ON THE COMPACTNESS OF MANIFOLDS*

XUE-MEI LI

Department of Mathematics, The Nottingham Trent University, Nottingham NG1 4BU, UK
Department of Mathematics, University of Connecticut, 196 Auditorium Road, Storrs, CT 06269-3008, USA
xml@math.ntu.ac.uk

FENG-YU WANG

Department of Mathematics, Beijing Normal University, Beijing 100875, China
wangfy@bnu.edu.cn

It is believed that the family of Riemannian manifolds with negative curvatures is much richer than that with positive curvatures. In fact there are many results on the obstruction of furnishing a manifold with a Riemannian metric whose curvature is positive. In particular any manifold admitting a Riemannian metric whose Ricci curvature is bounded below by a positive constant must be compact. Here we investigate such obstructions in terms of certain functional inequalities which can be considered as generalized Poincaré or log-Sobolev inequalities. A result of Salo-Coste is extended.

Keywords: Compactness; Riemannian manifolds; functional inequality; curvature.

1. Introduction

Let $M$ be a complete connected Riemannian manifold of dimension $d$. A basic topic in Riemannian geometry is the non-existence of Riemannian structures of particular properties on topological manifolds. One of the often studied question is to equip a manifold with certain curvature conditions. A classical result in this direction is Myers’s theorem [16] which says that a noncompact manifold does not admit Ricci curvature bounded below by a positive constant, say $K$. Furthermore an upper bound for the diameter $D$ of the manifold given: $D \leq \pi \sqrt{d - 1/K}$. Some effort have been made to extend Myers’ theorem and to understand the intrinsic meaning of the conditions imposed. See e.g. Bonnet [4] and Ambrose [1]. In Ambrose [1] it

*Research supported by NSF grant DMS-0072387, NNSFC for Distinguished Young Scholars (10025105), the 973-Project, TRAPOYIT and the Core Teachers Project, and the Guest Professor Programme of the University of Connecticut.
was shown that compactness follows if
\[ \int_0^\infty \text{Ric}(\gamma(t), \dot{\gamma}(t))dt = \infty \]
for each geodesic \( \gamma \) emanating from a fixed point and parameterized by arc length, allowing Ricci curvature being negative. In [10] Galloway showed, by a careful study of equations \( x'' + r(t)x = 0 \) of the Jacobi type being oscillatory, (1.1) can be replaced by the following
\[ \int_0^\infty \lambda \text{Ric}(\gamma(t), \dot{\gamma}(t))dt = \infty \]  
for some \( \lambda \in [0, 1) \), thus allowing quadratic decay of the Ricci curvature at the infinity. If furthermore the Ricci curvature is nonnegative, the manifold is compact if \( \lim \inf \text{Ric}(\gamma(t), \dot{\gamma}(t)) > (d-1)/4 \).

Another extension of Myers’ result was made by Li [15] using the stochastic positivity of Ricci curvature. More precisely, let \( \rho(x) \) denote the Riemannian distance between \( x \) and a fixed point \( p \), \( M \) is compact provided
\[ \kappa(x) = \inf \{ \text{Ric}(X, X) : X \in T_x M, |X| = 1 \} \geq \frac{-d}{(d-1)\rho(x)^2} \]
for big \( \rho(x) \) and
\[ \sup_{x \in K} \int_0^\infty \mathbb{E} \exp \left[ -\frac{1}{2} \int_0^t \kappa(x_s)ds \right] < \infty \] 
for any compact \( K \subset M \), where \( x_s \) denotes the Brownian motion on \( M \) starting from \( x \). Note that for compact manifolds, see [9], (1.2) is equivalent to the operator \(-\Delta + \frac{1}{2}\kappa(x)\) being positive.

Compactness was also studied by Salo-Coste [18] using the log-Sobolev inequality. He proved that a manifold \( M \) of finite volume is compact provided the Ricci curvature is bounded below and that there exists \( C_0 > 0 \) such that
\[ \mu(f^2 \log f^2) \leq \mu(f^2) \log \mu(f^2) + C_0 \mu(|\nabla f|^2), \quad f \in C_0^\infty(M), \] 
where \( \mu \) denotes the normalized volume measure. Estimates of \( D \) are presented by Salo-Coste [18] and Ledoux [13] in terms of \( C_0, d \) and the lower bound of the Ricci curvature.

