Abstract. Geometry arising from two diffusion operators (smooth semi-elliptic, second order differential operators) on different spaces but intertwined by a smooth map is described. Particular cases arise from Riemannian submersions when the operators are Laplace-Beltrami operators, from equivariant operators on the total space of a principal bundle, and for the operators on the diffeomorphism group arising from stochastic flows. Classical non-linear filtering problems also lead to such configurations. A basic tool is the, possibly, non-linear ”semi-connection” induced by this set up, leading to a canonical decomposition of the operator on the domain space. Topics discussed include: generalised Wietzenbock curvatures arising in the equivariant case, skew -product decompositions of diffusion processes, conditioned processes, classical filtering, decomposition of stochastic flows, and connections determined by stochastic differential equations.
Introduction

Filtering is the science of finding the law of a process given a partial observation of it. The main objects we study here are diffusion processes. These are naturally associated with second order linear differential operators which are semi-elliptic and so introduce a possibly degenerate Riemannian structure on the state space. In fact much of what we discuss is simply about two such operators intertwined by a smooth map, the “projection from the state space to the observations space”, and does not involve any stochastic analysis.

From the point of view of stochastic processes our purpose is to present and to study the underlying geometric structure which allows us to perform the filtering in a Markovian framework with the resulting conditional law being that of a Markov process. This geometry is determined by the symbol of the operator on the state space which projects to a symbol on the observation space. The projectible symbol induces a (possibly non-linear and partially defined) connection which lifts the observation process to the state space and gives a decomposition of the operator on the state space and of the noise. As is standard we can recover the classical filtering theory in which the observations are not usually Markovian by application of the Girsanov-Maruyama-Cameron-Martin Theorem.

This structure we have is examined in relation to a number of geometrical topics. In one direction this leads to a generalisation of Hermann’s theorem on the fibre bundle structure of certain Riemannian submersions. In another it gives a novel description of generalised Weitzenböck curvature. It also applies to infinite dimensional state spaces such as arise naturally for stochastic flows of diffeo-

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morphism defined by stochastic differential equations, and for certain stochastic partial differential equations.

Let $M$ be a smooth manifold. Consider a smooth second order semi-elliptic differential operator $L$ such that $L1 \equiv 0$. In a local chart, such an operator takes the following form

$$L = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + \sum b^i \frac{\partial}{\partial x^i}$$  \hspace{1cm} (1)$$

where the $a_{ij}$'s and $b^i$'s are smooth functions and the matrix $(a_{ij})$ is positive semi-definite.

Such differential operators are called diffusion operators. An elliptic diffusion operator induces a Riemannian metric on $M$. In the degenerate case we shall have to assume that the “symbol” of $L$ (essentially the matrix $[a_{ij}]$ in the representation (1)) has constant rank and so determines a sub-bundle $E$ of the tangent bundle $TM$ together with a Riemannian metric on $E$. In Elworthy-LeJan-Li [26] and [27] it was shown that a diffusion operator in Hörmander form, satisfying this condition, induces a linear connection on $E$ which is adapted to the Riemannian metric induced on $E$, but not necessarily torsion free. It was also shown that all metric connections on $E$ can be constructed by some choice of Hörmander form for a given $L$ in this way. The use of such connections has turned out to be instrumental in the decomposition of noise and calculation of covariant derivatives of the derivative flows.

A related construction of connections can extend to principal fibre bundles $P$, indeed to more general situations, such as foliated manifolds and stratified manifolds. An equivariant differential operator on $P$ induces naturally a diffusion operator on the base manifold. Conversely given a connection on $P$ one can lift horizontally a diffusion operator on the base manifold of the form of sum of squares of vector fields by simply lifting up the vector fields. It still need to be shown that the lift is independent of choices of its Hörmander form. Consider now a diffusion operator not given in Hörmander form. Since it has no zero order term we can associate with it an operator $\delta$ which send differential one forms to functions. In Proposition 1.2.1 a class of such operators are described, each of which determines a diffusion operator. Horizontal lifts of diffusion operators can then be defined in terms of the $\delta$ operator. This construction extends to situations where there is no equivariance and we have only partially defined and non-linear connections.
The connections discussed here arise in much more general situations, including for foliations though these are not discussed in this volume. We show that given a smooth $p : N \to M$: a diffusion operator $\mathcal{B}$ on $N$ which lies over a diffusion operator $\mathcal{A}$ on $M$ satisfying a "cohesiveness" property gives rise to a semi-connection, a partially defined, non-linear, connection which can be characterised by the property that, with respect to it, $\mathcal{B}$ can be written as the direct sum of the horizontal lift of its induced operator and a vertical diffusion operator. Of particular importance are examples where $p : N \to M$ is a principal bundle. In that case the vertical component of $\mathcal{B}$ induces differential operators on spaces of sections of associated vector bundles: we observe that these are zero-order operators, and can have geometric significance.

This geometric significance and the relationship between these partially defined connections and the metric connections determined by the Hörmander form as in [26] and [27] is seen when taking $\mathcal{B}$ to be the generator of the diffusion given on the frame bundle $GLM$ of $M$ by the action of the derivative flow of a stochastic differential equation on $M$. The semi-connection determined by $\mathcal{B}$ is then equivariant and is the adjoint of the metric connection induced by the SDE in a sense extending that of Driver [17] and described in [27]. The zero-order operators induced on differential forms as mentioned above turn out to be generalised Weitzenböck curvature operators, in the sense of [27], reducing to the classical ones when $M$ is Riemannian for particular choices of stochastic differential equations for Brownian motion on $M$. Our filtering then reproduces the conditioning results for derivatives of stochastic flows in [29] and [27].

Our approach is also applied to the case where $M$ is compact and $N$ is its diffeomorphism group, $Diff(M)$, with $P$ evaluation at a chosen point of $M$. The operator $\mathcal{B}$ is taken to be the generator of the diffusion process on $Diff(M)$ arising from a stochastic flow. However our constructions can be made in terms of the reproducing Hilbert space of vector fields on $M$ defined by the flow. From this we see that stochastic flows are essentially determined by a class of semi-connections on the bundle $p : Diff(M) \to M$ and smooth stochastic flows whose one point motions have a cohesive generator determine semi-connections on all natural bundles over $M$. Apart from these geometrical aspects of stochastic flows we also obtain a skew product decomposition which, for example, can be used to find conditional expectations of functionals of such flows given knowledge of the one point motion from our chosen point in $M$.

A feature of our approach is that in general we use canonical processes as solutions of martingale problems to describe our processes, rather than stochastic differential equations and semi-martingale calculus, unless we are explicitly deal-
ing with the latter. This leads to some new constructions, for example of integrals along the paths of our diffusions in Section 4.1, which are valid more generally than in the very regular cases we discuss here.

In more detail: In Chapter One we describe various representations of diffusion operators and when they are available. We also define the notion of such an operator being along a distribution. In Chapter Two we introduce the notion of semi-connection which is fundamental for what follows, show how these are induced by certain intertwined pairs of diffusion operators and how they relate to a canonical decomposition of such operators. We also have a first look at the topological consequences on \( p : N \to M \) of having \( B \) on \( N \) over some \( A \) on \( M \) which posses hypo-ellipticity type properties. This is a minor extension of part of Hermann’s theorem, [37], for Riemannian submersions. In Chapter Three we specialise to the case of principal bundles, introduce the example of derivative flow, and show how the generalised Wietzenbock curvatures arise.

It is not really until Chapter Four that stochastic analysis plays a major role. Here we describe methods of conditioning functionals of the \( B \)-process given information about its projection onto \( M \). We also use our decomposition of \( B \) and resulting decomposition of the \( B \)-process to describe the conditional \( B \)-process. In the equivariant case of principal bundles the decomposition of the process can be considered as a skew product decomposition. In Chapter 5 we show how our constructions can apply to classical filtering problems, where the projection of the \( B \)-process is non-Markovian. We can follow the classical approach and obtain, in Theorem 5.9, a version of Kushner’s formula for non-linear filtering in somewhat greater generality than is standard. This requires some discussion of analogues of innovations processes in our setting.

We return to more geometrical analysis in Chapter Six, giving further extensions of Hermann’s theorem and analysing the consequences of the horizontal lift of \( A \) commuting with \( B \), thereby extending the discussion in [7]. In particular we see that such commutativity, plus hypo-ellipticity conditions on \( A \), gives a bundle structure and a diffusion operator on the fibre which is preserved by the trivialisations of the bundle structure. This leads to an extension of the "skew-product" decomposition given in [24] for Brownian motions on the total space of Riemannian submersions with totally geodesic fibres. In fact the well known theory for Riemann submersions, and the special case arising from Riemannian symmetric spaces is presented in Chapter Seven.

Chapter Eight is where we describe the theory for the diffeomorphism bundle \( p : \text{Diff}(M) \to M \) with a stochastic flow of diffeomorphism on \( M \). Initially this is done independently of stochastic analysis and in terms of reproducing kernel
Hilbert spaces of vector fields on $M$. The correspondence between such Hilbert spaces and stochastic flows is then used to get results for flows and in particular skew-product decompositions of them.

In the Appendices we present the Girsanov Theorem in a way which does not rely on having to use conditions such as Novikov’s criteria for it to remain valid. This has been known for a long time, but does not appear to be as well known as it deserves. We also look at conditions for degenerate, but smooth, diffusion operators to have smooth Hörmander forms, and so to have stochastic differential equation representations for their associated processes. Finally we discuss semi-martingales and $\Gamma$-martingales along a sub-bundle of the tangent bundle with a connection.

For Brownian motions on the total spaces of Riemannian submersions much of our basic discussion, as in the first two and a half Chapters, of skew-product decompositions is very close to that in [24] which was taken further by Liao in [48]. A major difference from Liao’s work is that for degenerate diffusions we use the semi-connection determined by our operators rather than an arbitrary one, so obtaining canonical decompositions. The same holds for the very recent work of Lazaro-Cami & Ortega, [44] where they are motivated by the reduction and reconstruction of Hamiltonian systems and consider similar decompositions for semi-martingales. An extension of [24] in a different direction, to shed light on the Fadeev-Popov procedure for gauge theories in theoretical physics was given by Arnaudon & Paycha in [1]. Much of the equivariant theory presented here was announced with some sketched proofs in [25].

Key Words
semi-elliptic, second order differential operator, Hörmander forms, connection, semi-connection, diffusion processes, Girsanov theorem, intertwined diffusions, conditioned laws, filtering, Weitzenböck curvature, skew-product decomposition, stochastic flows, manifolds, Riemannian submersions, bundles, principal bundles, Diffeomorphism bundles.
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Chapter 1

Diffusion Operators

If $\mathcal{L}$ is a second order differential operator on a manifold $M$, denote by $\sigma^\mathcal{L} : T^*M \to TM$ its symbol determined by

$$df \left( \sigma^\mathcal{L}(dg) \right) = \frac{1}{2} \mathcal{L}(fg) - \frac{1}{2}(\mathcal{L}f)g - \frac{1}{2}f(\mathcal{L}g),$$

for $C^2$ functions $f, g$. We will often write $\sigma^\mathcal{L}(\ell_1, \ell_2)$ for $\ell_1 \sigma^\mathcal{L}(\ell_2)$ and consider $\sigma^\mathcal{L}$ as a bilinear form on $T^*M$. Note that it is symmetric. The operator is said to be semi-elliptic if $\sigma^\mathcal{L}(\ell_1, \ell_2) \geq 0$ for all $\ell_1, \ell_2 \in T_uM^*$, all $u \in M$, and elliptic if the inequality holds strictly. Ellipticity is equivalent to $\sigma^\mathcal{L}$ being onto.

**Definition 1.0.1** A semi-elliptic smooth second order differential operator $\mathcal{L}$ is said to be a diffusion operator if $\mathcal{L}1 = 0$.

### 1.1 Representations of Diffusion Operators

Apart from local representations as given by equation 1 there are several global ways to represent a diffusion operator $\mathcal{L}$. One is to take a connection $\nabla$ on $TM$. Recall that a connection on $TM$ gives, or is given by, a covariant derivative operator $\nabla$ acting on vector fields. For each $C^r$ vector field $U$ on $M$ it gives a $C^{r-1}$ section $\nabla_u U$ of $\mathbb{L}(TM; TM)$. In other words for each $x \in M$ we have a linear map $v \mapsto \nabla_v U$ of $T_xM$ to itself. This covariant derivative of $U$ in the direction $v$ satisfies the usual rules. In particular it is a derivation with respect to multiplication by differentiable functions $f : M \to \mathbb{R}$, so that $\nabla_v f U = df(v)U(x) + f(x) \nabla_v U$. Given any smooth vector bundle $\tau E \to M$
over \( M \) a \textit{connection on} \( E \) gives a similar covariant derivative acting on sections \( U \) of \( E \). This time \( v \to \nabla_v U \) is in \( \mathbb{L}(T_x M; E_x) \), where \( E_x \) is the fibre over \( x \) for \( x \in M \). Such connections always exist.

Then we can write

\[
\mathcal{L} f(x) = \text{trace}_{T_x M} \nabla (\sigma^\mathcal{L} (df)) + df(V^0(x))
\]

(1.1)

for some smooth vector field \( V^0 \) on \( M \). The trace is that of the mapping \( v \mapsto \nabla_v (\sigma^\mathcal{L} (df)) \) from \( T_x M \) to itself. To see this it is only necessary to check that the right hand side has the correct symbol since the symbol determines the diffusion operator up to a first order term.

If a smooth ‘square root’ to \( 2\sigma^\mathcal{L} \) can be found we have a Hörmander representation. The ‘square root’ is a smooth \( X : M \times \mathbb{R}^m \to TM \) with each \( X(x) \equiv X(x, -) : \mathbb{R}^m \to T_x M \) linear, such that

\[
2\sigma^\mathcal{L}_x = X(x)X(x)^* : T^*_x M \to T_x M.
\]

Thus there is a smooth vector field \( A \) with

\[
\mathcal{L} = \frac{1}{2} \sum_{j=1}^m L_{X_j} L_{X_j} + L_A,
\]

(1.2)

where \( L_V \) denotes Lie differentiation with respect to a vector field \( V \), so \( L_V f(x) = df_x(V(x)) \), and \( X^j(x) = X(x)(e_j) \) for \( \{e_j\} \) an orthonormal basis of \( \mathbb{R}^m \). \textbf{If} \( \sigma^\mathcal{L} \) \textbf{has constant rank such} \( X \textbf{ may be found}. \) Otherwise it is only known that locally Lipschitz square roots exist (see the discussions in Appendix A). In that case \( L_{X_j} L_{X_j} \) is only defined almost surely everywhere and the vector field \( A \) can only be assumed measurable and locally bounded. Nevertheless uniqueness of the martingale problem still holds (see below). Also there is still the hybrid representation, given a connection \( \nabla \) on \( TM \):

\[
\mathcal{L} f(x) = \frac{1}{2} \sum_{j=1}^m \nabla_{X^j(x)} (df)(X^j(x)) + df(V^0(x)).
\]

(1.3)

for \( V^0 \) locally Lipschitz.

The choice of a Hörmander representation for a diffusion operator, if it exists, determines a locally defined stochastic flow of diffeomorphisms \( \{\xi_t : 0 \leq t < \zeta\} \) whose one point motion solves the martingale problem for the diffusion operator.
1.2. THE ASSOCIATED FIRST ORDER OPERATOR

In particular on bounded measurable compactly supported $f : M \to \mathbb{R}$ the associated (sub)Markovian semigroup is given by $P_t f = E(f \circ \xi_t)$. See also Appendix II.

Despite the discussion above we can always write $\mathcal{L}$ in the following form:

$$\mathcal{L} = \sum_{ij=1}^N a_{ij}(\cdot)\mathbf{L}_{X^i} \mathbf{L}_{X^j} + \mathbf{L}_{X^0},$$

(1.4)

where $N$ is a finite number, $a_{ij}$ and $X^k$ are respectively smooth functions and smooth vector fields with $a_{ij} = a_{ji}$.

1.2 The Associated First Order Operator

Denote by $C^r \Lambda^p \equiv C^r \Lambda^p T^* M$, $r \geq 0$, the space of $C^r$ smooth differential $p$-forms on a manifold $N$. To each diffusion operator $\mathcal{L}$ we shall associate an operator $\delta^\mathcal{L}$, see Elworthy-LeJan-Li [26], [27] c.f. Eberle [19]. The horizontal lift of $\mathcal{L}$ will then be defined in terms of a lift of $\delta^\mathcal{L}$.

Proposition 1.2.1 For each diffusion operator $\mathcal{L}$ there is a unique smooth linear differential operator $\delta^\mathcal{L} : C^{r+1} \Lambda^1 \to C^r \Lambda^0$ such that

1. $\delta^\mathcal{L} (f \phi) = df \sigma^\mathcal{L}(\phi) + f \cdot \delta^\mathcal{L} (\phi)$

2. $\delta^\mathcal{L} (df) = \mathcal{L} f$.

Equivalently $\delta^\mathcal{L}$ is determined by either one of the following:

$$\delta^\mathcal{L} (f dg) = \sigma^\mathcal{L}(df, dg) + f \mathcal{L} g$$

(1.5)

$$\delta^\mathcal{L} (f dg) = \frac{1}{2} \mathcal{L} (fg) - \frac{1}{2} g \mathcal{L} f + \frac{1}{2} f \mathcal{L} g.$$  

(1.6)

Proof. Take a connection $\nabla$ on $TM$ then, as in (1.1), $\mathcal{L}$ can be written as $\mathcal{L} f = \text{trace} \nabla \sigma^\mathcal{L}(df) + \mathbf{L}_{V^0} f$ for some smooth vector field $V^0$. Set

$$\delta^\mathcal{L} \phi = \text{trace} \nabla (\sigma^\mathcal{L} \phi) + \phi(V^0).$$

Then $\delta^\mathcal{L} (df) = \mathcal{L} f$ and

$$\delta^\mathcal{L} (f \phi) = \text{trace} \nabla (f(\sigma^\mathcal{L} \phi)) + f \phi(V^0) = f \delta^\mathcal{L} \phi + df (\sigma^\mathcal{L} \phi).$$
CHAPTER 1. DIFFUSION OPERATORS

Note that a general $C^r$ 1-form $\phi$ can be written as $\phi = \sum_{j=1}^{k} f_j dq_j$ for some $C^r$ function $f_j$ and smooth $q_j$, for example, by taking $(g^1, \ldots, g^m) : M \to \mathbb{R}^m$ to be an immersion. This shows that (1) and (2) determine $\delta^\mathcal{L}$ uniquely. Moreover since $\mathcal{L}$ is a smooth operator so is $\delta^\mathcal{L}$.  

