weitzenböck. On a la formule de Weitzenböck :

$$\Delta\phi = \text{trace}\nabla^2\phi - \phi \circ \mathcal{R}. \quad (1)$$

On note \mathcal{R}^q , etc. les restrictions des opérateurs sur les formes différentielles d'ordre q et sur les q -vecteurs. Rappelons que $\mathcal{R}^1 = \text{Ric}^t$, le tenseur de Ricci.

On désigne par $\{P_t : t \geq 0\}$ le semi-groupe $e^{\frac{1}{2}t\Delta}$ agissant sur les formes différentielles de L^2 . Il agit donc sur les formes continues par la formule de Feynman-Kac d'Airault $P_t\phi = \mathbb{E}\phi(V_t)$, où V_t est la solution de l'équation covariante le long des trajectoires browniennes $\{x_t : t \geq 0\}$:

$$\frac{DV_t}{dt} = -\frac{1}{2}\mathcal{R}(V_t), \quad V_0 \in \wedge_{x_0}^*TM. \quad (2)$$

Note présentée par Jean-Michel BISMUT.

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On définit $W_t : \wedge^* TM \rightarrow \wedge^* TM$ par $V_t = W_t(V_0)$. Soit $\{\check{B}_t : t \geq 0\}$ l'anti-développement stochastique du mouvement brownien $\{x_t : t \geq 0\}$, et $//_t$ le transport parallèle de la connexion de Levi-Civita le long des trajectoires.

La formule de Bismut est, pour les fonctions bornées mesurables $f : M \rightarrow \mathbb{R}$:

$$dP_t f(v_0) = \frac{1}{t} \mathbb{E} f(x_t) \int_0^t \langle //_s d\check{B}_s, W_s \rangle. \quad (3)$$

On a donc la formule pour le gradient du noyau de la chaleur $p_t(x, y)$ sur les fonctions (voir [1]) :

$$\nabla_x \log p_t(x_0, y) = \mathbb{E} \left\{ \frac{1}{t} \int_0^t W_s^* (//_s d\check{B}_s) | x_t = y \right\}. \quad (4)$$

Nous généralisons ces formules aux cas des q -formes, $1 \leq q < n$.

THÉORÈME 1. – Soit ϕ une forme différentielle, bornée mesurable sur la variété M .

$$d(P_t \phi)(V_0) = \frac{1}{t} \mathbb{E} \phi \left(W_t^q \int_0^t (W_s^q)^{-1} \iota_{//_s d\check{B}_s} W_s^{q+1}(V_0) \right), \quad V_0 \in \wedge^* T_{x_0} M, \quad (5)$$

où $\iota_{//_s d\check{B}_s}$ désigne le produit intérieur (l'opérateur d'annihilation) :

$$\iota_{//_s d\check{B}_s} v^1 \wedge \cdots \wedge v^{q+1} = \sum_{j=1}^{q+1} (-1)^{j+1} \langle //_s d\check{B}_s, v^j \rangle v^1 \wedge \cdots \wedge \widehat{v^j} \wedge \cdots \wedge v^{q+1}.$$

Nous avons un corollaire analogue à (4) :

COROLLAIRE 2. – Pour le noyau de la chaleur $p_t^q(x_0, y)$ défini sur les q -formes :

$$d_x p_t^q(x_0, y)(V_0) = \frac{1}{t} p_t^q(x_0, y) \left(\mathbb{E} \left\{ W_t^q \int_0^t (W_s^q)^{-1} \iota_{//_s d\check{B}_s} W_s^{q+1}(V_0) | x_t = y_0 \right\} \right). \quad (6)$$

Nous remarquons que ces formules ne font pas intervenir les dérivées du tenseur de courbure. Dans [10], Norris donne une expression des dérivées covariantes de tel semi-groupes de la chaleur. Toutefois, ses formules font intervenir les dérivées du tenseur de courbure.

La formule (5) provient d'une formule non-intrinsèque de Li [9] :

$$d(P_t^q \phi) = \frac{1}{t} \mathbb{E} \int_0^t \langle //_s d\check{B}_s, T\xi_s(-) \rangle \wedge \phi(\wedge^q T\xi_t(-)), \quad (7)$$

voir aussi Elworthy–Li [7], en utilisant les techniques de Elworthy–Yor [5], comme dans Elworthy–Li [8]. Pour les détails voir la version anglaise.