The compactness of Riemannian manifolds with Ricci curvature bounded from below also follows from the following condition on the heat kernel below also follows from the following condition on the heat kernel \( p_t \):
\[ \int_M \frac{1}{p_t(x, y)}dy < \infty, \]
a result proved in Gong and Wang [11] and conjectured in Buler [5].

The purpose of this paper is to investigate the compactness of complete Riemannian manifolds in relations to certain functional inequalities which is in general weaker than the corresponding log-Sobolev inequalities. In some cases the Ricci curvature is allowed not to be bounded from below, see §3.
Let $L := \Delta + \nabla V$ be a $C^2$ function on the manifold with $Z := \int_M e^V dx$ finite. Consider the normalized measure $\mu := Z^{-1} e^V dx$ and the following functional inequality

$$
\mu(f^2) \leq r \mu((\nabla f)^2) + \beta(r) \mu(|f|^2), \quad r > r_0, f \in C_0^\infty(M),
$$

(1.4)

where $r_0 \geq 0$ is a constant and $\beta : (r_0, \infty) \to (0, \infty)$ is a decreasing function. This inequality was introduced in [19] and there it was shown that the essential spectrum $\sigma_{ess}(-L)$ of $-L$ satisfies $\sigma_{ess}(-L) \subset [\frac{1}{r_0}, \infty)$ if and only if (1.4) holds for some $\beta$. Note that (1.3) holds for some $C_0 > 0$ if and only if (1.4) holds for $r_0 = 0$ and $\beta(r) = \exp[c(1 + r^{-1})]$ for some $c > 0$. In fact (1.4) generalizes the concepts of Poincaré inequality, log-Sobolev inequality, Sobolev inequality, and Nash inequality.

In §2 we show the inequality (1.4) with $r_0 = 0$ together with a curvature-dimension condition implies the manifold is necessarily compact. Our proof is based on a spectrum argument. In section 3 we consider the following question: assume a functional inequality of type (1.4) holds what is the weakest possible condition on the curvature which implies the compactness of $M$. For example if (1.3) holds then the curvature condition

$$
\liminf_{\rho(x) \to \infty} \frac{0 \vee (-\kappa(x))}{\rho(x)^2} < \frac{1}{4(d - 1)^2 C_0^0}
$$

implies the compactness of $M$. This curvature condition is much weaker than the one used in Saloff-Coste [18], namely, the Ricci curvature is bounded below.

2. A Spectrum Argument

The basic idea is the following: if $\lambda_{ess} \equiv \lambda_{ess}(-L) := \inf \sigma_{ess}(-L)$ is positive then the first Dirichlet eigenvalue on geodesic balls of certain size is shown to have small uniform upper bound which forces the manifold to be compact. Let $D$ be an open connected open set of $M$. Denote by $\lambda_0(D)$ the first Dirichlet eigenvalue of $L \equiv \Delta + \nabla V$ on $D$, i.e.

$$
\lambda_0(D) \equiv \lambda_0(D, L) := \inf \{ \mu((\nabla f)^2) : \mu(f^2) = 1, f \in C_0^\infty(D) \},
$$

where $C_0^\infty(D) := \{ f \in C_0^\infty(M), \text{supp} f \subset D \}$, and $\mu(dx) = e^V dx$.

Let $B(x, r)$ denote the open geodesic ball around $x$ with radius $r$.

**Theorem 2.1.** If $M$ is not compact then

$$
\sup_{x \in M} \lambda_0(B(x, r)) \geq \lambda_{ess}
$$

for any $r > 0$ and any operator $L$ of the form $\Delta + \nabla V$, where $V$ is a $C^2$ function on $M$. Consequently if there is a $C^2$ function $V : M \to R$ and a positive number $r$ such that

$$
\lambda_0(r) := \sup_{x \in M} \lambda_0(B(x, r)) < \lambda_{ess},
$$

(2.1)

then $M$ is compact.
Proof. Suppose that \( M \) is noncompact. Set \( a = \frac{1}{2}(\lambda_{\text{ess}} - \lambda_0(r)) \). By Donnelly-Li’s decomposition principle [8], \( \sigma_{\text{ess}}(-L|_{D^c}) = \sigma_{\text{ess}}(-L) \) for compact sets \( D \). Thus, \( \lambda_0(D^c) \to \lambda_{\text{ess}} \) as \( D \) approaches \( M \). If \( a = \frac{1}{2}(\lambda_{\text{ess}} - \lambda_0(r)) > 0 \), then there is a compact domain \( D \) such that

\[
\lambda_0(D^c) \geq \lambda_{\text{ess}} - a = \frac{1}{2}\lambda_{\text{ess}} + \frac{1}{2}\lambda_0(r).
\]