**Remark 1.2.2** If the diffusion operator $\mathcal{L}$ has a representation

$$\mathcal{L} = \sum_{j=1}^{m} a_{ij} L_{X^j} L_{X^i} + L_{X^0}$$

for some smooth vector fields $X^i$ and smooth functions $a_{ij}, i, j = 0, 1, \ldots, m$ then

$$\delta^\mathcal{L} = \sum_{j=1}^{m} a_{ij} L_{X^j} \iota_{X^j} + \iota_{X^0},$$

where $\iota_A$ denotes the interior product of the vector field $A$ with a differential form. One can check directly that $\delta^\mathcal{L}(df) = \mathcal{L}f$ and that (1) holds. In particular in a local chart, for the representation given in equation (1) we see that $\delta^\mathcal{L}$ is given by

$$\delta^\mathcal{L} \phi = \sum_{j=1}^{m} a_{ij} \frac{\partial}{\partial x^i} \phi_j(x) + \sum b^i \phi_i(x)$$

where $\phi$ has the representation

$$\phi_x = \sum \phi_j(x) \, dx^i$$

### 1.3 Diffusion Operators Along a Distribution

Let $N$ be a smooth manifold. By a **distribution** $S$ in $N$ we mean a family $\{S_u : u \in N\}$ where $S_u$ is a linear subspace of $T_u N$; for example $S$ could be a subbundle of $TN$. Given such a distribution $S$ let $S^0 = \cup_u S^0_u$ for $S^0_u$ the **annihilator** of $S_u$ in $T^*_u N$.

**Definition 1.3.1**

Let $S$ be a distribution in $TN$. Denote by $C^r S^0$ the set of $C^r$ 1-forms which vanish on $S$. A diffusion operator $\mathcal{L}$ on $N$ is said to be **along** $S$ if $\delta^\mathcal{L} \phi = 0$ for $\phi \in C^1 S^0$. 

Suppose $L$ is along $S$ and take $\phi \in C^r S^0$. By Proposition 1.2.1 and the symmetry of $\sigma L$, $0 = (df)(\sigma L(\phi)) = \phi(\sigma L(df))$ giving $\phi_x \in \text{Image}[\sigma L^0]$. This proves Remark 1.3.2 (i):

**Remark 1.3.2**

(i) if $\delta L \phi = 0$ for all $\phi \in C^1 S^0$, then $\sigma L \phi = 0$ for all such $\phi$ and $\text{Image}[\sigma L^0] \subset \cap_{\phi \in C^1 S^0} \ker \phi_x$ for all $x \in N$.

(ii) If $S$ is a sub-bundle of $TN$ and $L$ is along $S$ then without ambiguity we can define $\delta L \phi$ for $\phi$ a $C^0$ section of $S^*$ by $\delta L \phi := \delta L \tilde{\phi}$ for any 1-form $\tilde{\phi}$ extending $\phi$. Recall that $S^*$ is canonically isomorphic to the quotient $T^* N/S^0$.

**Definition 1.3.3** If

$$S_x = \cap_{\phi \in C^1 S^0} \ker \phi_x$$

for all $x$ we say $S$ is a **regular distribution**.

Clearly sub-bundles are regular.

**Proposition 1.3.4**

(1) Let $S$ be a regular distribution of $N$ and $L$ an operator written in Hörmander form:

$$L = \frac{1}{2} \sum_{j=1}^{m} L_{Y_j} L_{Y_j} + L_{Y_0} \quad (1.7)$$

where the vector fields $Y_0$ and $Y_j, j = 1, \ldots, m$ are $C^0$ and $C^1$ respectively. Then $L$ is along $S$ if and only if $Y_i$ are sections of $S$.

(2) If $B$ is along a smooth sub-bundle $S$ of $TN$ then for any connection $\nabla^S$ on $S$ we can write $B$ as

$$B f = \text{trace}_{S_x} \nabla^S (\sigma^B(df)) + L_{X^0} f.$$ 

Also we can find smooth sections $X^0, \ldots, X^m$ of $S$ and smooth functions $a_{ij}$ such that

$$B = \sum_{i,j} a_{ij}(\cdot)L_{X_i} L_{X_j} + L_{X^0}.$$ 

**Proof.** For part (1), if $Y_i$ are sections of $S$, take $\phi \in C^1 S^0$ then

$$\delta L \phi = \frac{1}{2} \sum_{j=1}^{m} L_{Y_j} \phi(Y^j) + \phi(Y^0) = 0$$
and so $\mathcal{L}$ is along $S$.

Conversely suppose $\mathcal{L}$ is along $S$. Define a $C^1$ bundle map $Y : \mathbb{R}^m \to TN$ by $Y(x)(e) = \sum_{j=1}^m Y^j(x)e_j$ for $\{e_j\}_{j=1}^m$ an orthonormal base of $\mathbb{R}^m$. Then

$$2\sigma^\mathcal{L}_x = Y(x)Y(x)^*$$

and

$$\text{Image}[Y(x)] = \text{Image}[\sigma^\mathcal{L}_x] \subset S,$$

by Remark 1.3.2. Now

$$\delta^\mathcal{L}_x \phi = \frac{1}{2} \sum L_{Y^j}(\phi(Y^j)) + \phi(Y^0) = \phi(Y^0),$$

which can only vanish for all $\phi \in C^1 S^0$ if $Y^0$ is a section of $S$. Thus $Y^1, \ldots, Y^m$, and $Y^0$ are all sections of $S$.

For part (2), we use (1.1) and take $\nabla$ there to be the direct sum of $\nabla^S$ with an arbitrary connection on a complementary bundle, obtaining $\sigma^B$ has image in $S$ by Remark 1.3.2(i).

1.4 Lifts of Diffusion Operators

Let $p : N \to M$ be a smooth map and $E$ a sub-bundle of $TM$. Let $S$ be a sub-bundle of $TN$ transversal to the fibre of $p$, i.e. $VT_yN \cap S = \{0\}$ all $u \in N$ and such that $T_y p$ maps $S_y$ isomorphically onto $E_{p(y)}$, for each $y$.

**Lemma 1.4.1** Every smooth 1-form on $N$ can be written as a linear combination of sections of the form $\psi + \lambda p^*(\phi)$ for $\lambda : N \to \mathbb{R}$ smooth, $\phi$ a 1-form on $M$, and $\psi$ annihilates $S$. In particular any 1-form annihilating $VTN$ is of the form $\lambda p^*(\phi)$. If $E = TM$ then $\psi$ is uniquely determined.

**Proof.** Take Riemannian metrics on $M$ and $N$ such that the isomorphism between $S$ and $p^*(E)$ given by $Tp$ is isometric. Fix $y_0 \in N$. Take a neighbourhood $V$ of $p(y_0)$ in $M$ over which $E$ is trivializable. Let $\{v^1, v^2, \ldots, v^p\}$ be a trivialising family of sections over $V$. Set $U = p^{-1}(V)$. If $\phi^j = (v^j)^*$, the dual 1-form to $v^j$, $j = 1$ to $p$, over $V$ then $\{p^*(\phi^j)^\#, j = 1$ to $p\}$ gives a trivialization of $S$ over $U$. [Indeed $p^*(\phi^j)_y(-) = \phi^j_{p(y)}(T_y p - ) = ((T_y p)^*(v^j), -).$] Since any vector field over $V$ can therefore be written as one orthogonal to $S$ plus a linear combination of the $p^*(\phi^j)^\#$, by duality the result holds for forms with support in $U$. The global result follows using a partition of unity.

For the uniqueness note that if $E = TM$ then $TN = VTN + S$. 

\[\square\]
Proposition 1.4.2 Let $\mathcal{A}$ be a diffusion operator on $M$ along the sub-bundle $E$ of $TM$. There is a unique lift of $\mathcal{A}$ to a smooth diffusion generator $\mathcal{A}^S$ along the transversal bundle $S$. Write $\bar{\delta} = \delta^{A^S}$. Then $\mathcal{A}^S$ is determined by

(i) $\bar{\delta}(\psi) = 0$ if $\psi$ annihilates $S$.

(ii) $\bar{\delta}(p^* \phi) = (\delta^A \phi) \circ p$, for $\phi \in \Omega^1(M)$.

Moreover (iii) for $y \in N$ let $h_y : E_{p(y)} \to T_y N$ be the right inverse of $T_y p$ with image $S_y$. Then

(a) $\sigma^A_{y^*} = h_y \sigma^A h_y^*$

(b) If $\mathcal{A}$ is given by

$$\mathcal{A} = \sum_{i,j=1}^{N} a_{ij} L_{X_i} L_{X_j} + L_{X^0}$$

(1.8)

where $X^1, \ldots, X^N$ and $X^0$ are sections of $E$ then

$$\mathcal{A}^S = \sum_{i,j=1}^{N} (a_{ij} \circ p) L_{\bar{X}_i} L_{\bar{X}_j} + L_{\bar{X}^0}$$

(1.9)

for $\bar{X}_j(y) = h_y(X_j(p(y)))$.

Proof. Lemma 1.4.1 ensures that (i) and (ii) determine $\bar{\delta}$ uniquely as a smooth operator on smooth 1-forms if it exists. On the other hand we can represent $\mathcal{A}$ as in (1.8) and define $\mathcal{A}^S$ be (1.9). It is straightforward to check that then $\delta^{A^S}$ satisfies (i) and (ii).

By definition and the observation after (1.9) this must be the horizontal lift, if it is a diffusion generator. On the other hand if $\mathcal{A}$ is given by (1.8) we use it to define $\mathcal{A}^S$ by (1.9). It is easy to see that $\delta^{A^S}$ satisfies (i) and (ii) and so $\delta^{A^S} = \bar{\delta}$. From this $\bar{\mathcal{A}} = A^S$ and $A^S$ is a smooth diffusion generator.

In the terminology of section 1.3 $S_u = \ker[T_u p]$, sometimes written as $VT_u N$, is a distribution.

Definition 1.4.3 When an operator $\mathcal{B}$ is along the vertical distribution $\ker[T p]$ we say $\mathcal{B}$ is vertical, and when there is a horizontal distribution such as $\{H_u : u \in N\}$ as given by Proposition 2.1.2 below and $\mathcal{B}$ is along that horizontal distribution we say $\mathcal{B}$ is horizontal.
**Proposition 1.4.4** Let $B$ be a smooth diffusion operator on $N$ and $p : N \to M$ any smooth map, then the following conditions are equivalent:

1. The operator $B$ is vertical.

2. The operator $B$ has a expression of the form of $\sum_{j=1}^m a^{ij}L_{Y^i}L_{Y^j} + L_{Y^0}$ where $a^{ij}$ are smooth functions and $Y^j$ are smooth sections of the vertical tangent bundle of $TN$.

3. $B(f \circ p) = 0$ for all $C^2$ functions $f : M \to \mathbb{R}$.

**Proof.** (a). From (1) to (3) is trivial. From (3) to (1) note that every $\phi$ which vanishes on vertical vectors is a linear combination of elements of the form $fp^*(dg)$ for some smooth $g : M \to \mathbb{R}$ by Lemma 1.4.1. To show that $B$ is vertical we only need to show that $\delta B(fp^*(dg)) = 0$. But $B(g \circ p) = 0$ implies $\delta B(p^*(dg)) = 0$ and also $p^*(dg)\sigma B(p^*(dg)) = \frac{1}{2}B(g \circ p)^2 - (g \circ p)B(g \circ p) = 0$. By semi-ellipticity of $B$, $\sigma B(p^*(dg)) = 0$. Thus assertion (1) follows since $\delta B(fp^*(dg)) = df\sigma B(p^*(dg)) + f\cdot \delta B(p^*(dg))$ from Proposition 1.2.1(1), and so (1) and (3) are equivalent.

Equivalence of (1) and (2) follows from Proposition 1.3.4.

**Remark 1.4.5** (1) If $B$ is vertical, then by Proposition 1.2.1, for all $C^2$ functions $f_1$ on $N$ and $f_2$ on $M$, $B(f_1(f_2 \circ p)) = (f_2 \circ p)Bf_1$;

(2) If $B$ and $B'$ are both over a diffusion operator $A$ of constant rank nonzero rank such that $A$ is along the image of $\sigma A$, then $B - B'$ is not in general vertical, although $(B - B')(f \circ p) = 0$ for all $C^2$ function $f : M \to \mathbb{R}$, since it may not be semi-elliptic. For example take $p : \mathbb{R}^2 \to \mathbb{R}$ to be the projection $p(x, y) = x$ with $A = \frac{\partial^2}{\partial x^2}$, $B = \frac{\partial^3}{\partial x^2} + \frac{\partial}{\partial y}$. Let $B' = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x \partial y}$. Then $B$ is also over $A$ but $B - B' = -\frac{\partial^2}{\partial x \partial y}$ is not vertical.
Chapter 2
Decomposition of Diffusion Operators

Consider a smooth map \( p : N \to M \) between smooth manifolds \( M \) and \( N \). By a lift of a diffusion operator \( \mathcal{A} \) on \( M \) over \( p \) we mean a diffusion operator \( \mathcal{B} \) on \( N \) such that

\[
\mathcal{B}(f \circ p) = (\mathcal{A}f) \circ p
\]

for all \( C^2 \) functions \( f \) on \( M \). In this situation we adopt the following terminology:

**Definition 2.0.6** If (2.1) holds we say that \( \mathcal{B} \) is over \( \mathcal{A} \), or that \( \mathcal{A} \) and \( \mathcal{B} \) are intertwined by \( p \). A diffusion operator \( \mathcal{B} \) on \( N \) is said to be projectible (over \( p \)), or \( p \)-projectible, if it is over some diffusion operator \( \mathcal{A} \).

Recall that the pull back \( p^* \phi \) of a 1-form \( \phi \) is defined by

\[
p^*(\phi)_u = \phi_{p(u)}(Tp(-)) = (Tp)^*\phi_{p(u)}.
\]

For our map \( p : N \to M \), a diffusion operator \( \mathcal{B} \) is over \( \mathcal{A} \) if and only if

\[
\delta^B(p^*\phi) = (\delta^A\phi)(p),
\]

for all \( \phi \in C^1 \wedge^1 T^*M \).

### 2.1 The Horizontal Lift Map

**Lemma 2.1.1** Suppose that \( \mathcal{B} \) is over \( \mathcal{A} \). Let \( \sigma^B \) and \( \sigma^A \) be respectively the symbols for \( \mathcal{B} \) and \( \mathcal{A} \). Then

\[
(T_u p)\sigma^B_u (T_u p)^* = \sigma^A_{p(u)}, \quad \forall u \in N,
\]

(2.3)
\section*{CHAPTER 2. DECOMPOSITION OF DIFFUSION OPERATORS}

\textit{i.e.} the following diagram is commutative:

\[ \begin{array}{ccc}
T_u^*N & \xrightarrow{\sigma_u^B} & T_uN \\
(T_u p)^* & \Downarrow & \Downarrow \\
T_{p(u)}^*M & \xrightarrow{\sigma_{p(u)}^A} & T_{p(u)}M.
\end{array} \]

\textbf{Proof.} Let $f$ and $g$ be two smooth functions on $M$. Then for $u \in N$, $x = p(u)$,

\[(df_x) \sigma_x^A (dg_x) = \frac{1}{2} A(fg)(x) - \frac{1}{2} (f A g)(x) - \frac{1}{2} (g A f)(x) \]

\[= \frac{1}{2} B((fg) \circ p)(u) - \frac{1}{2} f \circ p B(g \circ p)(u) - \frac{1}{2} g \circ p B(f \circ p)(u)\]

\[= d (g \circ p)_u \sigma_u^B (d (f \circ p)_u) \]

\[= (dg \circ T_u p) \sigma_u^B (df \circ T_u p),\]

which gives the desired equality. \hfill \square

For $x$ in $M$, set $E_x := \text{Image}[^A_x \sigma_x] \subset T_x M$. If $\sigma^A$ has constant rank, \textit{i.e.} \(\dim[E_x]\) is independent of $x$, then $E := \bigcup_x E_x$ is a smooth sub-bundle of $TM$.

\textbf{Proposition 2.1.2} Assume $\sigma^A$ has constant rank and $B$ is over $A$. Then there is a unique, smooth, horizontal lift map $h_u : E_{p(u)} \rightarrow T_u N$, $u \in N$, characterised by

\[h_u \circ \sigma_{p(u)}^A = \sigma_u^B(T_u p)^*. \tag{2.4}\]

In particular

\[h_u(v) = \sigma_u^B((T_u p)^*(\alpha)) \tag{2.5}\]

where $\alpha \in T_{p(u)}^* M$ satisfies $\sigma_{p(u)}^A(\alpha) = v$.

\textbf{Proof.} Clearly (2.5) implies (2.4) by Lemma 2.1.1 and so it suffices to prove $h_u$ is well defined by (2.5). For this we only need to show $\sigma^B((T_u p)^*(\alpha)) = 0$ for every $\alpha$ in $\ker[\sigma_{p(u)}^A]$. Now $\sigma^A \alpha = 0$ implies that

\[(T p)^*(\alpha) \sigma^B((T p)^*(\alpha)) = 0,\]

by Lemma 2.1.1. Considering $\sigma^B$ as a semi-definite bilinear form this implies $\sigma_u^B(T_u p)^* \alpha$ vanishes as required. \hfill \square
Note that the vertical distribution \( \ker [T_p] \) is regular as \( \ker [T_p] \) is annihilated by all differential 1-forms of the form \( \theta \circ T_p \).

Let \( H_u = \text{Image}[h_u] \) and \( H = \sqcup_u H_u \). Set \( F_u = (Tu)^{-1}[E_{p(u)}] \) so we have a splitting
\[
F_u = H_u + VT_u N \tag{2.6}
\]
where \( VT_u N = \ker [T_u P] \) the ‘vertical’ tangent space at \( u \) to \( N \). In the elliptic case \( p \) is a submersion, the vertical tangent spaces have constant rank, and \( F := \sqcup_u F_u \) is a smooth sub-bundle of \( TN \). In this case we have a splitting of \( TN \), a \textbf{connection} in the terminology of Kolar-Michor-Slovak [42]. In general we will define a \textbf{semi-connection} on \( E \) to be a sub-bundle \( H_u \) of \( TN \) such that \( T_u p \) maps each fibre \( H_u \) isomorphically to \( E_{p(u)} \). In the equivariant case considered in Chapter 3 such objects are called \( E \)-connections by Gromov. For the case when \( : N \to M \) is the tangent bundle projection , or the orthonormal frame bundle note that the ”partial connections” as defined by Ge in [35] are rather different from the semi-connections we would have: they give parallel translations along \( E \)-horizontal paths which send vectors in \( E \) to vectors in \( E \), and preserve the Riemannian metric of \( E \), whereas the parallel transports of our semi-connections do not in general preserve the fibres of \( E \), nor any Riemannian metric, and they act on all tangent vectors.

\textbf{Lemma 2.1.3} Assume \( \sigma^A \) has constant rank and \( B \) is over \( A \). For all \( u \in N \) the image of \( \sigma^B_u \) is in \( F_u \).