On peut déduire le théorème 1 d'une formule d'intégration par partie plus générale sur les espaces de chemins pour les q -formes (voir [6]). Quand on remplace le mouvement brownien par le mouvement brownien avec dérive, la même démonstration fournit des formules avec des changements évidents (voir [7], [8]). De même pour le changement de Thalmaier, où $\frac{1}{t} \int_0^t$ est remplacé par $\frac{1}{h} \int_\delta^{\delta+h}$, $0 \leq \delta < \delta + h \leq t$ (voir [7]).

2. Les formules pour $d^* P_t \phi$. – Nous donnons un autre corollaire, suite à une conversation avec Bruce Driver. Driver [3] a démontré le cas particulier de $q = 1$ par une méthode différente. Il a également prouvé une formule intrigante faisant intervenir une intégrale stochastique rétrograde (voir [2]).

COROLLAIRE 3. – Soit ϕ une forme différentielle d'ordre q de L^2 . On a :

$$d^* P_t \phi = -\frac{1}{t} \mathbb{E} \phi \left(W_t^q \int_0^t (W_s^q)^{-1} (//_s d\check{B}_s \wedge W_s^{q-1}(-)) \right). \quad (8)$$

Voir la version anglaise pour une preuve.

3. *Les formules pour $\Delta P_t \phi$.* – La méthode utilisée dans [7] pour obtenir les dérivées d'ordre supérieur du semi-groupe de la chaleur agissant sur les fonctions peut être utilisée ici pour obtenir des résultats plus élégants :

COROLLAIRE 4

$$\begin{aligned} \Delta P_t \phi(V) &= \frac{4}{t^2} \mathbb{E} \phi \left(W_t \int_{t/2}^t (W_s)^{-1} dM_s(V) \right), \\ \text{où} \quad dM_s(V) &= \iota_{//_s} d\check{B}_s W_s \left(\int_0^{t/2} (W_r)^{-1} (//_r d\check{B}_r \wedge W_r(V)) \right) \\ &\quad + \iota_{//_s} d\check{B}_s W_s \left(\int_0^{t/2} (W_r)^{-1} (\iota_{//_r} d\check{B}_r W_r(V)) \right). \end{aligned} \quad (9)$$

Nous pouvons itérer cette méthode afin d'obtenir les formules pour $(\Delta)^r P_t \phi$, $r = 1, 2, \dots$. Elles permettent d'établir les formules pour $d_x p_t^q(\cdot, y)$, $d_x^* p_t^q(\cdot, y)$, $(\Delta_x)^r p_t^q(\cdot, y)$. En particulier, on déduit que $d_x p_t^q(\cdot, y)$, $d_x^* p_t^q(\cdot, y)$, $(\Delta_x)^r p_t^q(\cdot, y)$ et $d P_t \phi$, $d^* P_t \phi$, $(\Delta)^r P_t \phi$, $r = 1, 2, \dots$, sont uniformément bornés par une constante dépendant uniquement de t , des courbures de Weitzenböck \mathcal{R}^q , $\mathcal{R}^{q\pm 1}$ et du maximum de $|\phi|$.

1. Let M be a compact Riemannian manifold with Hodge–de Rham Laplacian Δ given the sign convention $\Delta = -(\delta\delta + \delta\delta)$. There is then the Weitzenböck formula:

$$\Delta \phi = \text{trace } \nabla^2 \phi - \phi \circ \mathcal{R}, \quad (1)$$

where $\mathcal{R} : \wedge^* TM \rightarrow \wedge^* TM$ is the Weitzenböck curvature term. Let \mathcal{R}^q , etc. denote the restriction to q -forms and q -vectors. Recall $\mathcal{R}^1 = \text{Ric}^\sharp$, the Ricci tensor.