Now for any \( r \) we can find \( x \) such that \( B(x; r) \cap D = \emptyset \). Thus by the domain monotonicity of the first Dirichlet eigenvalue

\[
\lambda_0(r) \geq \lambda_0(B(x, r)) \geq \lambda_0(D^c) \geq \frac{1}{2}\lambda_{\text{ess}} + \frac{1}{2}\lambda_0(r),
\]

which implies \( a \leq 0 \).

In the following we shall use (1.4) and upper bounds of \( L \) acting on distance functions to obtain (2.1). Let \( \rho_x \) be the Riemannian distance function from \( x \), and \( \text{cut}(x) \) the cut locus of \( x \).

Let us first recall a comparison lemma:

Lemma 2.2. Let \( \gamma \) be a positive continuous function on \((0, \infty)\) such that \( L\rho_x(y) \leq \gamma(\rho_x(y)) \) for any \( x \) and \( y \not\in \{x\} \cup \text{cut}(x) \). Define a measure \( \nu \) on \([0, \infty)\) with

\[

\nu(dr) = e^{\gamma(s)} ds \ dr.

\]

Let \( \Lambda^\gamma \) be the principal eigenvalue of \( L^\gamma := \frac{d^2}{dr^2} + \frac{\gamma(s)}{r^2} \). Then

\[

\limsup_{s \to \infty} \sup_{x \in M} \lambda_0(B(x, s)) \leq \Lambda^\gamma.
\]

Proof. Let \( \Lambda_{\text{ess}}^\gamma \) be the first eigenvalue of \( L^\gamma = \frac{d^2}{dr^2} + \gamma(r)\frac{d}{dr} \) on \([0, s]\) with Neumann boundary at 0 and Dirichlet boundary at \( s \), and \( h_s \) the corresponding (positive) eigenfunction. Then \( h_s \) is decreasing since it has no critical point on \((0, s)\) as shown in Chen-Wang [7] (Proposition 6.4).

Now for any \( x \), \( h_s \circ \rho_x \) is defined on \( B(x, s) \) and

\[

(\Delta + V)(h_s \circ \rho_x) = \Delta(h_s \circ \rho_x) + h_s'(\rho_x)\langle \nabla V, \nabla \rho_x \rangle
\]

\[

= h_s''(\rho_x) + h_s'\rho_x) L\rho_x
\]

\[

\geq h_s''(\rho_x) + h_s'(\rho_x) \gamma(\rho_x)
\]

\[

= -\Lambda_{\text{ess}}^\gamma h_s \circ \rho_x
\]

outside of the cut locus of \( x \). Since the cut locus of \( x \) has measure 0,

\[

(\Delta + \nabla V)(h_s \circ \rho_x) \geq -\Lambda_{\text{ess}}^\gamma h_s \circ \rho_x
\]
on $B(x, s)$ in the sense of distribution (see e.g. Appendix in Yau [21]). Therefore $\lambda_0(B(x, s)) \leq \Lambda^\gamma_0$ and

$$\limsup_{s \to \infty, x \in M} \lambda_0(B(x, s)) \leq \lim_{s \to \infty} \Lambda^\gamma_0 = \Lambda^\gamma. \quad \Box$$

**Theorem 2.3.** Suppose $\int_M e^{V(x)} dx < \infty$ and $L \rho \leq \gamma \circ \rho$. Let $\Lambda^\gamma$ be the principal eigenvalue of $\frac{\partial^2}{\partial x^2} + \gamma \frac{\partial}{\partial r}$ on $[0, \infty)$. Then the inequality (1.4) does not hold for any $r_0 < \frac{1}{K^\gamma}$ unless the manifold is compact.