\textbf{Proof.} Suppose \( \alpha \in T_u^* N \) with \( \sigma^B(\alpha) \notin F_u \). Then there exists \( k \) in the annihilator of \( E_{p(u)} \) such that \( k (Tu p \sigma^B(\alpha)) \neq 0 \). However
\[
k (Tu p \sigma^B(\alpha)) = \alpha (\sigma^B ((Tu p)^*(k))) = \alpha h_u \sigma^A_{p(u)}(k)
\]
by Proposition 2.1.2; while \( \sigma^A_{p(u)}(k) = 0 \) because for all \( \beta \in T^*_{p(u)} M \),
\[
\beta \sigma^A_{p(u)}(k) = k \sigma^A_{p(u)}(\beta) = 0
\]
giving a contradiction. \( \square \)

\textbf{Proposition 2.1.4} Let \( A \) be a diffusion operator on \( M \) with \( \sigma^A \) of constant rank. For \( i \in \{1, 2\} \), let \( p^i : N^i \to M \) be smooth maps and \( B^i \) be diffusion operators on \( N^i \) over \( A \). Let \( F : N^1 \to N^2 \) be a smooth map with \( p^2 \circ F = p^1 \). Assume \( F \)
interwines $B^1$ and $B^2$. Let $h^1, h^2$ be the horizontal lift maps determined by $A, B^1$ and $A, B^2$. Then

$$h^2_{F(u)} = T_u F(h^1_u), \quad u \in N^1; \quad (2.7)$$

i.e. the diagram

$$\begin{CD}
T_u N^1 @>>> T_u F @>>> T_{F(u)} N^2 \\
@VV{h^1_u}V @VV{h^2_{F(u)}}V @VV{E_{p^1(u)}}V \\
E_{p^1(u)} \end{CD}$$

commutes for all $u \in N$.

**Proof.** Since $F$ intertwines $B^1$ and $B^2$, Lemma 2.1.1 gives

$$\sigma_{F(u)}^{B^2} = T_u F \circ \sigma_{u}^{B^1} \circ (T_u F)^*.$$

Now take $\alpha \in T^*_{p^1(u)} M$ with $\sigma_{p^1(u)}^A(\alpha) = v$, some given $v \in E_{p^1(u)}$. From (2.5)

$$h^2_{F(u)}(v) = \sigma_{F(u)}^{B^2}((T p^2)^* \alpha) = T_u F \circ \sigma_{u}^{B^1} \circ (T_u F)^* (T p^2)^* \alpha = T_u F \circ \sigma_{u}^{B^1} (T_u p^1)^* \alpha = T_u h^1_u(v)$$

as required. \qed

**Definition 2.1.5** A diffusion operator $B$ on $N$ will be said to have **projectible symbol** for $p : N \rightarrow M$ if there exists a map $\eta : T^* M \rightarrow TM$ such that for all $u \in N$ the diagram:

$$\begin{CD}
T_u^* N @>{\sigma_u^B}>> T_u N \\
@AA{(T_u p)^*}A @A{T_u p}A \\
T_{p(u)}^* M @>{\eta_{p(u)}}>> T_{p(u)} M. \\
\end{CD}$$

commutes, i.e. if $(T_u p) \sigma_u^B (T_u p)^*$ depends only on $p(u)$.
In this case we also get a uniquely defined horizontal lift map as in Proposition 2.1.4 defined by equation (2.7) using $\eta$ instead of the symbol of $A$. This situation arises naturally in the standard non-linear filtering literature as described later see chapter 5.

### 2.2 Example: The Horizontal Lift Map of SDEs

Let us consider the horizontal lift connection in more detail when $B$ and $A$ are given by stochastic differential equations. For this write $A$ and $B$ in Hörmander form corresponding to factorisations $\sigma^A_x = X(x)X(x)^*$ and $\sigma^B_x = \tilde{X}(x)\tilde{X}(x)^*$ for

$$
X(x) : \mathbb{R}^m \to T_x M, \quad x \in M
$$

$$
\tilde{X}(u) : \mathbb{R}^{\tilde{m}} \to T_u N, \quad u \in N.
$$

Then $X(x)$ maps onto $E_x$ for each $x \in M$. Define $Y_x : E_x \to \mathbb{R}^m$ to be its right inverse:

$$
Y(x) = \left[ X(x) \big|_{\ker X(x)^\perp} \right]^{-1}.
$$

**Lemma 2.2.1** For each $u \in N$ there is a unique linear $\ell_u : \mathbb{R}^m \to \mathbb{R}^{\tilde{m}}$ such that $\ker \ell_u = \ker X(x)$ and the diagram

$$
\begin{array}{ccc}
T_u^* N & \xrightarrow{\tilde{X}(u)^*} & \mathbb{R}^{\tilde{m}} \\
\downarrow{(T_u p)^*} & & \downarrow{\ell_u} \\
T_x^* M & \xrightarrow{X(x)^*} & \mathbb{R}^m
\end{array}
$$

$$
\begin{array}{ccc}
\mathbb{R}^{\tilde{m}} & \xrightarrow{\tilde{X}(u)} & T_u N \\
\downarrow{T_u p} & & \downarrow{T_u p} \\
\mathbb{R}^m & \xrightarrow{X(x)} & T_x M
\end{array}
$$

commutes, for $x = p(u)$, i.e. $\sigma^A_x = T_u p \circ \sigma^B_x (T_u p)^*$ and $X(x) = T_u p \circ \tilde{X}(u) \circ \ell_u$. In particular the horizontal lift map is given by $h_u = \tilde{X}(u)\ell_u Y(p(u))$.

**Proof.** The larger square commutes by Lemma 2.1.1. For the rest we need to construct $\ell_u$. It suffices to define $\ell_u$ on $[\ker X(x)]^\perp$. Note that $[\ker X(x)]^\perp = \text{Image } X(x)^*$ in $\mathbb{R}^m$. We only have to show that $\alpha \in [\ker X(x)]^\perp$ implies

$$
\tilde{X}(u)^*(T_u p)^* \alpha = 0.
$$
In fact for such $\alpha$ the proof of part (i) of Proposition 2.1.2 is valid and therefore $(T_u p)^* \alpha \in \ker \sigma_B^P$. However since $\tilde{X}(u)$ is injective on the image of $\tilde{X}(u)^*$ we see $\ker \sigma_B^P = \ker \tilde{X}(u)$. Thus $\ell_u$ is defined with $\ker \ell_u = \ker X(x)$ and such that the left hand square of the diagram commutes. Since the perimeter commutes it is easy to see from the construction of $\ell_u$ that the right hand side also commutes. The uniqueness of $\ell_u$ with kernel equal that of $X(x)$ is clear since on $[\ker X(x)]^\perp \ell_u(e) = \tilde{X}(u)^*(T_u p)^* X(x)(e)$. 

Note. The horizontal lift of $X(x)$, which can be used to construct a Hörmander form representation $X^V$ of $A^H$, as in Proposition 2.3.5 and Theorem 3.2.1 below is given by:

$$X^V(u) : \mathbb{R}^m \to T_u P$$

$$X^V(u) = h_u X(u) = \tilde{X}(u) \ell_u$$

since $Y_x X(x)$ is the projection onto $\ker X(x)^\perp$. (In the terminology of Elworthy-LeJan-Li [27] $X^V$ does not involve the ‘redundant noise’.) Furthermore consider the special case that $\tilde{m} = m$ and also that $\tilde{X}$ and $X$ are $p$-related, i.e.

$$T_u p(\tilde{X}(u)e) = X(p(u))e, \quad u \in N, e \in \mathbb{R}^m.$$ 

Then $\ell_u$ is the projection of $\mathbb{R}^m$ onto $[\ker X(p(u))]^\perp$:

$$\ell_u = Y(p(u))X(p(u))$$

giving

$$h_u = \tilde{X}(u)Y(p(u)) \quad (2.8)$$

In this case the ‘diffusion coefficients’ $X^V$, above, is obtained from $\tilde{X}$ by restriction to the ‘relevant noise’ for $X$.

### 2.3 Lifts of Cohesive Operators & Decomposition

**Theorem**

A diffusion generator $\mathcal{L}$ on a manifold is said to be **cohesive** if

(i) $\sigma_x^\mathcal{L}, x \in X$, has constant non-zero rank and

(ii) $\mathcal{L}$ is along the image of $\sigma_x^\mathcal{L}$. 


2.3. LIFTS OF COHESIVE OPERATORS & DECOMPOSITION THEOREM

**Remark 2.3.1** From Theorem 2.1.1 in Elworthy-LeJan-Li [27] we see that if the rank of $\sigma_x^L$ is bigger than 1 for all $x$ then $L$ is cohesive if and only if it has a representation

$$L = \frac{1}{2} \sum_{j=1}^{m} L_{X_j} L_{X_j}$$

where $E_x = \text{span}\{X^1(x), \ldots X^m(x)\}$ has constant rank.

**Proposition 2.3.2** Let $B$ be a smooth diffusion operator on $N$ over $\mathcal{A}$ with $\mathcal{A}$ cohesive. The following are equivalent:

(i) $B = \mathcal{A}^H$

(ii) $B$ is cohesive and $T_u p$ is injective on the image of $\sigma_u^B$ for all $u \in N$.

(iii) $B$ can be written as

$$B = \frac{1}{2} \sum_{j=1}^{m} L_{X^0} L_{X_j} + L_{X^0}$$

where $X^0, \ldots, X^m$ are smooth vector fields on $N$ lying over smooth vector fields $X^0, \ldots, X^m$ on $M$, i.e. $T_u p(X^j(u)) = X^j(p(u))$ for $u \in N$ for all $j$.

**Proof.** If (i) holds take smooth $X^1, \ldots, X^m$ with $\mathcal{A} = \frac{1}{2} \sum_{j=1}^{m} L_{X_j} L_{X_j} + L_{X^0}$, by Proposition 1.3.4, and set $\hat{X}^j(u) = h_u X^j(p(u))$ to see (iii) holds. Clearly (iii) implies (ii) and (ii) implies (i), so the three statements are equivalent. \qed

**Definition 2.3.3** If any of the equivalent conditions of the proposition holds we say that $B$ has no vertical part.

Recall that is $S$ is a distribution, $S^0$ denotes the set of annihilators of $S$.

**Lemma 2.3.4** For $\ell \in H_u^0$ and $k \in (V_u T N)^0$, some $u \in N$ we have:

A. $\ell \sigma^B(k) = 0$

B. $\sigma^B(k) = \sigma^{\mathcal{A}^H}(k)$

C. $\sigma^{\mathcal{A}^H}(\ell) = 0$.

In particular $H_u$ is the orthogonal complement of $V T_u N \cap \text{Image}(\sigma_u^B)$ in $\text{Image}(\sigma_u^B)$ with its inner product induced by $\sigma_u^B$. 

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Proof. Set \( x = p(u) \). For part A and part B it suffices to take \( k = \phi \circ T_u p \) some \( \phi \in T_x^* M \). Then by (2.4), \( \sigma^B_u(\phi \circ T_u p) = h_u \circ \sigma^A_x(\phi) \) giving part A, and also part B by Proposition 1.4.2 (iii)(a) since \( \phi = h_u^*(\phi \circ T_u p) \), part C comes directly from Proposition 1.4.2 (iii)(a).

Theorem 2.3.5 For \( B \) over \( A \) with \( A \) cohesive there is a unique decomposition

\[
B = B^1 + B^V
\]

where \( B^1 \) and \( B^V \) are smooth diffusion generators with \( B^V \) vertical and \( B^1 \) over \( A \) having no vertical part. In this decomposition \( B^1 = A^H \), the horizontal lift of \( A \) to \( H \).

Proof. Set \( B^V = B - A^H \). To see that \( B^V \) is semi-elliptic take \( u \in N \) and observe that any element of \( T^*_u N \) can be written as \( \ell + k \) where \( \ell \in H^1_u \) and \( k \in (VT_u N)^0 \) by Lemma 2.3.4 and

\[
(\ell + k)\sigma^B(\ell + k) = \ell\sigma^B(\ell) \geq 0.
\]

Since \( B^V(f \circ p) = 0 \) any \( f \in C^2(M; \mathbb{R}) \) Proposition 1.4.4 implies \( B^V \) is vertical. Uniqueness holds since the semi-connections determined by \( B \) and \( B' \) are the same by Remark 1.3.2(i) applied to \( B^V \) and so by Proposition 2.3.2 we must have \( B^1 = A^H \).

For \( p \) a Riemannian submersion and \( B \) the Laplacian, Berard-Bergery and Bourguignon [7] define \( B^V \) directly by

\[
B^V f(u) = \Delta_{N_x}(f|_{N_x})(u) \quad \text{for} \quad x = p(u) \quad \text{and} \quad N_x = p^{-1}(x) \quad \text{with} \quad \Delta_{N_x} \quad \text{the Laplace-Beltrami operator of} \quad N_x.
\]

Example 2.3.6 1. Take \( N = S^1 \times S^1 \) and \( M = S^1 \) with \( p \) the projection on the first factor. Let

\[
B = \frac{1}{2}\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \tan \alpha \frac{\partial^2}{\partial x \partial y}.
\]

Here \( 0 < \alpha < \frac{\pi}{4} \) so that \( B \) is elliptic. Then \( A = \frac{1}{2} \frac{\partial^2}{\partial x^2} \) and \( B^V = \frac{1}{2}(1 - (\tan \alpha)^2) \frac{\partial^2}{\partial y^2} \) with \( A^H = \frac{1}{2}(\frac{\partial^2}{\partial x^2} + (\tan \alpha)^2 \frac{\partial^2}{\partial y^2}) + \tan \alpha \frac{\partial^2}{\partial x \partial y} \). This is easily checked since, with this definition \( A^H \) has Hörmander form

\[
A^H = \frac{1}{2} \left( \frac{\partial}{\partial x} + \tan \alpha \frac{\partial}{\partial y} \right)^2
\]
and so is a diffusion operator which has no vertical part. Also $B^V$ is clearly vertical and elliptic. Note that this is an example of a Riemannian submersion: several more of a similar type can be found in [7]. In this case the horizontal distribution is integrable and if $\alpha$ is irrational the foliation it determines has dense leaves.

2. Take $N = \mathbb{R}^3$ with Heisenberg group structure. This is defined by

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - yx')).$$

Let $X, Y, Z$ be the left-invariant vector fields which give the standard basis for $\mathbb{R}^3$ at the origin. As operators:

$$X(x, y, z) = \frac{\partial}{\partial x} - \frac{1}{2}y \frac{\partial}{\partial z}, \quad Y(x, y, z) = \frac{\partial}{\partial y} + \frac{1}{2}x \frac{\partial}{\partial z}, \quad Z(x, y, z) = \frac{\partial}{\partial z}.$$  

Take $B$ to be half the sum of the squares of $X, Y$, and $Z$. This is half the left invariant Laplacian:

$$B = \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(1 + \frac{1}{4}(x^2 + y^2)\right) \frac{\partial^2}{\partial z^2} + \frac{1}{2}(x \frac{\partial^2}{\partial y \partial z} - y \frac{\partial^2}{\partial x \partial z}) \right).$$

Take $M = \mathbb{R}^2$ and $p : \mathbb{R}^3 \to \mathbb{R}^2$ to be the projection on the first 2 coordinates. Then

$$A = \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \quad A^H = \frac{1}{2}(X^2 + Y^2);$$

$$B^V = \frac{1}{2} Z^2 = \frac{1}{2} \frac{\partial^2}{\partial z^2}.$$  

Note that the horizontal lift $\tilde{\sigma}$, of a smooth curve $\sigma : [0, T] \to M$ with $\sigma(0) = 0$, is given by

$$\tilde{\sigma}(t) = \left(\sigma^1(t), \sigma^2(t), \frac{1}{2} \int_0^t (\sigma^1(t)d\sigma^2(t) - \sigma^2(t)d\sigma^1(t)) \right). \quad (2.9)$$

Thus the “vertical” component of the horizontal lift is the area integral of the curve. Equation (2.9) remains valid for the horizontal lift of Brownian motion on $\mathbb{R}^2$ , or more generally for any continuous semi-martingale,
provided it is interpreted as a Stratonovich equation (or equivalently an
Ito equation in the Brownian motion case). This example is also that of
a Riemannian submersion. In this case the horizontal distributions are not
integrable. Indeed the Lie brackets satisfy \([X,Y] = Z\) and Hörmander’s
condition for hypoellipticity: a diffusion operator \(L\) satisfies Hörmander’s
condition if for some (and hence all) Hörmander form representation such
as in equation (1.7) the vector fields \(Y^1, \ldots, Y^m\) together with their iter-
at ed Lie brackets span the tangent space at each point of the manifold. For
an enjoyable discussion of the Heisenberg group and the relevance of this
e xample to “Dido’s problem” see [52]. See also [3],[9], and [36].

Recall that \(F \equiv \sqcup_u F_u = \sqcup_u (T_u p)^{-1}[E_p(u)]\), we can now strengthen Lemma
2.1.3 which states that \(\text{Image} \sigma_B \subset F_u\).

**Corollary 2.3.7** If \(B\) is over \(A\) with \(A\) cohesive, then \(B\) is along \(F\).

**Proof.** Since \(H_u \in F_u\) and \(VT_u N \subset F_u\) both \(B^1\) and \(B^V\) are along \(F\). \(\Box\)

### 2.4 Diffusion Operators with Projectible Symbols

Given \(p : N \to M\) as before, suppose now that we have a diffusion operator \(B\)
on \(M\) with a projectible symbol, c.f. Definition 2.1.5. This means that \(\sigma_B\) lies
over some positive semi-definite linear map \(\eta : T^*M \to TM\). Assume that \(\eta\) has
constant rank. We will show that in this case we also have a decomposition of \(B\).
To do this first choose some cohesive diffusion operator \(A\) on \(M\) with \(\sigma_A = \eta\). In
general there is no canonical way to do this, though if \(\eta\) were non-degenerate we
could choose \(A\) to be a multiple of the Laplace-Beltrami operator of the induced
metric on \(M\).

From above we also have an induced semi-connection with horizontal sub-
bundle \(H\), say, of \(TN\).

**Definition 2.4.1** We will say that \(B\) descends cohesively (over \(p\)) if it has a pro-
jectible symbol and there exists a horizontal vector field, \(b^H\), such that

\[
B - L_{b^H}
\]

is projectible over \(p\).
The following is a useful observation. Its proof is immediate from the two lemmas and proposition which are given after it:

**Proposition 2.4.2** If \( B \) descends cohesively then for each choice of \( A \) satisfying \( \sigma^A_{p(u)} = T_u p \sigma^B_u (T_u p)^* \) there is a horizontal vector field \( b^H \) such that \( B - L_{b^H} \) lies over \( A \).

**Lemma 2.4.3** Assume that \( \eta \) has constant rank. If \( f \) is a function on \( M \) let \( \tilde{f} = f \circ p \). For any choice of \( A \) with symbol \( \eta \) the map

\[
  f \mapsto B(\tilde{f}) - \widetilde{A}(f)
\]

is a derivation from \( C^\infty M \) to \( C^\infty N \) where any \( f \in C^\infty M \) acts on \( C^\infty N \) by multiplication by \( \tilde{f} \).