Let $\{P_t : t \geq 0\}$ be the semigroup $e^{\frac{1}{2}t\Delta}$ acting on L^2 -forms (by the essentially self-adjointness of Δ) and on bounded measurable forms via Airault's version of the Feynman–Kac formula:

$$P_t \phi(V_0) = \mathbb{E} \phi(V_t), \quad (2)$$

where V_t satisfies the covariant equation along Brownian paths $\{x_t : t \geq 0\}$:

$$\frac{DV_t}{\partial t} = -\frac{1}{2} \mathcal{R}(V_t), \quad V_0 \in \wedge^* T_{x_0} M. \quad (3)$$

We will write $V_t = W_t(V_0)$ to give $W_t : \wedge^* TM \rightarrow \wedge^* TM$. Let $\{\check{B}_t : t \geq 0\}$ be the stochastic anti-development of $\{x_t : t \geq 0\}$, a Brownian motion on $T_{x_0} M$, and denote by $//_t$ the parallel translation for the Levi-Civita connection along our paths.

Bismut's formula is for bounded measurable function $f : M \rightarrow \mathbb{R}$. It is

$$d P_t f(v_0) = \frac{1}{t} \mathbb{E} f(x_t) \int_0^t \langle //_s d\check{B}_s, W_s \rangle \quad (4)$$

which leads to his gradient formula for the heat kernel $p_t(x, y)$ on functions (*see* [1]):

$$\nabla_x \log p_t(x_0, y) = \mathbb{E} \left\{ \frac{1}{t} \int_0^t W_s^* (//_s d\check{B}_s) | x_t = y \right\}. \quad (5)$$

Here we extend these formulae to the case of q -forms, $1 \leq q < n$, to show:

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THEOREM 1.1. – For ϕ a bounded measurable q -form on M and $V_0 \in \wedge^q T_{x_0} M$

$$d(P_t \phi)(V_0) = \frac{1}{t} \mathbb{E} \phi \left(W_t^q \int_0^t (W_s^q)^{-1} \iota_{//_s d\check{B}_s} W_s^{q+1}(V_0) \right), \quad (6)$$

where $\iota_{//_s d\check{B}_s}$ denotes the interior product (annihilation operator) given by:

$$\iota_{//_s d\check{B}_s} v^1 \wedge \dots \wedge v^{q+1} = \sum_{j=1}^{q+1} (-1)^{j+1} \langle //_s d\check{B}_s, v^j \rangle v^1 \wedge \dots \wedge \widehat{v^j} \wedge \dots \wedge v^{q+1}.$$

As for (5) there is the corollary:

COROLLARY 1.2. – For the heat kernel $p_t^q(x_0, y)$ on q -forms:

$$d_x p_t^q(x_0, y)(V_0) = \frac{1}{t} p_t^q(x_0, y) \left(\mathbb{E} \left\{ W_t^q \int_0^t (W_s^q)^{-1} \iota_{//_s d\check{B}_s} W_s^{q+1}(V_0) | x_t = y_0 \right\} \right). \quad (7)$$

Note these formulae do not involve derivatives of the curvature tensor. Expressions for covariant derivatives of such heat semigroups have been obtained by Norris [10], however these do involve such derivatives.

Formula (6) is derived from a non-intrinsic formula in Li [9]:

$$d(P_t^q \phi) = \frac{1}{t} \mathbb{E} \int_0^t \langle //_s d\check{B}_s, T\xi_s(-) \rangle \wedge \phi(\wedge^q T\xi_t(-)), \quad (8)$$

see also Elworthy–Li [7], by techniques of Elworthy–Yor [5], as used in Elworthy–Li [8]. The details follow:

Proof of Theorem 1.1. – Taken an isometric embedding $\alpha : M \rightarrow \mathbb{R}^m$, some m , using Nash's Theorem. Let $X(x) : \mathbb{R}^m \rightarrow T_x M$ be the orthogonal projection, identifying $T_x M$ with its image in \mathbb{R}^m under $d\alpha$. Let $\{B_t : t \geq 0\}$ be a Brownian motion on \mathbb{R}^m . The *gradient Brownian system*

$$dx_t = X(x_t) \circ dB_t \quad (9)$$

has a solution flow $\{\xi_t(x) : t \geq 0, x \in M\}$ consisting of random diffeomorphisms $\xi_t : M \rightarrow M$ with derivatives denoted by $T_{x_0} \xi_t : T_{x_0} M \rightarrow T_{x_t} M$ for $x_t = \xi_t(x_0)$. Each solution $(x_t : t \geq 0)$ is a Brownian motion on M .