**Proof.** Suppose that the inequality (1.4) holds for some $r_0 < \frac{1}{K^\gamma}$ then by [19]

$$\lambda_{ess}(-L) \geq \frac{1}{r_0} > \Lambda^\gamma.$$

By the eigenvalue comparison lemma $L(\rho) \leq \gamma(\rho)$ implies that there exists $s > 0$ such that

$$\sup_{x \in M} \lambda_0(B(x, s)) \leq \Lambda^\gamma_0 < \lambda_{ess}(-L).$$

Theorem 2.1 now applies to imply the compactness of the manifold. \[ \Box \]

It is known that Ricci curvature bounded from below implies that $\Delta \rho \leq c(1 + \rho^{-1})$ for some constant $c$ and any $x \in M$. In general this is true for $L = \Delta + \nabla V$ if the following curvature dimension condition holds:

$$\Gamma_2(f, f) := \frac{1}{2} \left[ \langle \nabla f, \nabla f \rangle - \langle \nabla f, \nabla Lf \rangle \right] \geq -K |\nabla f|^2 + \frac{1}{n} (Lf)^2, \quad f \in C^\infty(M), \quad (2.2)$$

where $K \geq 0$, $n > 1$ are constants. This inequality is equivalent to that the Ricci curvature being bounded from below by $-K$ in the case that $L = \Delta$ and $n = d$, the dimension of the manifold. It was shown in Qian [17] that (2.2) implies that $L \rho \leq \gamma(\rho)$ outside of $\{x\} \cup \text{cut}(x)$ where

$$\gamma(r) = \sqrt{K(n-1)} \coth[r\sqrt{K/(n-1)}]. \quad (2.3)$$

This consideration leads to the following corollary:

**Corollary 2.4.** Assume (2.2) and $\int_M e^{V(x)} dx < \infty$. Then (1.4) cannot hold for any $r_0 < 4/K(n-1)$ unless the manifold is compact.

**Proof.** Cheeger’s inequality implies that the principal eigenvalue of $\frac{\partial^2}{\partial x^2} + \gamma \frac{\partial}{\partial r}$ is less than or equal to $K(n-1)/4$. See Chavel [6]. Theorem 2.3 now applies. \[ \Box \]

Let $P_t$ be the semigroup associated to the heat equation $\frac{\partial}{\partial t} = L$. We relate the spectrum of $-L$ to the kernel $p_t(x, x)$, with respect to the measure $\mu$, of the semigroup $P_t$.

**Proposition 2.5.** Assume $\int_M e^{V(x)} dx < \infty$. If $\int \mu_{x}(x) \mu(dx) < \infty$ for some $t > 0$, then $\lambda_{ess}(-\Delta - \nabla V) = \infty$, or equivalently, (1.4) holds for some function $\beta$.
with \( r_0 = 0 \). Consequently \( \int_M p_t(x, x) \mu(dx) < \infty \) for some \( t > 0 \) and the curvature dimension inequality (2.2) together imply that the manifold is compact.

**Proof.** The relation of the super Poincaré inequality (1.4) and the essential spectrum of \((-L)\) is given by Theorem 2.1 in [19]. We shall show \( \lambda_{\text{ess}}(-L) = \infty \). For any \( f \) with \( \mu(f^2) \leq 1 \), one has \( (p_t f(x))^2 \leq p_{2t}(x, x), t > 0, x \in M \). Therefore, if \( \int p_t(x, x) \mu(dx) < \infty \) then \( P_{t/2} \) is \( L^2(\mu) \)-uniformly integrable and hence \( P_t \) is compact in \( L^2(\mu) \), see e.g. Theorem 2.3 in [20]. Thus, the proof is complete since \( \sigma_{\text{ess}}(L) = \emptyset \) if \( P_t \) is compact.

**Corollary 2.6.** Assume (2.2) and \( \int_M e^{k(x)} \mu(dx) = \infty \). Let \( \rho := \rho_{x_0} \) for a fixed \( x_0 \in M \). Then \( M \) is compact provided one of the following holds:

(1) \( K = 0 \) and \( \mu(\rho^n) < \infty \).