**Proof.** The map is clearly linear and for smooth \( f, g : M \to \mathbb{R} \) we have

\[
  \eta(df, dg) = \sigma^B(df, dg)
\]

so by definition of symbols:

\[
  B(\tilde{f} \tilde{g}) - \widetilde{A}(f \tilde{g}) = B(\tilde{f})\tilde{g} + B(\tilde{g})\tilde{f} - \widetilde{A}(f)\tilde{g} - \widetilde{A}(g)\tilde{f}
\]

as required. \( \square \)

Let \( \mathcal{D} \) denote the space of derivations from \( C^\infty M \) to \( C^\infty N \) using the above action. Note that for \( p^* TM \to N \) the pull back of \( TM \) over \( p \), the space \( C^\infty \Gamma p^* TM \) of smooth sections of \( p^* TM \) can be considered as the space of smooth functions \( V : N \to TM \) with \( V(u) \in T_{p(u)} M \) for all \( u \in N \). We can then define

\[
  \Theta : C^\infty \Gamma p^* TM \to \mathcal{D}
\]

by

\[
  \Theta(V)(f)(u) = df_{p(u)}(V(u)).
\]

**Lemma 2.4.4** Assume that \( \eta \) has constant rank The map \( \Theta : C^\infty \Gamma p^* TM \to \mathcal{D} \) is a linear bijection.
Proof. Let \( d \in D \). Fix \( u \in N \). The map from \( C^\infty M \) to \( \mathbb{R} \) given by \( f \mapsto df(u) \) is a derivation at \( p(u) \), here the action of any \( f \in C^\infty M \) on \( \mathbb{R} \) is multiplication by \( f(p(u)) \), and so corresponds to a tangent vector, \( V(u) \) say, in \( T_{p(u)}M \). Then \( df(u) = df_{p(u)}(V(u)) \). By assumption \( df(u) \) is smooth in \( u \), and so by suitable choices of \( f \) we see that \( V \) is smooth. Thus \( \Theta(V) = d \) and \( \Theta \) has an inverse. \( \square \)

From these lemmas we see there exists \( b \in C^\infty \Gamma p^*TM \) with the property that

\[
(\mathcal{B} \tilde{f} - \tilde{\mathcal{A}}f)(u) = df_{p(u)}(b(u))
\]

for all \( u \in N \) and \( f \in C^\infty M \). Assume that \( b \) has image in the subbundle \( E \) of \( TM \) determined by \( \eta \). Using the horizontal lift map \( h \) determined by \( \mathcal{B} \) define a vector field \( b^H \) on \( N \):

\[
b^H(u) = h_u(b(u)).
\]

**Proposition 2.4.5** Assume that \( \eta \) has constant rank and that \( b \) has image in the subbundle \( E \) determined by \( \eta \). The vector field \( b^H \) is such that \( \mathcal{B} - b^H \) is over \( \mathcal{A} \).

Proof. For \( f \in C^\infty M \),

\[
(\mathcal{B} - b^H)(\tilde{f}) = \tilde{\mathcal{A}}f + df(b(-)) - df \circ Tp(b^H(-)) = \tilde{\mathcal{A}}f
\]

using the fact that \( Tp(b^H(-)) = b(-) \). \( \square \)

We can now extend the decomposition theorem:

**Theorem 2.4.6** Let \( \mathcal{B} \) be a diffusion operator on \( N \) which descends cohesively over \( p : N \to M \). Then \( \mathcal{B} \) has a unique decomposition:

\[
\mathcal{B} = \mathcal{B}^H + \mathcal{B}^V
\]

into the sum of diffusion operators such that

(i) \( \mathcal{B}^V \) is vertical

(ii) \( \mathcal{B}^H \) is cohesive and \( T_u\mathcal{B} \) is injective on the image of \( \sigma_u^{\mathcal{B}^H} \) for all \( u \in N \).

With respect to the induced semi-connection \( \mathcal{B}^H \) is horizontal.
2.5. **HORIZONTAL LIFT OF PATHS & COMPLETENESS OF SEMI-CONNECTIONS**

*Proof.* Using the notation of the previous proposition we know that $B - b^H$ is over a cohesive diffusion operator $A$. By Theorem 2.3.5 we have a canonical decomposition

$$B - b^H = B^1 + B^V,$$

leading to

$$B = (b^H + B^1) + B^V.$$  

If we set $B^H = b^H + B^1$ we have a decomposition as required. On the other hand if we have two such decompositions of $B$ we get two decompositions of $B - b^H$. Both components of the latter must agree by the uniqueness in Theorem 2.3.5, and so we obtain uniqueness in our situation. \qed

Extending Definition 2.3.3 we could say that a diffusion operator $B^H$ satisfying condition (ii) in the theorem has no vertical part.

Note that if we drop the hypothesis that $b^H$ is horizontal, or equivalently that $b$ in Proposition 2.4.5 has image in $E$, we still get a decomposition by taking an arbitrary lift of $b$ to be $b^H$ but we will no longer have uniqueness.

### 2.5 Horizontal lift of paths & completeness of semi-connections

A semi-connection on $p : N \to M$ over a sub-bundle $E$ of $TM$ gives a procedure for horizontally lifting paths on $M$ to paths on $N$ as for ordinary connections but now we require the original path to have derivatives in $E$; such paths may be called $E$-horizontal.

**Definition 2.5.1** A Lipschitz path $\tilde{\sigma}$ in $N$ is said to be a horizontal lift of a path $\sigma$ in $M$ if

- $p \circ \tilde{\sigma} = \sigma$

- The derivative of $\tilde{\sigma}$ almost surely takes values in the horizontal subbundle $H$ of $TN$.

Note that a Lipschitz path $\sigma : [a, b] \to M$ with $\dot{\sigma}(t) \in E_{\sigma(t)}$ for almost all $a \leq t \leq b$ has at most one horizontal lift from any starting point $u_a$ in $p^{-1}(\sigma(a))$. To see this first note that any such lift must satisfy

$$\dot{\tilde{\sigma}}(t) = h_{\dot{\tilde{\sigma}}(t)} \dot{\tilde{\sigma}}(t). \quad (2.11)$$
CHAPTER 2. DECOMPOSITION OF DIFFUSION OPERATORS

This equation can be extended to give an ordinary differential equation on all of $N$. For example take a smooth embedding $j : M \to \mathbb{R}^m$ into some Euclidean space. Set $\beta(t) = j(\sigma(t))$. Let $X(x) : \mathbb{R}^m \to E_x$ be the adjoint of the restriction of the derivative $T_xj$ of $j$ to $E_x$, using some Riemannian metric on $E$. Then $\sigma$ satisfies the differential equation

$$\dot{x}(t) = X(x(t))(\dot{\beta}(t))$$

(2.12)

and it is easy to see that the horizontal lifts of $\sigma$ are precisely the solutions of

$$\dot{u}(t) = h_u X(p(u(t)))(\dot{\beta}(t))$$

starting from points above $\sigma(a)$ and lasting until time $b$.

In the generality in which we are working there may not be any such solutions, for example because of “holes” in $N$. We define the semi-connection to be complete if every Lipschitz path $\sigma$ with derivatives in $E$ almost surely, has a horizontal lift starting from any point above the starting point of $\sigma$.

Note that completeness is assured if the fibres of $N$ are compact, or if an $X$, with values in $E$, and $\beta$, can be found so that $\sigma$ is a solution to equation (2.12) and there is a complete metric on $N$ for which the horizontal lift of $X$ is bounded on the inverse image of $\sigma$ under $p$. In particular the latter will hold if $p$ is a principal bundle and we have an equivariant semi-connection as in the next chapter. It will also hold if there is a complete metric on $N$ for which the horizontal lift map $h_u \in \mathbb{L}(E_{p(u)}; T_u N)$ is uniformly bounded for $u$ in the image of $\sigma$.

2.6 Topological Implications

Although our set up of intertwining diffusions with a cohesive $A$ seems quite general it implies strong topological restrictions if the manifolds are compact and more generally. Here we partially extend the approach Hermann used for Riemannian submersions in [37] with a more detailed discussion in Chapter 6 below.

For this let $D^0(x)$ be the set of points $z \in M$ which can be reached by Lipschitz curves $\sigma : [0, t] \to M$ with $\sigma(0) = x_0$ and $\sigma(t) = z$ with derivative in $E$ almost surely. Its closure $D'(x)$ relates to the propagation set for the maximum principle for $A$, and to the support of the $A$- diffusion as in Stroock-Varadhan [66], see Taira[70].

Theorem 2.6.1 For $B$ and $A$ as before with $A$ cohesive take $x_0 \in M$ and $z \in D^0(x_0)$. Assume the induced semi-connection is complete. Then if $p^{-1}(x_0)$ is a
submanifold of \( N \) so is \( p^{-1}(z) \) and they are diffeomorphic. Also if \( z \) is a regular value of \( p \) so is \( x \).

**Proof.** Let \( \sigma : [0, T] \rightarrow M \) be a Lipschitz \( E \)-horizontal path from \( x \) to \( z \). There is a smooth factorisation \( \sigma_x^A = X(x)X(x)^* \) for \( X(x) \in \mathcal{L}(\mathbb{R}^m; T_x M) \), \( x \in M \). Take the horizontal lift \( \tilde{X} : \mathbb{R}^t \rightarrow TN \) of \( X \).

By the completeness hypothesis the time dependent ODE on \( N \),
\[
\frac{dy_s}{ds} = \tilde{X}(y_s)X\left(\sigma(s)\left|_{\ker X(x_0)}\right.\right)^{-1}(\dot{\sigma}(s))
\]
will have solutions from each point above \( \sigma(0) \) defined up to time \( T \) and so a flow giving the required diffeomorphism of fibres. Moreover, by the usual lower semi-continuity property of the "explosion time", this holonomy flow gives a diffeomorphism of a neighbourhood of \( p^{-1}(x) \) in \( N \) with a neighbourhood of the fibre above \( z \). The diffeomorphism commutes with \( p \). Thus if one of \( x \) and \( z \) is a regular value so is the other.

**Corollary 2.6.2** Assume the conditions of the theorem and that \( E \) satisfies the standard Hörmander condition that the Lie algebra of vector fields generated by sections of \( E \) spans each tangent space \( T_y M \) after evaluation at \( y \). Then \( p \) is a submersion all of whose fibres are diffeomorphic.

**Proof.** The Hörmander condition implies that \( D^0(x) = M \) for all \( x \in M \) by Chow’s theorem (e.g. see Sussmann [69] or [36]). In [36] Gromov shows that under this condition any two points of \( M \) can be joined by a smooth \( E \)-horizontal curve.

**Corollary 2.6.3** Assume the conditions of the theorem and that \( D^0(x) \) is dense in \( M \) for all \( x \in M \) and \( p : N \rightarrow M \) is proper. Then \( p \) is a locally trivial bundle over \( M \).

**Proof.** Take \( x \in M \). The set \( \text{Reg}(p) \) of regular values of \( p \) is open by our properness assumption. It is also non-empty, even dense in \( M \), by Sard’s theorem, and so since \( D^0(x) \) is dense, there exists a regular value \( z \) which is in \( D^0(x) \). It follows from the theorem that \( x \in \text{Reg}(p) \), and so \( p \) is a submersion. However it is a well known consequence of the inverse function theorem that a proper submersion is a locally trivial bundle.

Note that we only need \( \text{Reg}(p) \) to be open, rather than \( p \) proper, to ensure that \( p \) is a submersion. The density of \( D^0(x) \) can hold because of global behaviour, for example if \( M \) is a torus and \( E \) is tangent to the foliation given by an irrational flow.
Equivariant Diffusions on Principal Bundles

Let $M$ be a smooth finite dimensional manifold and $P(M,G)$ a principal fibre bundle over $M$ with structure group $G$ a Lie group. Denote by $\pi : P \to M$ the projection and $R_a$ right translation by $a$. Consider on $P$ a diffusion generator $B$, which is equivariant, i.e. for all $f \in C^2(P;\mathbb{R})$,

$$Bf \circ R_a = B(f \circ R_a), \quad a \in G.$$ 

Set $f^a(u) = f(ua)$. Then the above equality can be written as $Bf^a = (Bf)^a$. The operator $B$ induces an operator $A$ on the base manifold $M$. Set

$$Af(x) = B(f \circ \pi)(u), \quad u \in \pi^{-1}(x), f \in C^2(M),$$ 

which is well defined since

$$B(f \circ \pi)(u \cdot a) = B((f \circ \pi)^a)(u) = B((f \circ \pi))(u).$$

3.1 Invariant Semi-connections on Principal Bundles

Definition 3.1.1 Let $E$ be a sub-bundle of $TM$ and $\pi : P \to M$ a principal $G$-bundle. An invariant semi-connection over $E$, or principal semi-connection in the terminology of Michor, on $\pi : P \to M$ is a smooth sub-bundle $H^E TP$ of $TP$ such that
(i) $T_u\pi$ maps the fibres $H^E T_u P$ bijectively onto $E_{\pi(u)}$ for all $u \in P$.

(ii) $H^E T P$ is $G$-invariant.

Notes.

1. Such a semi-connection determines and is determined by, a smooth horizontal lift:

$$h_u : E_{\pi(u)} \to T_u P$$

such that (i) $T_u\pi \circ h_u(v) = v$, for all $v \in E_x \subset T_x M$;

(ii) $h_{u\cdot a} = T_u R_a \circ h_u$.

2. The action of $G$ on $P$ induces a homomorphism of the Lie algebra $g$ of $G$ with the algebra of left invariant vector fields on $P$: if $A \in g$,

$$A^*(u) = \frac{d}{dt} \bigg|_{t=0} u \exp(tA), \quad u \in P,$$

and $A^*$ is called the fundamental vector field corresponding to $A$.

Using the splitting (2.6) of $F_u$ our semi-connection determines, (and is determined by), a ‘semi-connection one-form’ $\varpi \in \mathcal{L}(H + VTN; g)$ which vanishes on $H$ and has $\varpi(A^*(u)) = A$.

3. Let $F$ be an associated vector bundle to $P$ with fibre $V$. An $E$ semi-connection on $P$ gives a covariant derivative $\nabla_w Z \in F_x$ for $w \in E_x$, $x \in M$ where $Z$ is a section of $F$. This is defined, as usual for connections, by

$$\nabla_w Z = u(d(\tilde{Z})(h_u(w))),$$

$u \in \pi^{-1}(x)$. Here $\tilde{Z} : P \to V$ is

$$\tilde{Z}(u) = u^{-1}Z(\pi(u))$$

considering $u$ as an isomorphism $u : V \to F_{\pi(u)}$. This agrees with the ‘semi-connections on $E$’ defined in Elworthy-LeJan-Li [27] when $P$ is taken to be the linear frame bundle of $TM$ and $F = TM$.

**Theorem 3.1.2** Assume $\sigma^A$ has constant rank. Then $\sigma^B$ gives rise to an invariant semi-connection on the principal bundle $P$ whose horizontal map is given by (2.5).
3.1. INVARIANT SEMI-CONNECTIONS ON PRINCIPAL BUNDLES

**Proof.** It has been shown that $h_u$ is well defined by (2.5). Next we show $h_u$ defines a semi-connection. As noted earlier, $h$ defines a semi-connection if (i) $T_u \pi \circ h_u(v) = v, v \in E_x \subset T_x M$ and (ii) $h_{u.a} = T_u R_a \circ h_u$. The first is immediate by Lemma 2.1.1 and for the second observe $\pi \circ R_a = \pi$. So $T \pi \circ T R_a = T \pi$ and $(T \pi)^* = (T R_a)^* \cdot (T \pi)^*$ while the following diagram

\[
\begin{array}{ccc}
T^*_u P & \overset{\sigma_u^B}{\longrightarrow} & T_u P \\
\downarrow (T_u R_a)^* & & \downarrow T_u R_a \\
T^*_u a P & \overset{\sigma_{u.a}^B}{\longrightarrow} & T_u a P
\end{array}
\]

commutes by equivariance of $B$. Therefore

\[
T_u R_a \circ h_u = T_u R_a \cdot \sigma_u^B (T_u \pi)^* \circ (\sigma_x^A)^{-1}
\]

\[
= T_u R_a \cdot \sigma_u^B (T_u R_a)^* \circ (T_{u.a} \pi)^* \circ (\sigma_x^A)^{-1}
\]

\[
= \sigma_{u.a}^B \circ (T_{u.a} \pi)^* \circ (\sigma_x^A)^{-1} = h_{u.a}.
\]

\[\square\]

Curvature forms and holonomy groups etc for semi-connections are defined analogously to those associated two connections, we note the following:

**Proposition 3.1.3** In the situation of Proposition 2.1.4 suppose $A$ is elliptic, $p^1$, $p^2$ are principal bundles with groups $G^1$ and $G^2$ respectively, and $F$ is a homomorphism of principal bundles with corresponding homomorphism $f : G^1 \to G^2$. Let $\Gamma^1$ and $\Gamma^2$ be the semi-connections on $N^1$, $N^2$ determined by $B^1$ and $B^2$. Then

(i) $\Gamma^2$ is the unique semi-connection on $p^2 : N^2 \to M$ such that $T F$ maps the horizontal subspaces of $TN^1$ into those of $TN^2$.

(ii) If $\omega^j, \Omega^j$ are the semi-connection and curvature form of $\Gamma^j$, for $j = 1, 2$, then

\[F^* (\omega^2) = f_\ast \circ \omega^1\]

and

\[F^* (\Omega^2) = f_\ast \circ \Omega^1\]
for \( f_* : g_1 \rightarrow g_2 \) the homomorphism of Lie algebras induced by \( f \).

(iii) Moreover \( f : G^1 \rightarrow G^2 \) maps the \( \Gamma^1 \) holonomy group at \( u \in N^1 \) onto the \( \Gamma^2 \) holonomy group at \( F(u) \) for each \( u \in N^1 \) and similarly for the restricted holonomy groups.

Proof. Proposition 2.1.4 assures us that \( TF \) maps horizontal to horizontal. Uniqueness together with (ii), (iii) come as in Kobayashi-Nomizu [41] (Proposition 6.1 on p79).

\[ \square \]

3.2 Decompositions of Equivariant Operators

Take a basis \( A_1, \ldots, A_n \) of \( \mathfrak{g} \) with corresponding fundamental vector fields \( \{ A^*_i \} \).

Write the semi-connection 1-form as \( \varpi = \sum \varpi^k A_k \) so that \( \varpi^k \) are real valued, partially defined, 1-forms on \( P \).

In our equivariant situation we can give a more detailed description of the decomposition in Proposition 2.3.5.