Define $J_t : \wedge^{q+1} TM \rightarrow \wedge^q TM$ by:

$$J_t(V) = \int_0^t \langle T\xi_t -, //_r d\check{B}_r \rangle \wedge (\wedge^q T\xi_t(-)), \quad t > 0. \quad (10)$$

We see that formula (8) can be written as:

$$d(P_t \phi)(V) = \frac{1}{t} E_\phi(J_t(V)), \quad t > 0,$$

and J_t satisfies

$$DJ_t = \iota_{X(x_t) dB_t} \wedge^{q+1} T\xi_t + d \wedge^q (\nabla X(-) dB_t) - \frac{1}{2} \mathcal{R}^q(J_t) dt,$$

where the term involving \mathcal{R}^q comes from the Stratonovich correction. (See Elworthy–Yor [5].) Next set

$$\bar{J}_t = E \{J_t | \mathcal{F}_t^{x_0}\},$$

where \mathcal{F}^{x_0} is the filtration generated by $\{x_s\}$ and the conditional expectation is understood to be $//_t E \{ //_{t-1}^{-1} J^t | \mathcal{F}_t^{x_0} \}$. Using the decomposition of B_t into relevant and “redundant” noise as in Elworthy–Yor [5], we have:

$$d \wedge^q \nabla X (-) dB_t = d \wedge^q \nabla X (-) //_{t-1}^{-1} d\beta_t$$

where $//_{t-1}^{-1}$, $t \geq 0$, consists of random isometries of \mathbb{R}^m and $\{\beta_t : t \geq 0\}$ is a Brownian motion on $\text{Ker } X(x_0)$ independent of $\mathcal{F}_t^{x_0}$. From this \bar{J}_t satisfies

$$D\bar{J}_t = \iota_{X(x_t)} dB_t W_t^{q+1} - \frac{1}{2} \mathcal{R}^q(\bar{J}_t) dt. \quad (11)$$

For a general discussion, see Elworthy–Le Jan–Li [4]. Solve the equation (11) to get:

$$\bar{J}_t = W_t^q \int_0^t (W_s^q)^{-1} (\iota_{X(x_s)} dB_s) W_s^{q+1}.$$

The required result follows. \square

Theorem 1.1 can also be deduced from a more general integration by parts formula for q -forms on path space (see [6]). The obvious variation when the Brownian motion is replaced by one with a drift Z also holds with the same proof: here $\{W_s\}$ is replaced by $\{W_s^Z\}$ as in [7], [8]. So also does Thalmaier’s variant when $\frac{1}{t} \int_0^t$ is replaced by $\frac{1}{h} \int_\delta^{\delta+h}$ for $0 \leq \delta < \delta + h \leq t$ in (17), (15), and (16).

2. *Formulae for $d^*P_t\phi$.* – We give the next corollary following conversations with Bruce Driver. Driver [3] proved it in the special case $q = 1$, by a different method. Also in [2], he gave an intriguing different formula in that case involving a backward stochastic integral.

COROLLARY 2.1. – *Let ϕ be a L^2 q -form. Then:*

$$d^*P_t\phi = -\frac{1}{t} \mathbb{E}\phi \left(W_t^q \int_0^t (W_s^q)^{-1} (//_s d\check{B}_s \wedge W_s^{q-1}(-)) \right). \quad (12)$$

Proof. – We can assume that ϕ is smooth. Also by going to the double cover if necessary we can assume that M is oriented with volume element $d\text{vol}$ considered as a n -form. The Hodge star construction then gives $*$: $\wedge T^*M \rightarrow \wedge T^*M$ and $*$: $\wedge TM \rightarrow \wedge TM$ such that $\alpha \wedge * \beta = \langle \alpha, \beta \rangle d\text{vol}$ and $v_1 \wedge * v_2 = \langle v_1, v_2 \rangle (d\text{vol})^\sharp$. Then $*\phi(V) = (-1)^q \langle n-q \rangle \phi(*V)$, for $V \in \wedge^q T_x M$, $** = (-1)^q \langle n-q \rangle \text{Id}$, and $d^*\phi = (-1)^n \langle q-1 \rangle + 1 * d * \phi$.