(2) \( K > 0 \) and

\[
\mu\left(\rho^{n/2} \exp \left[ \frac{1}{2} \sqrt{nK}(\sqrt{2} + 1) \rho \right] \right) < \infty.
\]

**Proof.** By Proposition 2.5, in both cases we only need to prove that \( \int_M p_t(x, x) \mu(dx) < \infty \) holds for some \( t > 0 \). First observe, by Corollary 2 in [2] (see [3] for more details),

\[
p_t(x, x) \exp \left[ \left( \frac{\rho_{x}(y) + \sqrt{nK} s}{4s} \right)^2 - \frac{\sqrt{2nK}}{2} \min \left\{ (\sqrt{2} - 1)\rho_{x}(y), \frac{\sqrt{2}nK}{2} s \right\} \right] \leq \left( \frac{t + x}{t} \right)^{n/2} p_{t+s}(x, y), \quad t, s > 0, x, y \in M.
\]

For part (1), take \( s = \rho(x)^2 + 1 \) in (2.4) and integrate both sides over \( y \) with respect to \( \mu \) to obtain

\[ cp_t(x, x) \mu(B(x_0, 1)) \leq \left( \frac{t + \rho(x)^2 + 1}{t} \right)^{n/2} \]

for some \( c > 0 \). Thus

\[
\int_M p_t(x, x) \mu(dx) \leq c_1 (1 + t^{-n/2}) < \infty
\]

for some \( c_1 > 0 \) and all \( t > 0 \).

For part (2) take \( s = (\rho(x) + 1)/\sqrt{nK} \) in (2.4) to see

\[
p_t(x, x) \leq c(t)(\rho(x) + 1)^{n/2} \exp \left[ \frac{\sqrt{2nK}}{2} \left( \sqrt{2} + 1 \right) \rho(x) \right]
\]

for some \( c(t) > 0 \). Hence \( \int_M p_t(x, x) \mu(dx) < \infty \) for all \( t > 0 \).

So far we conclude that (1.4) together with the curvature-dimension condition (2.2) implies the compactness of \( M \). Below we show that (1.4) alone, with a good enough function \( \beta \), also implies the compactness of the manifold.
Proposition 2.7. Assume \( \int_M e^{V(x)} \, dx < \infty \). If (1.4) holds for \( r_0 = 0 \) some \( \beta \) satisfying
\[
C(\delta) := \int_1^\infty \frac{1}{r^2} \log \beta \left( \frac{1}{\delta r^2} \right) \, dr < \infty
\]
for some \( \delta > 1 \), then \( M \) is compact with diameter
\[
D \leq \inf_{\delta > 1} \left\{ \log \delta \mu(e^\rho) \delta^{n-1} + C(\delta) \right\}.
\]
Conversely, if \( M \) is compact then (1.4) holds for \( r_0 = 0 \) and \( \beta(r) = c(1 + r^{-d/2}) \) for some \( c > 0 \), hence (2.5) holds for all \( \delta > 1 \).

Proof. The first assertion follows from Theorem 6.1 in [19], while the second assertion follows from Corollary 3.3 in [19] by the Sobolev inequality on compact manifolds.

3. A Measure-Curvature Argument

In this section we shall assume that the essential spectrum of \(-L\) is empty, i.e. \( \lambda_{ess} = \infty \). Recall that according to [19] this is equivalent to the super Poincaré inequality
\[
\mu(f^2) \leq r \mu(|\nabla f|^2) + \beta(r) \mu(|f|^2), \quad r > 0, f \in C_0^\infty(M)
\]
a decreasing function \( \beta : (0, \infty) \to (0, \infty) \). Consider the following generalized curvature dimension inequality:
\[
\Gamma_2(f, f)(x) \geq -(n-1)(k \circ \rho)|\nabla f|^2 + \frac{1}{n}(Lf)^2, \quad f \in C_0^\infty(M),
\]
where \( \rho := \rho_{x_0} \) for a fixed point \( x_0 \), \( n > 1 \) and \( k \) is an increasing function from \((0, \infty)\) to \((0, \infty)\). When \( L = \Delta \) and \( n = d \) is the dimension of the manifold, (3.2) is equivalent to \( \text{Ric}_x \geq -(d-1)k \circ \rho(x), x \in M \). We allow \( k \) to be unbounded. Now (3.1) implies decay of \( \mu(\rho > r) \) while (3.2) provides a lower bound of \( \mu(\rho > r) \). The two together with appropriate choices of \( \beta \) and \( k \) should force the manifold to be compact.