\textbf{Theorem 3.2.1} Let \( B \) be an equivariant operator on \( P \) and \( A \) be the induced operator on the base manifold. Assume that \( A \) is cohesive and let \( B = A^H + B^V \) be the decomposition of Proposition 2.3.5. Then \( B^V \) has a unique expression of the form

\[ \sum \alpha^{ij} \mathcal{L}_{A^*_i} \mathcal{L}_{A^*_j} + \sum \beta^k \mathcal{L}_{A^*_k}, \]

where \( \alpha^{ij} \) and \( \beta^k \) are smooth functions on \( P \), given by

\[ \alpha^{ij} = \varpi^k \left( \sigma^B (\varpi^l) \right), \quad \beta^k = \delta^B (\varpi^l) \]

for \( \varpi \) the semi-connection 1-form on \( P \). Define \( \alpha : P \rightarrow \mathfrak{g} \otimes \mathfrak{g} \) and \( \beta : P \rightarrow \mathfrak{g} \) by

\[ \alpha(u) = \sum \alpha^{ij}(u) A_i \otimes A_j, \quad \beta(u) = \sum \beta^k(u) A_k. \]

These are independent of the choices of basis of \( \mathfrak{g} \) and are equivariant:

\[ \alpha(ug) = (ad(g) \otimes ad(g)) \alpha(u) \]

and

\[ \beta(ug) = ad(g) \beta(u). \]

Proof. Since every vertical vector field is a linear combination of the fundamental vertical vector fields, Proposition 1.4.4, shows that

\[ B^V = \sum \alpha^{ij} \mathcal{L}_{A^*_i} \mathcal{L}_{A^*_j} + \sum \beta^k \mathcal{L}_{A^*_k} \]
for certain functions $\alpha^{ij}$, $\beta^k$. For $f, g : P \to \mathbb{R}$ setting $\sigma := \sigma^{B-A^H}$,

$$
\frac{1}{2} \sum \alpha^{ij} L_{A^*_i} L_{A^*_j}(fg) - \frac{1}{2} \sum g \alpha^{ij} L_{A^*_i} L_{A^*_j}(f) - \frac{1}{2} \sum f \alpha^{ij} L_{A^*_i} L_{A^*_j}(g) = \sum \alpha^{ij} L_{A^*_i} L_{A^*_j}(fg) - \frac{1}{2} \sum \alpha^{ij} df(A^*_i) dg(A^*_j).
$$

Since $\varpi(A^*_k) = A_k$, we see that $\varpi^k(A^*_i) = \delta_{ik}$ and $\varpi^k(\sigma(\varpi^{\ell})) = \sum \alpha^{ij} \delta_{ik} \delta_{j\ell} = \alpha^{k\ell}$.

Since $A^H$ is horizontal $\sigma^{A^H}$ has image in the horizontal tangent bundle and so is annihilated by $\varpi^k$. Thus

$$
\alpha^{k\ell} = \varpi^k(\sigma(\varpi^{\ell})) = \sum \alpha^{ij} \delta_{ik} \delta_{j\ell} = \alpha^{k\ell}.
$$

Note that by the characterisation, Proposition 1.2.1,

$$
\delta^B \varpi = \sum \alpha^{ij} L_{A^*_i} L_{A^*_j} + \sum \beta^k L_{A^*_k}.
$$

Since $\varpi^{\ell}(A^*_i)$ is identically 1, it follows that $\delta^B(\varpi) = \beta^{\ell}$. Again $\delta^{A^H}(\varpi^{\ell}) = 0$ and so

$$
\beta^{\ell} = \delta^B(\varpi^{\ell})
$$

as required.

For the last part $\alpha$ and $\beta$ can be considered as obtained from the extension of the symbol $\sigma^B$ and $\delta^B$ to $\mathfrak{g}$-valued two and one forms respectively: $\alpha = \varpi(-)\sigma^B \varpi(-)$ and $\beta = \delta^B(\varpi(-))$. To make this precise consider $\sigma^B_u$ as a bilinear form and so as a linear map

$$
\sigma^B_u : T_u^* P \otimes T_u^* P \to \mathbb{R}.
$$

The extension is the trivial one given by

$$
\sigma^B_u \otimes 1 \otimes 1 : T_u^* P \otimes T_u^* P \otimes \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R} \otimes \mathfrak{g} \otimes \mathfrak{g} \simeq \mathfrak{g} \otimes \mathfrak{g}
$$

using the identification of $T_u^* P \otimes \mathfrak{g}$ with $L(T_u P; \mathfrak{g})$. Similarly the extension of $\delta^B$ is

$$
\delta^B_u \otimes 1 : T_u^* P \otimes \mathfrak{g} \to \mathbb{R} \otimes \mathfrak{g} \simeq \mathfrak{g}.$$

Thus
\[
\alpha(u)(\omega \otimes \omega) = (\sigma^B \otimes 1 \otimes 1)(P_{23}\omega \otimes \omega)
\]
where \(P_{23} : T^*P \otimes g \otimes T^*P \otimes g \to T^*P \otimes T^*P \otimes g \otimes g\) is the standard permutation and \(\beta_u(\omega) = (\delta^B_u \otimes 1)(\omega)\).

The equivariance of \(\varpi\)
\[(R_g)^*\varpi = \text{ad}(g^{-1})(\varpi), \quad g \in G\]
is equivalent to the invariance of \(\varpi\) when considered as a section of \(T^*M \otimes g\) under \(TR_g \otimes \text{ad}(g) : T^*M \otimes g \to T^*M \otimes g, \quad g \in G\).

\(\square\)

**Remark 3.2.2**  
(a) For any equivariant operator of the form \(B = \sum_{i,j} \alpha^{ij} L_{A_i}^* L_{A_j} + \sum \beta^k L_{A_k}\) with \((\alpha^{ij}(u))\) positive semi-definite for each \(u \in P\) we can define maps \(\alpha\) and \(\beta\) by (3.2). Note that \(\alpha(u)\) is essentially the symbol of \(B\) restricted to the fibre \(P_{\pi(u)}\) through \(u\):
\[
\sigma^B_u|_{P_{\pi(u)}} : T^*P_{\pi(u)} \to T_uP_{\pi(u)}
\]
with \(\varpi_u\) identifying \(T_uP_{\pi(u)}\) with \(g\). Similarly \(\beta\) determines \(\delta^B\) on a basis of sections of \((VTP)^*\).

(b) Let \(\{u_t : 0 \leq t \leq \zeta\}\) be a \(B\)-diffusion on \(P\). By (3.3), \(2\alpha^{kl}(u_t)\) is the derivative of the bracket \(\left\langle \int_0^t \varpi^k_{u_s} \circ du_s, \int_0^t \varpi^l_{u_s} \circ du_s \right\rangle\) of the integrals of \(\omega^k\) and \(\omega^l\) along \(\{u_t : 0 \leq t < \zeta\}\). See chapter 4 below for a detailed discussion. Thus \(\alpha(u_t)\) is the derivative of the tensor quadratic variation:
\[
\alpha(u_t) = \frac{1}{2} \frac{d}{dt} \int_0^t \left( \varpi_{u_t} \circ du_t \otimes \varpi_{u_t} \circ du_t \right).
\]

Moreover by (3.4) and Lemma 4.1.2 below \(\int_0^t \beta(u_s)ds\) is the bounded variation part of \(\int_0^t \varpi_{u_s} \circ du_s\).

(c) If we fix \(u_0 \in P\) and take an inner product on \(g\) we can diagonalise \(\alpha(u_0)\) to write
\[
\alpha(u_0) = \sum_n \mu_n A_n \otimes A_n
\]
where \( \{A_n : n = 1, \ldots, \dim(g)\} \) is an orthonormal basis. The \( \mu_n \) are the eigenvalues of \( \alpha(u_0)^\# : g \rightarrow g \) obtained using the isomorphism:

\[
g \otimes g \rightarrow L(g; g) \\
an \otimes b \mapsto (a \otimes b)^\#
\]

where \((a \otimes b)^\#(v) = \langle b, v \rangle a\).

Note that for \( g \in G \), \( \alpha(u_0 \cdot g) = \sum_n \mu_n \text{ad}(g) A_n \otimes \text{ad}(g) A_n \). When the inner product is \( \text{ad}(G) \)-invariant then \( \{\text{ad}(g) A_n\}_{n=1}^{\dim(g)} \) is still orthonomal and the \( \{\mu_n\}_n \) are the eigenvalues of \( \alpha(u_0 \cdot g)^\# \). They are therefore independent of the choice of \( u_0 \) in a given fibre, (but depend on the inner product chosen).

### 3.3 Derivative Flows and Adjoint Connections

Let \( A \) on \( M \) be given in Hörmander form

\[
A = \frac{1}{2} \sum_{j=1}^m \mathcal{L}_{X_j} \mathcal{L}_{X_j} + \mathcal{L}_A
\]

for some smooth vector fields \( X_1, \ldots, X_m, A \). As before let \( E_x = \text{span}\{X_1(x), \ldots, X_m(x)\} \) and assume \( \dim E_x \) is constant, denoted by \( p \), giving a sub-bundle \( E \subset TM \). The vector fields \( \{X_1(x), \ldots, X_m(x)\} \) determine a vector bundle map

\[
X : \mathbb{R}^m \rightarrow TM
\]

with \( \sigma^A = X(x)X(x)^* \).

We can, and will, consider \( X \) as a map \( X : \mathbb{R}^m \rightarrow E \). Let \( Y_x \) be the right inverse \( [X(x)]_{\text{ker}X(x)^*}]^{-1} \) of \( X(x) \) and \( \langle , \rangle_x \) the inner product, induced on \( E_x \) by \( Y_x \). Then \( X \) projects the flat connection on \( \mathbb{R}^m \) to a metric connection \( \tilde{\nabla} \) on \( E \) defined by

\[
\tilde{\nabla}_U V = X(x) d[y \mapsto Y_y(U(y))(v)], \quad U \in C^1 \Gamma E, v \in T_y M, \quad (3.6)
\]

(In [27] we have studied the properties of this construction together with the SDE induced by \( X \), and there \( \tilde{\nabla} \) is referred as the LW connection for the SDE.) Moreover any connection \( \nabla \) on a subbundle \( E \) of \( TM \) has an adjoint semi-connection \( \nabla' \) on \( TM \) over \( E \) defined by

\[
\nabla'_V U = \nabla_U V + [U, V], \quad U \in \Gamma E, V \in \Gamma TM.
\]
Let $\pi : GLM \to M$ be the frame bundle of $M$, so $u \in \pi^{-1}(x)$ is a linear isomorphism $u : R^n \to T_xM$. It is a principal bundle with group $GL(n)$. If $g \in GL(n)$ and $\pi(u) = x$ then $u \cdot g : R^n \to T_xM$ is just the composition of $u$ with $g$.

Any smooth vector field $A$ on $M$ determines smooth vector fields $A_{TM}$ and $A_{GL}$ on $TM$ and $GLM$ respectively as follows: Let $t \in (-\epsilon, \epsilon)$ be a (partial) flow for $A$ and $T_\eta_t$ its derivative. Then $v \mapsto T_\eta_t(v)$ is a partial flow on $TM$ and $u \mapsto T_\eta_t \circ u$ one on $GLM$. Let $A_{TM}$ and $A_{GL}$ be the vector fields generating these flows. In fact $A_{TM}$ is $\tau \circ TA : TM \to TTM$ where $\tau : TTM \to TTM$ is the canonical twisting map:

$$\tau(x, v, w, v') = (x, v, v', w)$$

in local coordinates.

Using this, the choice of our Hörmander form representation induces a diffusion operator $B$ on $GLM$ by setting

$$B = \frac{1}{2} \sum L_{(X^i)GL} L_{(X^j)GL} + L_{A_{GL}}.$$ 

Then $\pi$ intertwines $B$ and $A$. For $w \in E_x$, set

$$Z^w(y) = X(y)Y_x(w).$$

**Theorem 3.3.1** Assume the diffusion operator $A$ given by (3.5) is cohesive and let $B$ be the operator on $GLM$ determined by $A$. Let $E$ be the image of $\sigma^A$, a vector bundle.

(a) The semi-connection $\nabla$ induced by $B$ is the adjoint of $\bar{\nabla}$ given by (3.6). Consequently $\nabla_w V = L_Z^w V$ for any vector field $V$ and $w \in E_x$.

(b) For $u \in GLM$, identifying $\mathfrak{gl}(n)$ with $\mathcal{L}(R^n, R^n)$,

$$\alpha(u) = \frac{1}{2} \sum \left( u^{-1} (-) \bar{\nabla}_{u(-)} X^p \right) \otimes \left( u^{-1} (-) \bar{\nabla}_{u(-)} X^p \right),$$

$$\beta(u) = -\frac{1}{2} \sum u^{-1} \bar{\nabla}_{u(-)} X^p X^p \bar{\nabla}_{u(-)} A - \frac{1}{2} u^{-1} \text{Ric}^# u(-) + u^{-1} \bar{\nabla}_{u(-)} A.$$ 

Here $\text{Ric}^# : TM \to E$ is the Ricci curvature of $\bar{\nabla}$ considered as an operator from $TM$ to $E$, defined by

$$\text{Ric}^#(v) = \sum_{j=1}^m \tilde{R}(v, X^j(x)) X^j(x)$$

for $\tilde{R}$ the curvature operator of $\bar{\nabla}$. 

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Proof. The first part can be deduced from the stochastic flow results in chapter 8 but we give a direct proof here. Let $\pi^e_t$ be the flow of $X(\cdot)(e)$. It induces a linear map $\tilde{X}(u) : \mathbb{R}^m \rightarrow T_uGLM$ on the general linear bundle $GLM$:

$$\tilde{X}(u)(e) = \frac{d}{dt}(TS^e_t \circ u)|_{t=0}, \quad u \in GLM.$$  \hfill (3.7)

We can apply lemma 2.2.1 with $\tilde{R}^m = \mathbb{R}^m$ and so $\ell_u = Y(p(u))X(p(u))$. If $x = p(u)$ and $e \perp \ker[X(x)]$ then the horizontal lift map $h_u$ defined by Theorem 3.1.2 is

$$h_u (X(x)(e)) = \tilde{X}(u)(\ell_u(e)) = \frac{d}{dt}\bigg|_{t=0} (T\pi^e_t \circ u).$$

Note this will not hold in general if $e \in \ker[X(x)]$.

Let $\sigma : [0, T] \rightarrow M$ be a $C^1$ curve with $\dot{\sigma}(t) \in E_{\sigma(t)}$ each $t$. Then

$$Z^{\dot{\sigma}(t)}(x) := X(x)Y_{\sigma(t)}\dot{\sigma}(t).$$

Let $S^{\sigma}_{s,t}$ be the flow, from time $s$ to time $t$, of the time dependent vector field $Z^{\dot{\sigma}(t)}$. Now $S^{\sigma}_{s,t}(\sigma(s)) = \sigma(t)$ for $0 \leq s \leq t \leq T$. Also, for any torsion free connection and any $v \in T_{\sigma(s)}M$

$$\frac{D}{dt}TS^{\sigma}_{s,t}(v) = \nabla Z^{\dot{\sigma}(t)}(TS^{\sigma}_{s,t}(v)) \bigg|_{t=s} = \nabla_v Z^{\dot{\sigma}(s)}.$$

Thus

$$\frac{D}{dt}TS^{\sigma}_{0,t}(v) = \nabla_{TS^{\sigma}_{0,t}}Z^{\dot{\sigma}(t)}.$$  \hfill (3.7)

If $\varpi$ is the connection form of this torsion free connection then

$$\varpi\left(\frac{D}{dt}TS^{\sigma}_{0,t} \circ u_0\right) = [e \mapsto (TS^{\sigma}_{0,t} \circ u_0)^{-1}\frac{D}{dt}TS^{\sigma}_{0,t}(u_0(e))]$$

$$= [e \mapsto (TS^{\sigma}_{0,t} \circ u_0)^{-1}\nabla_{TS^{\sigma}_{0,t}u_0(e)}Z^{\dot{\sigma}(t)}]$$

$$= \varpi\left(h_{TS^{\sigma}_{0,t}u_0}(\dot{\sigma}(t))\right)$$

by (3.7), showing that the vertical parts of $\frac{d}{dt}(TS^{\sigma}_{0,t} \circ u_0)$ and $h_{TS^{\sigma}_{0,t}u_0}(\dot{\sigma}(t))$ equal.
On the other hand, using this auxiliary connection, the horizontal parts of 
\( \frac{d}{dt} (TS_{0,t}^\sigma \circ u_0) \) and \( h_{TS_{0,t}^\sigma \circ u_0}(\dot{\sigma}(t)) \) are both equal to the horizontal lift of \( \dot{\sigma}(t) \). Thus
\[
\frac{d}{dt} (TS_{0,t}^\sigma \circ u_0) = h_{TS_{0,t}^\sigma \circ u_0}(\dot{\sigma}(t))
\]
and so \( \{TS_{0,t}^\sigma \circ u_0 : 0 \leq t \leq T\} \) is the horizontal lift of \( \{\sigma(t) : 0 \leq t \leq T\} \) with respect to the semi-connection induced by \( B \). However by Lemma 1.3.4 in Elworthy-LeJan-Li [27], \( TS_{0,t}^\sigma(v) \) of \( S_{0,t}^\sigma \) is the parallel translation of \( v \) along \( \sigma \) by the adjoint semi-connection \( \dot{\nabla} \) of the LeJan-Watanabe connection on \( E \) associated to \( X \) and \( \{TS_{0,t}^\sigma \circ u_0 : 0 \leq t \leq T\} \) is the horizontal lift of \( \{\sigma(t) : 0 \leq t \leq T\} \) with respect to \( \dot{\nabla} \). This proves the first claim. And \( \dot{\nabla} \omega V = L_{\omega}V \) by Lemma 1.3.4 of Elworthy-LeJan-Li [27].

For the last part let \( \varpi : H \oplus VTGLM \rightarrow g = L(\mathbb{R}^n; \mathbb{R}^n) \) be the semi-connection 1-form. For \( u_0 \in GLM \), set \( u_t = T\xi_t \circ u_0 \) where \( \{\xi_t\} \) is a local flow for the stochastic differential equation
\[
dx_t = X(x_t) \circ dB_t + A(x_t)dt
\]
on \( M \) where \( \{B_t\} \) is a Brownian motion on \( \mathbb{R}^n \). (This defines the derivative flow on \( GLM \).)

As for ordinary connections
\[
\varpi(\circ du_t) = u_t^{-1} \tilde{D} \frac{dt}{dt} (u_t-) \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n).
\]

Here, on the right hand side \( u_t \) is differentiated as a process of linear maps \( u_t \in \mathcal{L}(\mathbb{R}^n; T_{x_t}M) \) over \( (x_t) \). [It suffices to check the equality for \( C^1 \) curves \( (u_t) \) with \( x_t = \pi(u_t) \) having \( \tilde{x}_t \in E_{x_t}, t \geq 0 \). For this we can write \( u_t = \tilde{x}_t \cdot g_t \) for \( \tilde{x}_t \) a horizontal lift of \( \{x_t\} \) and \( g_t \in G \). Then observe that \( \frac{dt}{dt}(u_t-) = \tilde{x}_t \frac{dt}{dt}(\tilde{x}_t^{-1}u_t-) \).]