Furthermore, $*P_t = P_t*$, $*W_t = W_t*$, and $*\iota_u V = (-1)^{q-1} u \wedge (*V)$ any $u \in T_x M$. Thus $d^*P_t\phi(-) = (-1)^n \langle q-1 \rangle + 1 * d * (P_t\phi)(-) = (-1)^q d * (P_t\phi)(*-) = (-1)^q d(P_t * \phi)(*-)$. Consequently,

$$\begin{aligned} d^*P_t\phi(-) &= (-1)^q \frac{1}{t} \mathbb{E}(*\phi) \left(W_t^{n-q} \int_0^t (W_s^{n-q})^{-1} \iota_{X(x_s)} dB_s W_s^{n-q+1}(*-) \right) \\ &= (-1)^q \frac{1}{t} (-1)^q \langle n-q \rangle \mathbb{E}(\phi) \left(W_t^q \int_0^t (W_s^q)^{-1} [(-1)^{n-q} X(x_s) dB_s \wedge W_s^{q-1}(**-)] \right) \\ &= -\frac{1}{t} \mathbb{E}\phi \left(W_t^q \int_0^t (W_s^q)^{-1} (X(x_s) dB_s \wedge W_s^{q-1}(-)) \right) \end{aligned} \quad (9)$$

\square

giving (12).

3. *Formulae for $\Delta P_t \phi$.* – The method used in [7] to obtain higher order derivatives for heat semigroups on functions can be applied to the present situation, with neater results. First, note that for a $(q + 1)$ -form ψ equation (6) at time $t/2$ is equivalent to

$$dP_{t/2} \psi(V) = \frac{2}{t} \mathbb{E} \psi_{\xi_t^{t/2}(x_0)} \left(W_t^{\frac{1}{2}} \int_{t/2}^t (W_s^{t/2})^{-1} \iota_{X(\xi_s^{t/2}(x_0))} dB_s W_s^{t/2}(V) \right), \quad (13)$$

where $\xi_s^{t/2}$ and $W_s^{t/2}$, $t/2 \leq s < \infty$, refer to the respective flows starting at time $t/2$ and evaluated at time s . Since $d^* dP_t \phi(V) = dP_{t/2}^* (dP_{t/2}) \phi(V)$ equations (13), (8), the Markov property, and the semigroup property of stochastic flows and the development map yield

$$d^* dP_t \phi(V) = -\frac{4}{t^2} \mathbb{E} \phi_{x_t} \left(W_t \int_{t/2}^t (W_s)^{-1} \iota_{//_s} d\check{B}_s W_s \int_0^{t/2} (W_r)^{-1} //_r d\check{B}_r \wedge W_r(V) \right).$$

Adding on the corresponding formula for $d^* dP_t \phi(V)$ we obtain:

COROLLARY 3.1. – We have $\Delta P_t \phi(V) = \frac{4}{t^2} \mathbb{E} \phi \left(W_t \int_{t/2}^t (W_s)^{-1} dM_s(V) \right)$, where

$$dM_s(V) = \iota_{//_s} d\check{B}_s W_s \left(\int_0^{t/2} (W_r)^{-1} (//_r d\check{B}_r \wedge W_r(V)) \right) + //_s d\check{B}_s \wedge W_s \left(\int_0^{t/2} (W_r)^{-1} (\iota_{//_r} d\check{B}_r W_r(V)) \right).$$

This can be iterated to give formulae for $(\Delta)^r P_t \phi$, $r = 1, 2, \dots$. These together with the previous ones lead to formulae for the corresponding operators acting on the heat kernels; as for (7), and show that $d_x p_t^q(\cdot, y)$, $d_x^* p_t^q(\cdot, y)$, $(\Delta_x)^r p_t^q(\cdot, y)$ and $dP_t \phi$, $d^* P_t \phi$, $(\Delta)^r P_t \phi$, $r = 1, 2, \dots$, all have uniform bounds for fixed positive t , depending only on t and bounds for the Weitzenböck curvatures \mathcal{R}^q , $\mathcal{R}^{q \pm 1}$ (and the uniform bound of ϕ).

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