Theorem 3.1. Assume \( \int_M e^{V(x)} \, dx < \infty \). The manifold \( M \) is compact if (3.2) holds and
\[
\limsup_{r \to \infty} \frac{-\log \mu(\rho > r)}{(n-1)r\sqrt{k(2r+3)}} > 1.
\]

Proof. Assume that \( M \) is noncompact. For any \( r > 0 \) there exists \( x_r \in M \) such that \( \rho(x_r) = r + 1 \). Apply (3.2) to see
\[
\Gamma_2(f, f)(x) \geq -(n-1)k(2r+3)|\nabla f|^2(x) + \frac{1}{n}(Lf)^2(x), \quad f \in C^\infty(M), x \in B(x_r, r+2).
\]
On the other hand, see Qian [17],
\[ L \rho_{x_r} \leq (n-1)\sqrt{k(2r+3)} \coth \left( \sqrt{k(2r+3)} \rho_{x_r} \right) \]
on \( B(x_r, r+2) \setminus \{ x_r \} \cup \text{cut}(x_r) \). This implies, by a standard argument as in Lemma 2.2 in [11], that
\[ \mu(B(x_r, r+2)) \leq \mu(B(x_r, 1))(r+2)^n \exp \left[ (n-1)(r+1)\sqrt{k(2r+3)} \right]. \]
Consequently
\[ \mu(\rho > r) \geq \mu(B(x_r, 1)) \geq \mu(B(x_0, 1))(r+2)^{-n} \exp \left[ - (n-1)(r+1)\sqrt{k(2r+3)} \right] \]
contradicting with (3.3).

**Corollary 3.2.** Assume (3.1), (3.2) and \( \int_M e^{V(x)} \, dx < \infty \). Then \( M \) is compact if (3.3) holds with \( \mu(\rho > r) \) replaced by \( p_c(r) \) for any \( c > 0 \) defined below:
\[ p_c(r) := \inf_{\lambda, \delta > 1} \exp \left\{ (c-r)\lambda + \lambda \int_1^\lambda \frac{1}{s^2} \log \left[ \frac{\delta}{\delta - 1} \left( \frac{1}{\delta s^2} \right) \right] \, ds \right\}. \]

**Proof.** By Theorem 6.1 in [19] (3.1) implies that \( \mu(e^\rho) < \infty \) and
\[ \mu(\exp[\lambda \rho]) \leq \mu(e^\rho)^\lambda \exp \left[ \lambda \int_1^\lambda \frac{1}{r^2} \log \left[ \frac{\delta}{\delta - 1} \left( \frac{1}{\delta r^2} \right) \right] \, dr \right]. \]
Therefore, \( \mu(\rho > r) \leq p_c(r) \) for \( c := \log \mu(e^\rho) \). The proof is complete by Theorem 3.1.

**Corollary 3.3.** Assume \( \int_M e^{V(x)} \, dx < \infty \) and the super Poincaré inequality (3.1) holds for the function \( \beta(r) = c_1 \exp(c_2 r^{-\alpha}) \), where \( c_1, c_2, \alpha > 0 \) are constants, and (3.2) holds. Then the manifold is compact in each of the following situations:

1. \( \alpha < 1/2 \).
2. \( \alpha = 1/2 \) and \( \limsup_{r \to \infty} \frac{r}{\log k(r)} > 2c_2 \).
3. \( \alpha > 1/2 \) and \( \limsup_{r \to \infty} \frac{r^{2/(2\alpha - 1)}}{k(r)} > (n-1)^2 \left( \frac{2\alpha}{2\alpha - 1} \right)^{4\alpha/(2\alpha - 1)} (c_2)^{2/(2\alpha - 1)}. \)

**Proof.** Part (1) is covered by Proposition 2.7. For part (2) and (3) we only need to verify that (3.3) holds for \( p_c(r) \) defined in the previous corollary. Note that for any \( \sigma_1 > \sigma_2 > 1 \), there exists \( c_3 > 0 \) such that for all \( \lambda \geq 1 \),
\[ \lambda \int_1^\lambda \frac{1}{r^2} \log \left[ \frac{\sigma_2}{\sigma_2 - 1} \beta \left( \frac{1}{(\sigma_2 r^2)} \right) \right] \, dr \leq \begin{cases} c_3 + \frac{c_2 \sigma^2}{2\alpha(2\alpha - 1)} \lambda^{2\alpha}, & \text{if } \alpha > 1/2, \\ c_3 + \frac{c_2 \sqrt{\sigma_1}}{\lambda} \log \lambda, & \text{if } \alpha = 1/2. \end{cases} \]
Then, for any \( c > 0 \) and any \( \sigma > 1 \),
\[ \log p_c(r) \leq \begin{cases} -r^{2\alpha/(2\alpha - 1)} \left( \frac{2\alpha-1}{2\alpha} \right)^{2\alpha/(2\alpha - 1)} (c_2)^{1/(2\alpha-1)}, & \text{if } \alpha > 1/2, \\ - \exp[r/(c_2 \sigma)], & \text{if } \alpha = 1/2, \\ - \exp[r/(c_2 \sigma)], & \text{if } \alpha < 1/2, \end{cases} \]
for \( c_2 \sigma > 1 \).
Thus each of (2) and (3) implies (3.3) for \( p_c(r) \) in place of \( \mu(\rho > r) \). The result now follows from Corollary 3.2.