However as in [27],
\[
u_t^{-1} \tilde{D} \frac{dt}{dt} (u_t-) = u_t^{-1} \tilde{\nabla} \circ dB_t + u_t^{-1} \tilde{\nabla} u_t - Adt.
\]

From this the formula for \( \alpha(u) \) follows by Remark 3.2.2(b). For \( \beta(u) \) we need to identify the bounded variation part of \( \int_0^t \varpi(\circ du_t) \). For this write
\[
u_t^{-1} \tilde{\nabla} \circ dB_t = u_t^{-1} T_{x_t} \xi_t^{-1} \tilde{X}^t_0 \circ \frac{dt}{dt} \tilde{\nabla}_{T\xi_t \circ u_0} X \circ dB_t
\]
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where \( \hat{\mathcal{N}} \) is the parallel translation along \( \{ \xi_s(x_0) : 0 \leq s \leq t \} \) using our semi-connection, which is the adjoint of \( \nabla \) by Theorem 3.3.1. As in [27]

\[
\hat{\mathcal{N}}_{T\xi_{t}u_0}X \circ dB_t = \hat{\mathcal{N}}_{T\xi_{t}u_0}X dB_t - \frac{1}{2} \hat{\mathcal{N}}_{T\xi_{t}u_0}X dB_t - \frac{1}{2} \text{Ric}^\#(T\xi_{t}u_0 - ...) dt
\]

while

\[
u_0^{-1}T\xi_{t}^{-1}\hat{\mathcal{N}}_{t} = \nu_0^{-1} - \int_0^t \nu_0^{-1}T\xi_{s}^{-1}\hat{\mathcal{N}}_{s}X \circ dB_s - \int_0^t \nu_0^{-1}T\xi_{s}^{-1}\hat{\mathcal{N}}_{s}X dB_s\]

giving the formula claimed for \( \beta \).

Example: Gradient Brownian SDE

An isometric immersion \( j : M \to \mathbb{R}^m \) of a Riemannian manifold \( M \) determines a stochastic differential equation on \( M \):

\[
dx_t = X(x_t) \circ dB_t
\]

where \( X(x) : \mathbb{R}^m \to T_xM \) is the orthogonal projection and \( B \) is a Brownian motion on \( \mathbb{R}^m \). More precisely

\[
X(x)(e) = \nabla [y \mapsto \langle j(y), e \rangle](x).
\]

It is well known that the solutions of the SDE are Brownian motions on \( M \), see [21],[63], [22], and the equation is often called a "gradient Brownian SDE". Moreover the LW connection given by equation (3.6) is the Levi-Civita connection, (by the classical construction of the latter), see [27]. Since the adjoint of the Levi-Civita connection is itself, Theorem 3.3.1, shows that our connection induced on \( GLM \) by the derivative flow of a gradient Brownian system is also the Levi-Civita connection. Almost by definition,

\[
\langle \nabla_vX^p, w \rangle_{\mathbb{R}^m} = \langle a(v, w), e_p \rangle_{\mathbb{R}^m}
\]

where \( a : TM \times TM \to \mathbb{R}^m \) is the second fundamental form of the immersion with

\[
\nabla_vX(e) = A(v, n_x e) \quad v \in T_xM, x \in M, e \in \mathbb{R}^m
\]

for \( n_x : \mathbb{R}^m \to T_xM^\perp \) the projection and \( A : TM \oplus T^\perp \to TM \) the shape operator given by

\[
\langle A(v, e), w \rangle_{\mathbb{R}^m} = \langle a(v, w), e_p \rangle_{\mathbb{R}^m}.
\]
Here $TM^{\perp}$ refers to the normal bundle of $M$ and $T_xM^{\perp}$ to the normal space at $x$ to $M$, though we are considering its elements as being in the ambient space $\mathbb{R}^m$. Thus the vertical operator in the decomposition of the generator of the derivative flow on $GLM$ for gradient flows is given by Theorem 3.3.1 with

$$\alpha(u) = \frac{1}{2} \sum_{j=1}^{m-n} u^{-1} A(u-, l^j) \otimes u^{-1} A(u-, l^j)$$

$$\beta(u) = -\frac{1}{2} \sum_{j=1}^{m-n} A(A(u-, l^j), l^j) - \frac{1}{2} u^{-1} Ric^\#(u-)$$

at a frame $u$ over a point $x$. Here $l^1, ..., l^{m-n}$ denotes an orthonormal base for $T_xM^{\perp}$.

For the standard embedding of $S^n$ in $\mathbb{R}^{n+1}$ we have

$$\alpha(u, v) = \langle u, v \rangle_x$$

for $u, v \in T_xS^n$. Also the Ricci curvature is given by $Ric^\#(v) = (n-1)v$ for all $v \in TM$. Thus for the standard gradient SDE on $S^n$, at any frame $u$ we have

$$\alpha(u) = \frac{1}{2} Id \otimes Id \quad (3.11)$$

$$\beta(u) = -\frac{1}{2} n Id. \quad (3.12)$$

### 3.4 Associated Vector Bundles & Generalised Weitzenböck Formulae

As before let $\pi : P \to M$ be a smooth principal $G$-bundle and $\rho : G \to \mathbb{L}(V; V)$ a $C^\infty$ representation of $G$ on some separable Banach space $V$. There is then the (possibly weakly) associated vector bundle $\pi^\rho : F \to M$ where $F = P \times V / \sim$ for the equivalence relation given by $(u, e) \sim (ug, \rho(g^{-1})e)$ for $u \in P$, $e \in V$, $g \in G$. If $[(u, e)] \in F$ denotes the equivalence class of $(u, e)$ we can identify any $u \in P$ with a linear isomorphism

$$\tilde{u} : V \to F_{\pi(u)}$$

by

$$\tilde{u}(e) = [(u, e)]. \quad (3.13)$$
Consider the set of smooth maps from $P$ to $V$, equivariant by $\rho$:

$$M_\rho(P; V) = \{\text{smooth } Z : P \to V, Z(ug) = \rho(g)^{-1}Z(u), \ u \in P, g \in G\}.$$ 

There is the standard bijective correspondence $\mathfrak{F}^\rho$ between $M_\rho(P, V)$ and $\Gamma(F)$, the space of smooth sections of $F$ defined by

$$\mathfrak{F}^\rho(Z)(x) = \bar{u}[Z(u)], \quad u \in \pi^{-1}(x), Z \in M_\rho(P; V).$$ 

Via this map, an equivariant diffusion generator $B$ on $P$ induces a differential operator $B^\rho \equiv \mathfrak{F}^\rho(B)$ on $\Gamma(F)$, of order at most 2, by

$$\mathfrak{F}^\rho(B)(\mathfrak{F}^\rho(Z)) = \mathfrak{F}^\rho[B(Z)], \quad Z \in M_\rho(P; V). \quad (3.14)$$

Here $\mathcal{B}$ has been extended trivially to act on $V$-valued functions. Note that the definition makes sense since,

$$B(Z)(ug) = B(Z \circ R_g)(u) = B(\rho(g)^{-1}Z)(u) = \rho(g)^{-1}B(Z)(u).$$

For such a representation $\rho$ let

$$\rho_* : g \to \mathcal{L}(V; V)$$

be the induced representation of the Lie algebra $g$ (the derivative of $\rho$ at the identity).

**Theorem 3.4.1** When $\mathcal{B}$ is a vertical equivariant diffusion generator the induced operator on sections of any associated vector bundle is a zero order operator. With the notation of Theorem 3.2.1, the zero order operator in $\Gamma(F)$ induced by $\mathcal{B}$ is represented by $\lambda^\rho : P \to \mathcal{L}(V; V)$ for

$$\lambda^\rho(u) = \rho_*(\beta(u)) + \text{Comp} \circ (\rho_* \otimes \rho_*)(\alpha(u)), \quad u \in P \quad (3.15)$$

for $\text{Comp} : \mathcal{L}(V; V) \otimes \mathcal{L}(V; V) \to \mathcal{L}(V; V)$ the composition map $A \otimes B \mapsto AB$.

**Proof.** The operator $\mathcal{B}^\rho$ is a zero order operator if $\mathcal{F}^\rho(\mathcal{B})(S)(x_0) = \mathcal{F}^\rho(\mathcal{B})(S')$ whenever two sections $S$ and $S'$ of $F$ agree at $x_0$. This holds if $\mathcal{B}(fZ) = f\mathcal{B}(Z)$ for any invariant function $f : P \to R$ and $V$-valued function $Z$ on $P$. But this holds by Remark 1.4.5.

For the representation (3.15), suppose $Z : P \to V$ is equivariant:

$$Z(u \circ g) = \rho(g)^{-1}Z(u), \quad g \in G.$$
Then
\[ L_{A_j^*}(Z)(u) = \frac{d}{dt} Z(u \cdot e^{A_j^* t})|_{t=0} \]
\[ = \frac{d}{dt} \rho(e^{-A_j t}) Z(u)|_{t=0} \]
\[ = -\rho_* (A_j) Z(u). \]
Iterating we have
\[ B(Z)(u) = \sum \alpha^{ij}(u) \rho_* (A_j) \rho_* (A_i) Z(u) + \sum \beta_k \rho_* (A_k) Z(u) \]
proving (3.15).

From this theorem we easily have the following estimate, which combined with the discussions below, when applied to the associated bundle \( \wedge F \) to the orthonormal bundle, shows that the Weitzenböck curvature is positive if the curvature is.

**Corollary 3.4.2** If \( \rho \) is an orthogonal representation, i.e. \( (\rho_*(\alpha))^* = -\rho_*(\alpha) \) for all \( \alpha \in g \), then \( \lambda^\rho(v,v) \leq 0 \) for all \( v \in V \).

**Proof.** Write \( \alpha = \sum \mu_k A_k \otimes A_k \) where \( \{A_k\} \) is as in Remark 3.2.2(c). Then for \( v \in F \),
\[ \langle \text{Comp} \circ (\rho_\otimes \rho_*)_*(\alpha(u))(v), v \rangle \]
\[ = -\sum \mu_k \langle \rho_*(A_k)(v), \rho_*(A_k)(v) \rangle \leq 0, \]
since \( \mu_k \leq 0 \). The result follows from (3.15) since \( \rho_*(\beta(u)) \) is skew symmetric.

The situation of Corollary 3.4.2 arises when considering the derivative flow for an SDE on a Riemannian manifold whose flow consists of isometries; for example canonical SDE’s on symmetric spaces as in [27].

Quantitative estimates can be obtained by some representation theory. For example suppose \( G = O(n) \) with \( \rho \) the standard representation on \( R^n \). Consider the representation \( \wedge^k \rho \) on \( \wedge^k R^n \).

We use the following conventions, as in [27]. Let \( V \) be an \( N \) dimensional real inner product space. For \( 1 \leq i \leq n \),
\[ a_1 \wedge \cdots \wedge a_n = \frac{1}{n!} \sum_\pi \text{sgn}(\pi) a_{\pi(1)} \otimes \cdots \otimes a_{\pi(n)}, \]
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\[ t_v(u_1 \wedge \cdots \wedge u_q) = \sum_{j=1}^{q} (-1)^{j+1} \langle v, u_j \rangle u_1 \wedge \cdots \wedge \hat{u}_j \wedge \cdots \wedge u_q \]  

(3.16)

\[ \langle \otimes a_i, \otimes b_i \rangle = n! \prod_i \langle a_i, b_i \rangle, \]  

and \( \langle \wedge a_i, \wedge b_i \rangle = \det(\langle a_i, b_j \rangle) \). Let \( \wedge V \) stand for the exterior algebra of \( V \) and \( a_j^* \) the “creation operator” on \( \wedge V \) given by \( a_j^* v = e_j \wedge v \) for \( (e_1, \ldots, e_N) \) an orthonormal basis for \( \wedge V \). Let \( a_j \) be its adjoint, the “annihilation operator” given by \( a_j = \imath_{e_j} \). Note the commutation law:

\[ a_i a_j + a_j a_i = d_{ij} \]  

(3.17)

For linear forms we have the corresponding operators: \( (a^j)^* \phi(v) = \phi(a^j v) \) and \( (a^j \phi)(v) = \phi(a^j v) \). In particular \( a^j \phi(v) = \phi(e_j \wedge v) \) and \( (a^j)^* \phi(v) = e_j^\ast \wedge \phi \).

If \( A : V \to V \) is a linear map on \( V \), there are the operators \( \wedge A \) and \( (dA)(A) \) on \( \wedge V \), which restricted to \( \wedge^p V \) are:

\[ (dA)(A) (u_1 \wedge \cdots \wedge u_p) = \sum_{i=1}^{p} u_1 \wedge \cdots \wedge u_{j-1} \wedge Au_j \wedge u_{j+1} \wedge \cdots \wedge u_p, \]

and also

\[ \wedge A(u_1 \wedge \cdots \wedge u_p) = Au_1 \wedge \cdots \wedge Au_p. \]

Note that since \( \alpha(u) \) is symmetric, \( (\rho_+ \otimes \rho_-) \alpha(u) : V \otimes V \to V \otimes V \) has

\[ (\rho_+ \otimes \rho_-) \alpha(u)(v^1 \wedge v^2) = \sum_{i,j} \alpha^{ij}(u) \rho_-(A_i) \otimes \rho_+(A_j)(v^1 \wedge v^2) \]  

(3.18)

\[ = \sum_{i,j} \alpha^{ij}(u) A_i v^1 \wedge A_j v^2. \]  

(3.19)

and so \( (\rho_+ \otimes \rho_-) \alpha(u) \) restricts to a map of \( \wedge^2 V \) to itself.

**Corollary 3.4.3** Take the Hilbert-Schmidt inner product on \( \mathfrak{so}(n) \) and let \( 0 = \mu_1(x) \leq \cdots \leq \mu(x) \frac{1}{2} n(n-1) \) be the eigenvalues of \( \alpha \) on the fibre \( p^{-1}(x) \), \( x \in M \), as described in Remark 3.2.2(c). Then for all \( V \in \wedge^k \mathbb{R}^n \),

\[ -\frac{1}{2} k(n-k) \mu_{\frac{1}{2} n(n-1)}(x) \leq \langle \wedge^k(u)V, V \rangle \leq -\frac{1}{2} k(n-k) \mu_1(x). \]

**Proof.** Following Humphreys [38], §6.2, consider the bilinear form \( \beta \) on \( \mathfrak{so}(n) \) given by

\[ \beta(A, B) = \text{trace} \left( (d \wedge^k)(A), (d \wedge^k)(B) \right) = \frac{(n-2)!}{(k-1)!(n-k-1)!} \text{trace}(AB) \]
by a short calculation using elementary matrices. By Remark 3.2.2(c) since our inner product on $so(n)$ is $\text{ad}(O(n))$-invariant we can write

$$\alpha(u) = \sum_{l=1}^{\frac{1}{2}n(n-1)} \mu_l(x) A_l(u) \otimes A_l(u)$$

with $x = p(u)$ and \{A_l(u)\}_l an orthonormal base for $so(n)$ at each $u \in P$.

For each $u \in P$, set

$$A'_l(u) = \frac{(k-1)!(n-k-1)!}{(n-2)!} A_l(u)$$

to ensure $\beta(A'_l(u), A_j(u)) = \delta_{lj}$ for each $u$.

Then

$$\langle \text{Comp} \circ (\rho_s^k \otimes \rho_s^k)(\alpha(u)V, V) \rangle = \sum \mu_l(x) \langle (d\wedge^k)A_l(u) \circ (d\wedge^k)A_l(u)V, V \rangle$$

$$= \left[ \frac{(k-1)!(n-k-1)!}{(n-2)!} \right]^{-1} \langle (d\wedge^k)A_l(u) \circ (d\wedge^k)A'_l(u)V, V \rangle$$

$$\leq -\frac{(n-2)!}{(k-1)!(n-k-1)!} \langle c_{\wedge^k} V, V \rangle,$$

where

$$c_{\wedge^k} = (d\wedge^k)A_l(u) \circ (d\wedge^k)A'_l(u),$$

the Casimir element of our representation $d\wedge^k$ of $so(n)$. Since the representation is irreducible, (for example see [10] Theorem 15.1 page 278),

this element is a scalar, and we have, see Humphreys [38]

$$c_{\wedge^k} = \dim so(n) \dim \wedge^k R^n = \frac{1}{2} n(n-1) \cdots (n-k+1) k!.$$

Thus $\lambda_{\wedge^k}(u) \leq -\frac{1}{2} k(n-k) \mu_1$. The lower bound follows in the same way. \hfill \Box

When $B$ has an equivariant Hörmander form representation the zero order operator $F^\rho(V)$ can be given in a simple way by (3.20) below. This was noted for the classical Weitzenböck curvature terms using derivative flows in Elworthy [23].

**Proposition 3.4.4** Suppose $B$ lies over a cohesive operator $A$ and has a smooth Hörmander form: $B = \frac{1}{2} \sum \mathcal{L}_{Y^j} \mathcal{L}_{Y^j} + \sum \beta_k \mathcal{L}_{Y^o}$ with the vector fields $Y^j, j = 1, 2, \ldots n$.
1, . . . , m, being G-invariant. Let \((\eta^j_t)\) be the flow of \(Y^j\). For a representation \(\rho\) of \(G\) with associated vector bundle \(\pi^\rho : F \rightarrow M\) the zero order operator \(\mathcal{F}^\rho(B^V)\) corresponding to the vertical component of \(B\) is given by

\[
\mathcal{F}^\rho(B^V)(x_0) = \frac{1}{2} \sum_{j=1}^m \left. \frac{D^2}{dt^2} \eta^j_t(u_0) \right|_{t=0} \circ \left. (\bar{u}_0)^{-1} \right\} + \left. \frac{D}{dt} \eta^j_t(u_0) \right|_{t=0} \circ \left. (\bar{u}_0)^{-1} \right\} \tag{3.20}
\]

for any \(u_0 \in \pi^{-1}(x_0)\).

**Proof.** Set \(u^j_t = \eta^j_t(u_0) \in P\) and \(\sigma(t) = \pi(u^j_t)\) so \(\bar{u}^j_t \in \mathcal{L}(V; F_{\sigma(t)})\). From Remark 3.2.2(b)

\[
\alpha(u_0) = \frac{1}{2} \sum_{j=1}^m \varpi(Y^j(u_0)) \otimes \varpi(Y^j(u_0))
\]

and so

\[
(\rho_* \otimes \rho_*) \alpha(u_0) = \frac{1}{2} \sum_{j=1}^m \left. (\bar{u}_0)^{-1} \frac{D}{dt} \bar{u}^j_t \right|_{t=0} \otimes \left. (\bar{u}_0)^{-1} \frac{D}{dt} \bar{u}^j_t \right|_{t=0}
\]

as in the proof of Theorem 3.3.1.

Also from equation (3.4)

\[
\beta(u_0) = \frac{1}{2} \sum_{j=1}^m \mathcal{L}_{Y^j} \left( \varpi(Y^j(-)) \right)(u_0) + \frac{1}{2} \left( \varpi(Y^0(-)) \right)(u_0).
\]

Let \((\parallel_t)\) denotes parallel translation in \(F\) along \(\sigma\). Then

\[
\rho_* \mathcal{L}_{Y^j} \left( \varpi(Y^j(-)) \right)(u_0) = \left. \frac{d}{dt} \rho_* \varpi \left( Y^j(u^j_t) \right) \right|_{t=0}
\]

\[
= \left. \frac{d}{dt} \left( (\bar{u}_0)^{-1} \frac{D}{dt} \bar{u}^j_t \right) \right|_{t=0}
\]

\[
= \left. \frac{d}{dt} \left( (\parallel_t^{-1} \bar{u}_0)^{-1} \parallel_t^{-1} \frac{D}{dt} \bar{u}^j_t \right) \right|_{t=0}
\]

\[
= -\left. \bar{u}_0^{-1} \frac{D}{dt} \bar{u}^j_t \right|_{t=0} \circ \left. \bar{u}_0^{-1} \frac{D}{dt} \bar{u}^j_t \right|_{t=0} + \left. \bar{u}_0^{-1} \frac{D^2}{dt^2} \bar{u}^j_t \right|_{t=0}
\]

leading to the required result via Theorem 3.4.1.
To examine particular examples we will need to have detailed information about the zero order operators determined by a vertical diffusion generator. For this suppose $B$ is vertical and given by

$$B = \frac{1}{2} \sum \alpha_{ij} L_{A_i^*} L_{A_j^*} + \sum \beta_k L_{A_k^*}$$

for $\alpha : P \rightarrow g \otimes g$ and $\beta : P \rightarrow g$ as in Theorem 3.2.1 and (3.2).

Motivated by the Weitzenböck formula for the Hodge-Kodaira Laplacian on differential forms, see Corollary 3.4.8 below, [64], [15], we shall examine in more detail the case of the exterior power $\wedge^{p} : G \rightarrow L(\wedge^p V; \wedge^p V)$ of a fixed representation $\rho$ showing that $\lambda^{\wedge^p}$ has expressions in terms of annihilation and creation operators which are structurally the same as those of the Weitzenböck curvature (which are shown to be a special case in Corollary 3.4.8).

Lemma 3.4.5 If $B$ is a vertical operator on $P$ and $(e_i, i = 1, 2, \ldots, N)$ is an orthonormal basis of $V$, the zero order operator on the associated bundle $\wedge F \rightarrow M$ is represented by $\lambda^{\wedge^p} : P \rightarrow L(\wedge^p V; \wedge^p V)$ with

$$\lambda^{\wedge^p}(u) = \frac{1}{2} \sum_{i,j,k,l=1}^N \langle (\rho \otimes \rho) \alpha(u)(e_j \otimes e_l), e_i \otimes e_k \rangle a_i^* a_j a_k^* a_l$$

$$+ \sum_{i,j=1}^N \langle \rho \beta(u)e_j, e_i \rangle a_i^* a_j, \quad u \in P$$

Proof. Recall that if $A \in L(V; V)$ then

$$d\Lambda(A) = \sum_{i,j=1}^N \langle Ae_j, e_i \rangle a_i^* a_j,$$  \hspace{1cm} (3.21)

e.g. see Cycon-Froese-Kirsch-Simon [15]. Consequently

$$d\Lambda(\rho \beta(u)) = \sum_{i,j=1}^N \langle \rho \beta(u)e_j, e_i \rangle a_i^* a_j \hspace{1cm} (3.22)$$

On the other hand by Theorem 3.2.1 and (3.2), we can represent $\alpha$ as:

$$\alpha(u) = \sum_{n,m} a_{n,m}(u) A_n \otimes A_m$$
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where \( \{ A_i \}_{i=1}^N \) is a basis of \( \mathfrak{g} \). So

\[
\text{Comp} \circ (\wedge \rho \otimes \wedge \rho)(\alpha(u)) = \sum_{m,n} a_{n,m}(u) d\Lambda(\rho_* A_m) \otimes d\Lambda(\rho_* A_n)
\]

\[
= \sum_{m,n} a_{n,m}(u) d\Lambda(\rho_* A_m) \circ d\Lambda(\rho_* A_n)
\]

\[
= \sum_{m,n} a_{n,m}(u) \sum_{i,j,k,l=1}^N \langle \rho_* A_m e_j, e_i \rangle \langle \rho_* A_n e_l, e_k \rangle a_i^* a_j a_k^* a_l
\]

\[
= \frac{1}{2} \sum_{m,n} a_{n,m}(u) \sum_{i,j,k,l=1}^N \langle (\rho_* \otimes \rho_*) \alpha(u)(e_j \otimes e_i), e_l \otimes e_k \rangle a_i^* a_j a_k^* a_l
\]

since our convention for the inner product on tensor products gives

\[
\langle u_1 \otimes v_1, u_2 \otimes v_2 \rangle = 2 \langle u_1, u_2 \rangle \langle v_1, v_2 \rangle.
\]

The desired conclusion follows.

Theorem 3.4.6

Let \( R(u) : \wedge^2 V \to \wedge^2 V \) be the restriction of \( (\rho_* \otimes \rho_*) \alpha(u) : V \otimes V \to V \otimes V \), then

\[
\lambda^\rho(u) = - \sum_{i < k, j < l} \langle R(u)(e_j \wedge e_l), e_i \wedge e_k \rangle a_i^* a_j a_k^* a_l
\]

\[
+ \frac{1}{2} \sum_{i,j,l=1}^N \langle (\rho_* \otimes \rho_*) \alpha(u)(e_j \otimes e_l), e_i \otimes e_k \rangle a_i^* a_j a_k^* a_l + \sum_{i,j} \langle \rho_* \beta(u) e_j, e_i \rangle (a_i)^* a_j.
\]

This can be rewritten as:

\[
\lambda^\rho(u) = - \sum_{i < k, j < l} \langle R(u)(e_j \wedge e_l), e_i \wedge e_k \rangle a_i^* a_j a_k^* a_l + \frac{1}{2} d(\wedge (Z^\rho(u))) + d(\wedge (\rho_* \beta(u))).
\]

(3.23)

where \( Z^\rho(u) \in L(V; V) \) is defined by

\[
\langle Z^\rho(v_1), v_2 \rangle = \sum_{j=1}^N \langle (\rho_* \otimes \rho_*) \alpha(u)(e_j \otimes v_1), v_2 \otimes e_j \rangle_{V \otimes V}.
\]
Proof. This follows from Lemma 3.4.5 since
\[
\frac{1}{2} \sum_{i,j,k,l=1}^{N} \langle (\rho_\ast \otimes \rho_\ast) \alpha(u)(e_j \otimes e_l), e_i \otimes e_k \rangle a_i^* a_j^* a_k^* a_l \\
= -\frac{1}{2} \sum_{i,j,k,l=1}^{N} \langle (\rho_\ast \otimes \rho_\ast) \alpha(u)(e_j \otimes e_l), e_i \otimes e_k \rangle a_i^* a_j^* a_k^* a_l \\
+ \frac{1}{2} \sum_{i,j,l=1}^{N} \langle (\rho_\ast \otimes \rho_\ast) \alpha(u)(e_j \otimes e_l), e_i \otimes e_j \rangle a_i^* a_l \\
= -\sum_{j<l, i<k} \langle R(u)(e_j \wedge e_l), e_i \otimes e_k \rangle a_i^* a_k^* a_j^* a_l \\
+ \frac{1}{2} \sum_{i,j,l=1}^{N} \langle (\rho_\ast \otimes \rho_\ast) \alpha(u)(e_j \otimes e_l), e_i \otimes e_j \rangle a_i^* a_l.
\]

\[\square\]

Remark 3.4.7 (a) Note that the second term in (3.23) in general depends on the symmetric part of \((\rho_\ast \otimes \rho_\ast)(\alpha(u))\) as well as on \(R\).

(b) If we write
\[
\alpha(u) = \sum \mu_k(u) A_k(u) \otimes A_k(u)
\]
as in Remark 3.2.2(c), Then \(Z^\rho(u)\) in (3.23) has
\[
Z^\rho(u) = 2 \sum_k \mu_k(u) \left( \rho_\ast(A_k(u)) \rho_\ast(A_k(u)) \right).
\]

Corollary 3.4.8 For the derivative process in \(GLM\) of a cohesive generator \(A\) given in Hörmander form without a drift, the zero order operator induced by the vertical diffusion on the exterior bundles \(\wedge TM\) is the generalized Weitzenböck curvature given by:
\[
-\frac{1}{2} d^q \left( \text{Ric}^\# \right)(V) - \sum_{\substack{1 \leq i, k \leq n, 1 \leq j \leq p \leq l}} R_{ikjl} a_i^* a_j^* a_k^* a_l V
\]
for all \(V \in \wedge^q TM\). Here \(R_{ikjl} = \langle R(e_i, e_k) e_l, e_j \rangle\), \(1 \leq i, k \leq n, 1 \leq j, l \leq p\) for \(R\) the curvature tensor of the associated connection.
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Proof. By Theorem 3.3.1,

$$\alpha(u) = \frac{1}{2} \sum \left( u^{-1} \nabla_{u(-)} X^p \right) \otimes \left( u^{-1} \nabla_{u(-)} X^p \right), \quad u \in GLM.$$  

By Corollary C.5 in [27] the restriction of $\alpha$ to anti-symmetric tensors is $\frac{1}{2} R$.

By the relation between the curvature tensor and the curvature operator:

$$\langle R(u \wedge v), w \wedge z \rangle = \langle R(u, v) z, w \rangle,$$

the first term in $\lambda^\rho(u)$ of Lemma 3.4.5 is:

$$-2 \sum_{i<k,j<l} \langle R(u(e_j \wedge e_i), e_i \wedge e_k) a_i^* a_j^* a_j a_i = -2 \sum_{i<k,j<l} R_{jikl} a_i^* a_j^* a_j a_i.$$  

By (ii) of Remark 3.4.7, the second term is

$$\frac{1}{2} d \wedge^q \left( \sum_{p=1}^m u^{-1} \nabla_{u(-)} X^p X^p \right).$$

The required result follows since

$$\beta(u) = -\frac{1}{2} \sum_{p=1}^m u^{-1} \left( \nabla_{u(-)} X^p X^p \right) - \frac{1}{2} u^{-1} \left( \text{Ric}^\# u(-) \right).$$

Corollary 3.4.8 reflects the results in [27], Theorem 2.4.2, concerning Weitzenböck formula for Hörmander form operators on differential forms. In particular it gives another approach to the result that when $\nabla$ is the Levi-Civita connection, as holds for gradient stochastic differential equations, the generator induced on differential forms by the derivative process is the Hodge-Kodaira Laplacian up to a first order term.

Note that if $B$ is the operator on $GLM$ determined by the Hörmander form (3.5) of $A$ then for a representation $\rho : GL(M) \to \mathcal{L}(V; V)$ with associated $\pi^\rho : GL(n) \to \mathcal{L}(V; V)$ the induced operator $\mathcal{F}^\rho(B)$ on sections of $\pi^\rho$ is also given by the ‘Hörmander form’ $\frac{1}{2} \sum_j \mathcal{L}_{X^j} \mathcal{L}_{X^j} + \mathcal{L}_A$, where for any $C^1$ vector field $Y$ on $M$ and any $C^1$ section $U$ of $\pi^\rho$ the Lie derivative $\mathcal{L}_Y U \in \mathcal{F}$ is given by

$$(\mathcal{L}_Y U)(x) = u \frac{d}{dt} \left( T_{\eta_t^Y} U \right) \bigg|_{t=0}.$$
for $x \in M$, $u$ a frame at $x$, and $(\eta_t^Y)$ the flow of $Y$, using the notation of (3.13). Indeed by (3.13), for $Z(u) = \bar{u}U(\pi(u))$, so $U = \mathcal{F}^\rho(Z)$,

$$\mathcal{F}^\rho(B)(U) = \mathcal{F}^\rho\left[\left(\frac{1}{2} \sum_j \mathcal{L}_{(X_j)^{GL}} \mathcal{L}_{(X_j)^{GL}} + \mathcal{L}_{A^{GL}}\right)(Z)\right]$$

while $\mathcal{L}_{(X_j)^{GL}}(Z)(u) = \frac{d}{dt}Z(T\eta_t^{X_j} \circ u)\bigg|_{t=0}$ so that

$$\mathcal{F}^\rho\left[\mathcal{L}_{(X_j)^{GL}}(Z)\right](x) = \bar{u} \frac{d}{dt}Z(T\eta_t^{X_j} \circ u)\bigg|_{t=0} = \mathcal{L}_{X_j}(U)(x).$$

This representation of $\mathcal{F}^\rho(B)$ was noted in the case of the operator induced on differential forms by a stochastic flow on differential forms in [27], and for the case of the Hodge-Kodaira Laplacian in Elworthy [23].

**Example 3.4.9** Let $P$ be the orthonormal frame bundle for a Riemannian metric on $M$. Let $C : \mathbb{R}^n \to \mathbb{R}^n$ be a symmetric map, define

$$\alpha = \sum_{i,j} \text{trace}(C(A_i^{-}), A_j^{-}) A_i \otimes A_j,$$

where $\{A_i = \sqrt{2}e_i \wedge e_j\}$ is an orthonormal basis of $\mathfrak{so}(n)$. Then

$$\text{Comp} \circ \alpha = -\frac{1}{4} (\text{trace } C) \text{id} + \frac{1}{4}(2 - n) C.$$

Let $\text{Ric}^\# : TM \to TM$ be the Ricci curvature (for the Levi-Civita connection, say). When applied to $C(u) = u \text{Ric}^\#(u^{-1} - )$ for $u \in P$ with Ric positive, we see it defines a vertical operator on the orthonormal frame bundle with coefficients $\alpha$ as given above, $\beta = 0$. Its associated zero order term on vertical fields is then $\frac{1}{4}(2 - n) \text{Ric}^\#(u^{-1} - ) - \frac{1}{4} k$, where $k$ is the scalar curvature.

**Proof.** First observe that

$$\alpha = \frac{1}{2} (\text{d} \otimes \text{d} C) \left( \sum_i A_i \otimes A_i \right).$$

Then we use the elementary fact about elementary matrices $\{E_{ij}\}$:

$$E_{ij} C E_{ij} = C_{ij} E_{ij},$$

and take the basis of $g$ to be $\{\sqrt{2}e_i \wedge e_j, i < j\}$. Recall that $e_i \wedge e_j = 1/2(E_{ij} - E_{ji})$.  

$\square$
Remark 3.4.10 We have seen in Corollary 3.4.8 that there is zero order operator on the associated bundle \( \bigwedge F \to M \) represented by the Weitzenböck curvature of a given connection. On the other hand given a curvature operator \( R \) of a metric connection, or more generally an operator which has the same symmetry properties as a curvature tensor, is there a canonical vertical diffusion operator on \( GLM \) which induces zero order operators on differential forms which have the form of the Weitzenböck curvatures of \( R \)? A vertical operator with such a zero order term always exist since we can take \( R \) in a diagonal form:

\[
R(u) = \sum_{n=1}^{N} A_n(u) \wedge A_n(u),
\]

for some \( A_n : GLM \to \mathfrak{gl}(n) \) which are \( \text{ad}(G) \)-invariant, e.g. by taking an isometric embedding (e.g. see [27]). In this case let \( (e^j) \) be a basis of \( E_{\pi(u)} \):

\[
\alpha(u) = \frac{1}{2} \sum_{n=1}^{N} A_n(u) \otimes A_n(u),
\]

\[
\beta(u) = -\frac{1}{2} \sum_{n=1}^{N} (A_n(u))^2 - \frac{1}{2} \sum_{j=1}^{p} R(-, e^j) e^j,
\]

see Remark 3.4.7(b). Then \( \alpha \) is positive and we can define an operator with its coefficients \( \alpha \) and \( \beta \) given as above.

For a discussion of the representation of \( R \) in the form of (3.25) see Kobayashi-Nomizu [41] (Notes 17 and 18). In particular there is a discussion there of the number \( N \) required and of a rigidity theorem originating from Chern, See also Berger-Bryant-Griffiths [8].

When \( M \) is Riemannian with positive semi-definite curvature operator \( R : \bigwedge^2 TM \to \bigwedge^2 TM \) there is a canonical construction. For this take the orthonormal frame bundle \( \pi : OM \to M \), with \( G = O(n) \). We will use the isomorphism of \( \bigwedge^n \mathbb{R}^n \) with \( \mathfrak{so}(n) \) under which \( e_p \wedge e_q \) corresponds to \( \frac{1}{2}(E_{[p,q]} - E_{[q,p]}) \) for \( e_1, \ldots, e_n \) a fixed basis of \( \mathbb{R}^n \) and \( E_{[p,q]} \) the elementary matrix so \( E_{[p,q]}(v) = v_q e_p \).

Set \( A_{[p,q]} = \frac{1}{\sqrt{2}}[E_{[p,q]} - E_{[q,p]}] \) so \( \{ A_{[p,q]} : 1 \leq p < q \leq n \} \) forms an orthonormal basis for \( \mathfrak{so}(n) \). Define

\[
\alpha : OM \to \mathfrak{so}(n) \times \mathfrak{so}(n)
\]

by

\[
\alpha(u) = \sum_{1 \leq p \leq q \leq n, 1 \leq p' \leq q' \leq n} \langle R(\bigwedge^2(u)(e_p \wedge e_q), \bigwedge^2(u)(e_{p'} \wedge e_{q'})), \pi(u) A_{[p,q]} \otimes A_{[p',q']} \rangle.
\]
Our representation $\rho$ is just the identity map and, by (3.19) and Bianchi’s identity, the restriction of $\alpha(u) : \mathbb{R}^n \otimes \mathbb{R}^n \rightarrow \mathbb{R}^n \otimes \mathbb{R}^n$ to $\wedge^2 \mathbb{R}^n$ is just $\mathcal{R}$ itself. In the notation of (3.23) we see

$$\langle Z^\rho(v^1), v^2 \rangle = -4 \text{Ric}(v^1, v^2).$$

If we take $\beta = 0$, we obtain from (3.23) that

$$\lambda^{\wedge}(u) = - \sum_{i<k,j<l} R_{jli}a_i^* a_k^* a_j a_l - 2 (d \wedge) \text{Ric}^\#.$$

To get the full Weitzenböck term, extend $\alpha$ over GLM by equivariance and define $\beta(u)$, for $u \in \text{GLM}$, by $\beta(u) = \frac{3}{2} u^{-1} \text{Ric}^\#(u)$ as in (3.26).
Chapter 4

Projectible Diffusion Processes

Let \( M^+ \) be the Alexandrov one point compactification of a smooth manifold \( M \). Consider the space \( C_{y_0}^+ \) of processes \((y_t)\) with life time \( \zeta \) on \( N^+ \) such that \( t \to y_t \) is continuous with \( y_t = \Delta \) when \( t \geq \zeta \). Let \( \mathcal{L} \) be a diffusion operator on \( M \) and let \( \{P_{y_0}, y_0 \in M^+\} \) be the family of \( \mathcal{L} \)-diffusion measures in the sense of [39], i.e. the solution to the martingale problem on \( \mathcal{C}(M^+) \) so the canonical process \((y_t, 0 \leq t < \zeta)\) with the system of diffusion measures \( \{P^\mathcal{L}_{y_0}, y_0 \in N^+\} \) is a strong Markov process on \( M^+ \). Denote by \( E \) mathematical expectation with respect to the measure \( P_{y_0} \). We may add to these notations the relevant subscripts or superscripts indicating the diffusion operator or the Markov process concerned, e.g. \( \{P^\mathcal{L}_{y_0}\}, \zeta^\mathcal{L}, E^\mathcal{L}_{y_0} \) or even \( E^{y_0} \).