It is known from [19] that (3.1) is equivalent to an \( F \)-Sobolev inequality (see [19] for details). In particular we consider the following generalized log-Sobolev inequality

\[
\mu(f^2 \log(f^2 + 1))^\delta \leq C_1 \mu(\|\nabla f\|^2) + C_2, \quad f \in C_0^\infty(M), \mu(f^2) = 1,
\]

(3.4)

where \( \delta, C_1, C_2 > 0 \) are constants. This leads to the next corollary. When \( \delta \neq 1 \), we will reduce the inequality to (3.1) to apply Corollary 3.3. But when \( \delta = 1 \) we will use a Herbst's argument to obtain estimates of \( \mu(\rho > r) \) directly from (3.4). Certainly in the latter case the first method also applies, but the resulting condition (3) is worse than (4) below.

**Corollary 3.4.** Assume (3.2), (3.4) and \( \int_M e^{V(x)} \, dx < \infty \). \( M \) is compact provided at least one of the following holds.

1. \( \delta > 2 \).
2. \( \delta = 2 \) and \( \lim_{r \to \infty} \frac{r \log k(r)}{\log(1 + r)} > 2 \sqrt{C_1} \).
3. \( \delta < 2 \) and \( \lim_{r \to \infty} \frac{r^{2\delta/(2-\delta)} \log k(r)}{k(r)} > \frac{(n-1)^2 C_1^2 (2-\delta)^4}{(2-\delta)^{4/(2-\delta)}} \).
4. \( \delta = 1 \) and \( \lim_{r \to \infty} \frac{r^2}{k(r)} > 4(n-1)^2 C_2^2 \).

**Proof.** We shall apply Corollary 3.3 by converting (3.4) to (3.1). Letting \( F(t) := (\log(t + 1))^\delta \), we have

\[
F^{-1}(t) = \exp[t^{1/\delta}] - 1 \leq \exp[t^{1/\delta}], \quad t > 0.
\]

By (3.4) and the proof of Theorem 3.1 in [19], we obtain

\[
(t - C_2) \mu(f^2) \leq t \sqrt{\exp[t^{1/\delta}] \mu(f^2)} + C_1 \mu(\|\nabla f\|^2)
\]

for all \( t > 0 \) and all \( f \in C_0^\infty(M) \) with \( \mu(|f|) = 1 \). This implies that

\[
\mu(f^2) \leq \frac{C_1}{(1 - \epsilon)t - C_2} \mu(\|\nabla f\|^2) + \frac{t \exp[t^{1/\delta}]}{4\epsilon}.
\]

Taking \( t = (C_1 r^{-1} + C_2)/(1 - \epsilon) \), we obtain (3.1) for

\[
\beta(r) = \frac{C_1 r^{-1} + C_2}{4\epsilon(1 - \epsilon)} \exp \left[ \frac{(C_1 r^{-1} + C_2)^{1/\delta}}{1 - \epsilon} \right], \quad r > 0,
\]

for any \( \epsilon \in (0, 1) \). The required result now follows, for \( \delta \neq 1 \) from Corollary 3.3. If \( \delta = 1 \) then (3.4) implies

\[
\mu(f^2 \log f^2) \leq C_1 \mu(\|\nabla f\|^2) + C_2, \quad f \in C_0^\infty(M), \mu(f^2) = 1,
\]

(3.5)

for some \( C_1, C_2 > 0 \). By an argument due to Herbst (cf. p. 148 in [14]), (3.5) implies \( \mu(\rho > r) \leq c \exp[-r^2 / C_1] \) for some constant \( c > 0 \). Part (4) now follows from Theorem 3.1.
References