For \( y_0 \in M \) and \( f \in C^\infty_c(M) \), the space of smooth functions on \( M \) with compact support, let

\[
M_t^{df} := P_t^{df, \mathcal{L}} := f(y_{t\wedge \zeta}) - f(y_0) - \int_0^{t\wedge \zeta} \mathcal{L}f(y_s)ds
\]

(4.1)

Then \( (M_t^{df} : 0 \leq t < \infty) \) is a martingale on the probability space \((\mathcal{C}(M), P^\mathcal{L}_{y_0})\) with respect to the \( \{\mathcal{F}_t^{y_0}\} \), where \( \mathcal{F}_t^{y_0} = \sigma\{y_s; 0 \leq s \leq t\} \). Moreover it has bracket

\[
\langle M^{df} \rangle_t = 2 \int_0^{t\wedge \zeta} \sigma^\mathcal{L}((df)_{y_s}, (df)_{y_s})ds.
\]

This definition extends to the case of \( C^2 \) functions \( f \) but then \( M_t^{df} \) is only defined for \( 0 \leq t < \zeta^\mathcal{L} \) and is a local martingale.
4.1 Integration of predictable processes

**Proposition 4.1.1** Let \( \tau \) be a stopping time with \( \tau < \zeta \) and let \( \{ \alpha_t : 0 \leq t < \tau \} \) be a \( \mathcal{F}_t^\mathcal{P} \) predictable process in \( T^*_y M \) such that \( \alpha_t \in T^*_y M \) for each \( t \in [0, \tau) \), and for each compact subset of \( M \) we have

\[
\int_0^\tau \chi_K(y_s)\alpha_s(\sigma^\mathcal{L}\alpha_s) \, ds < \infty
\]

almost surely.

Then there is a unique local martingale \( \{ M^\alpha_t : 0 \leq t < \tau \} \) such that for all \( f \in C_\infty^c M \),

\[
\langle M^\alpha, M^{df} \rangle_t = 2 \int_0^t \sigma^\mathcal{L} (\alpha_s, (df)_y) \, ds, \quad t < \zeta.
\]

(4.2)

**Proof.** We can write

\[
\alpha_t = \sum_{j=1}^m g^j_t \cdot df_j(y_t),
\]

(4.3)

where the functions \( g^j_t \) are predictable real valued processes, e.g. by taking \( (f_1, \ldots, f_m) : M \to \mathbb{R}^m \) to be an embedding and \( g^j_t = \alpha_t \circ X^j \), for \( X(x) = \sum_{i=1}^m X^i(x) e_i \) the projection from \( \mathbb{R}^m \) to \( T_x M \). Using a partition of unity, at the cost of having an infinite, but locally finite sum, we can assume that the \( f_j \) in the representation are all in \( C_\infty^c M \). Define

\[
M^\alpha_t := \sum_j \int_0^t g^j_s dM^{df_j}_s.
\]

(4.4)

Clearly (4.2) holds. For uniqueness suppose \( K \) is a local martingale orthogonal to \( M^{df} \) for all \( f \in C_\infty^c M \).

Then \( K \) vanishes since the martingale problem for \( \mathcal{L} \) is well posed by an argument attributed to Dellacherie (see Rogers-Williams [63], the end of the proof of theorem 2.5.1). In fact it it were not zero we could take a suitable stopping time \( \tau \) to ensure \( (1 + K^{0}_{\tau \wedge t}) P^\mathcal{L}_{x_0} \) solves the martingale problem up to time \( t \) since

\[
K^0_{\tau \wedge t} M^{df}_s \equiv K^0_{\tau \wedge t} \left( f(x_s) - f(x_0) - \int_0^s \mathcal{L} f(x_s) ds \right), \quad 0 \leq s \leq t
\]

is a uniformly integrable martingale. \( \square \)
4.1. INTEGRATION OF PREDICTABLE PROCESSES

We will often write

$$M^\alpha_t = \int_0^t \alpha_s \, d\{y_s\}$$  \hspace{1cm} (4.5)

bringing out the fact it is the martingale part of the Stratonovitch integral \(\int_0^t \alpha_s \circ dy_s\) of \((\alpha_t)\) along the diffusion process \((y_t)\) when that integral is defined, e.g. when \((\alpha_t)\) is a continuous semi-martingale. Indeed

**Lemma 4.1.2** Let \(\alpha\) be a \(C^2\) 1-form then

$$M^\alpha_t = \int_0^t \alpha_{ys} \circ dy_s - \int_0^t (\delta^L \alpha)(y_s) \, ds, \quad 0 \leq t < \zeta. \hspace{1cm} (4.6)$$

**Proof.** This is clear for an exact 1-form. Suppose \(\lambda : M \to \mathbb{R}\) is \(C^2\) and \(\alpha\) is exact, then for \(t < \zeta\),

\[
\begin{align*}
M^{\lambda \alpha}_t &= \int_0^t \lambda(y_s) \, dM^\alpha_s = \int_0^t \lambda(y_s) \circ dM^\alpha_s - \frac{1}{2} \left\langle \int_0^t d\lambda(y_s) \, dy_s, M^\alpha_t \right\rangle \\
&= \int_0^t \lambda(y_s) \alpha_{ys} \circ dy_s - \int_0^t \lambda(y_s) (\delta^L \alpha)(y_s) \, ds - \frac{1}{2} \langle M^{d\lambda}, M^\alpha \rangle_t \\
&= \int_0^t \lambda(y_s) \alpha_{ys} \circ dy_s - \int_0^t \delta^L (\lambda \alpha)(y_s) \, ds
\end{align*}
\]

since \(M^{d\lambda}\) is the martingale part of \(\lambda(y_s)\) and

$$\langle M^{d\lambda}, M^\alpha \rangle_t = 2 \int_0^t \sigma^L (d\lambda_s, \alpha_s) \, ds.$$  

This proves the result for general \(\alpha\) by taking a suitable representation. \(\square\)

Let \(S_x\) be the image of \(\sigma^L_x\) in \(T_xM\) and let \(S := \cup_x S_x\). By a predictable \(S^*\)-valued process \((\alpha_t)\) over \((y_t : 0 \leq t < \zeta)\) we mean a process \((\alpha_t : 0 \leq t)\) such that

(i) \(\alpha_t \in S^*_{y_t}\) for all \(0 \leq t < \zeta\)

(ii) \((\alpha_t \circ \sigma^L_{y_t}, 0 \leq t < \zeta)\) is a predictable process in \(TM\), canonically identified with \(T^{**}M\).

Note that condition (ii) is equivalent to
(ii’) there exists a predictable \((\tilde{\alpha}_t)\) in \(T^*M\) over \((y_t)\) such that \(\tilde{\alpha}_t|_{S_{y_t}} = \alpha_t\) for all \(0 \leq t < \zeta\).

That (ii’) implies (ii) is immediate. To see (ii) implies (ii’) first note that \(\alpha_t \circ \sigma_{y_t} \in S_{y_t}\) for each \(t\) since \(\alpha_t \circ \sigma_{y_t} = \sigma_{y_t}(\tilde{\alpha}_t)\) for any extension \(\tilde{\alpha}_t\) of \(\alpha_t\) to \(T^*y_t\). We can then choose a measurable selection \(\tilde{\alpha}_t\) in \(T^*y_t\) with \(\sigma_{y}(\tilde{\alpha}_t) = \alpha_t \circ \sigma_{y_t}\). This process \(\tilde{\alpha}_t\) will satisfy the requirements of (ii’) since

\[
\tilde{\alpha}_t \sigma_{y_t} = \sigma_{y_t} \tilde{\alpha}_t = \alpha_t \sigma_{y_t}.
\]

In fact (4.7) is a reflection of the fact that \(\sigma_{y_t}\) extends to a linear isomorphism \(\sigma_y : S^*_y \rightarrow S_y\) canonically. In particular \(\sigma_{y}(\alpha_t)\) is well defined.

**Definition 4.1.3** If \((\alpha_t)\) satisfies (i) and (ii) we will say it is in \(L^2\) if

\[
\int_0^t \alpha_s \sigma_{y_t}(\alpha_s)ds < \infty
\]

for all \(t \geq 0\), and will say it is in \(L^2_{L^2,loc}\) if for any compact subset \(K\) of \(M\)

\[
E \int_0^{t \wedge \zeta} \chi_K(y_s) \alpha_s (\sigma_{y_t} \alpha_s) ds < \infty
\]

for all \(t \geq 0\).

**Remark 4.1.4** Suppose the processes associated to diffusion operators \(L\) and \(L + L_b\) are both non-explosive, where \(b\) is a locally bounded measurable vector field on \(M\). Assume that there exists a \(T^*M\)-valued process \(b^#\) defined on the canonical probability space \(C_{y_0}M\) such that \(P^L\)-almost surely:

1. \(2\sigma_L(b^#) = b(y_s)\)
2. \(\int_0^t b^# \sigma_L(b^#)ds < \infty\)

Then, by the GMCM-theorem, as in the Appendix section 9.1, we have on \(C([0, T]; M),\)

\[
P^{L + L_b} = Z_t P^L
\]

where \(Z_t = \exp\{M_t^{b^#} - \int_0^t b^# \sigma_L(b^#)ds\}\). In an obvious notation, for suitable \(\alpha\), as canonical processes we have, almost surely,

\[
\int_0^t \alpha_s d\{y_s\}^L = \int_0^t \alpha_s d\{y_s\}^{L + L_b} - \int_0^t \alpha(b(u_s))ds.
\]
Lemma 4.1.5 Suppose \( \sigma^L \) has image in a subset \( S \) of \( TM \). Then \((M_\alpha^\sigma)\) depends only on the restriction of \( \alpha_s \) in \( \mathcal{L}(T_{y_t}M; \mathbb{R}) \) to \( S_{y_t}, 0 \leq s < \zeta \). In particular (4.2) defines uniquely a local martingale for each predictable \( S^* \)-valued process \((\alpha_t)\) over \((y_t)\) for which the right hand side of (4.2) is always finite almost surely.

Proof. For \( T^*M \)-valued \( \mathcal{F}_s^{y_0} \) predictable processes \((\alpha^1_t, 0 \leq t < \zeta)\) and \((\alpha^2_t, 0 \leq t < \zeta)\) over \((y_t, 0 \leq t < \zeta)\) which agree on \( S \) we see

\[
\langle M^\alpha^1 - M^\alpha^2, M^d f \rangle_t = 2 \int_0^{t \wedge \zeta} \sigma(\alpha^1_s - \alpha^2_s, (df)_y) ds = 0
\]

for all \( f \in C_\infty^c M \). Therefore \( M^\alpha^1 = M^\alpha^2 \). On the other hand this also shows that if \( \alpha_s \in S_{y_t}^* \) for all \( s \), we can use condition (ii)' above to choose a predictable process \( \{\bar{\alpha}_s : 0 \leq s < \zeta\} \) with values in \( T^*M \) over \((y_t)\) and set \( M^\alpha = M_{\bar{\alpha}} \) without ambiguity.

Example 4.1.6 Canonical Brownian motion associated to a cohesive diffusion. For simplicity assume that our \( L \)-diffusion from a given point \( y_0 \) is non-explosive.

If \( L \) is cohesive with sub-bundle \( E \) of \( TM \), take a metric connection \( \Gamma \) for \( E \), using the metric determined by \( 2\sigma^L \). Let

\[
\alpha_s(\sigma) := (\|_{s})^{-1} : E_{\sigma(s)} \rightarrow E_{y_0}
\]

be the inverse of parallel translation, \( \|_{s} \), along \( \sigma \) from \( E_{\sigma(0)} \) to \( E_{\sigma(s)} \), for \( \mathbb{P}^{y_0} \) almost all paths \( \sigma \) in \( M \). Each component of this with respect to an orthonormal basis for \( E_{y_0} \) clearly lies in \( L_2^E \). With the obvious extension of our notation to the vector space valued case define an \( E_{y_0} \)-valued process \( B_t : t \geq 0 \) by

\[
B_t = M_t^\alpha = \int_0^t (\|_{s})^{-1} d\{y_s\}.
\]

It is easy to check from its quadratic variation that it is a Brownian motion on the inner product space \( E_{y_0} \). Moreover (as described in [27]) it has the same filtration as the canonical process on \( \mathcal{C}_{y_0}M \) up to sets of measure zero. It is the martingale part of the stochastic anti-development \( \int_0^t (\|_{s})^{-1} d\{y_s\} \) of our \( L \)-diffusion from \( y_0 \).

The use of a different metric connection would change it by a random rotation, so this process is defined on the canonical probability space \( \{\mathcal{C}_{y_0}M, \mathcal{F}^{y_0}, \mathbb{P}^{y_0}\} \) and up to such rotations depends only on it. We have, for \( \alpha \) as usual:

\[
\int_0^t \alpha_s d\{y_s\} = \int_0^t (\alpha_s \circ \|_{s}) dB_s.
\]
CHAPTER 4. PROJECTIBLE DIFFUSION PROCESSES

Using the definitions in the Appendix 9.3 we see that if our diffusion process \( y \) is a \( \Gamma \)-martingale then
\[
\int_0^t \alpha_s d\{y_s\} = (\Gamma) \int_0^t \alpha_s dy_s.
\]
(4.9)

Note that there is always some metric connection \( \Gamma \) on \( E \) for which a cohesive diffusion process is a \( \Gamma \)-martingale, by section 2.1 of [27].

4.2 Horizontality and filtrations

We can characterise horizontality of a diffusion operator or process in terms of filtrations using the following lemma:

**Lemma 4.2.1** Suppose \( p : N \to M \) is a smooth map, \( B \) a smooth diffusion operator over a smooth diffusion operator \( A \), and also

1. \( \sigma^A \) and \( \sigma^B \) have constant rank and
2. the filtration generated by \( u \) and \( p(u) \) agree up to sets of \( P^\|B\|u_0 \)-measure zero for some \( u_0 \in N \).

Then \( \text{rank} \sigma^B_u = \text{rank} \sigma^A_{p(u)} \), all \( u \in N \).

**Proof.** Set \( p = \text{rank} \sigma^A_x \) and \( \tilde{p} = \text{rank} \sigma^B_u \). By assumption \( p \) and \( \tilde{p} \) do not depend on \( x \in M \) and \( u \in N \). Take connections on \( \text{Image} \sigma^B \) and \( \text{Image} \sigma^A \) which are metric for the metrics induced by the symbols. Extend these connections to \( TN \) and \( TM \). The martingale part of the stochastic anti-development of \( (u \cdot) \) will be a Brownian motion stopped at \( \zeta^B \) of dimension \( \tilde{p} \) and that of \( (p(u) \cdot) \) will be one of dimension \( p \). By (ii) these have the same filtration up to sets of measure zero. But this implies \( p = \tilde{p} \) by the martingale representation theorem, as required.

**Proposition 4.2.2** The following are equivalent for \( B \) over \( A \) when \( A \) is cohesive:

(a) \( B = A^H \)

(b) \( B \) is cohesive and the filtration generated by its associated diffusion \( (u \cdot) \) agrees with that of \( p(u) \cdot \) up to sets of \( P^\|B\|u_0 \)-measure zero for given \( u_0 \in N \).

**Proof.** If (b) holds, Lemma 4.2.1 shows that \( \text{Image} \sigma^B_u = H_u \) for each \( u \in N \), since by (2.4) we always have \( H_u \subset \text{Image} \sigma^B_u \). Thus (b) implies criterion (ii) of Proposition 2.3.2. Also (b) follows from (iii) of Proposition 2.3.2 by considering the stochastic differential equation driven by horizontal lifts \( \tilde{X}^0, \ldots, \tilde{X}^m \).
4.3 The Filtering Equation

Let $p : N \to M$ be a smooth surjective map. Suppose that $B$ is over $A$. However we do not assume $\sigma^A$ of constant rank. Let $\{P^B_{u_0}\}$ and $\{P^A_{x_0}\}$ be, respectively, the solutions to the martingale problem for $B$ and $A$ on the canonical spaces $C(M^+)$ and $C(N^+)$. Denote by $(u_t)$ and $(x_t)$ the corresponding canonical processes with life time $\zeta^N$ and $\zeta^M$ respectively. Note that $\zeta^B \leq \zeta^A \circ p$ almost surely with respect to $P^B_{u_0}$. We shall assume that the paths of the diffusion on $N$ do not explode before their projections on $M$ do, more precisely $\zeta^M \circ p = \zeta^N$ almost surely with respect to $P^B_{u_0}$ for each $u_0$, equivalently,

**Assumption S.**

\[ C^p_{u_0} M^+ := \{ \sigma : [0, \infty) \to M^+ : \lim_{t \to \zeta^B_{u_0}} p(u_t) = \Delta \text{ when } \zeta^B < \infty \} \]

has full $P^B_{u_0}$ measure for each $u_0 \in N$.

Denote by the following the filtrations induced by the processes indicated:

\[ \mathcal{F}^u_{t_0} = \sigma(u_s, 0 \leq s \leq t), \quad \mathcal{F}^{x_0} = \sigma(x_s, 0 \leq s < \infty) \]

\[ \mathcal{F}^p_{(u_0)} = \sigma(p(u_s), 0 \leq s \leq t), \quad \mathcal{F}^{x_0} = \sigma(x_s, 0 \leq s < \infty) \]

**Proposition 4.3.1** Under Assumption S. $p^A_{u_0}(P^B_{u_0}) = P^A_{p(u_0)}$ and $P^B_{x_0}(f \circ p) = P^A_{t}(f \circ p)$ for all $f \in C_\infty^c(M)$.

**Proof.** If $p(u_0) = x_0$, $f \in C_\infty^c(M)$, we only need to show that $M_t^{d(f,A)}$ is a martingale with respect to $p^A_{u_0}(P^B_{u_0})$. Using Assumption S,

\[ M_t^{d(f,A)}(p(u)) = f(p(u_t)) - f(p(u_0)) - \int_0^t A f \circ p(u_s))ds \]

\[ = f(p(u_t)) - f(p(u_0)) - \int_0^t (B(f \circ p))(u_s)ds \]

\[ = M_t^{d(f \circ p,B)} \]

is a martingale with respect to $(\mathcal{F}^{u_0}_{t_0})$ and $P^B_{u_0}$. Take $s \leq t$ and let $G$ be a $\mathcal{F}^{x_0}_s$-measurable function. Then

\[ E^{p^A_{u_0}} \{ M_t^{d(f,A)} G \} = E^{P^B_{u_0}} \{ M_t^{d(f,A)}(p(u_0))G(p(u_0)) \} \]

\[ = E^{P^B_{x_0}} \{ M_t^{d(f \circ p,B)} G \circ p \} = E^{P^B_{u_0}} \{ M_s^{d(f \circ p,B)} G \circ p \} \]

\[ = E^{p^A_{u_0}} \{ M_s^{d(f,A)} G \} \]
and the result follows from the uniqueness of the martingale problem. □

We will need the following elementary lemma:

**Lemma 4.3.2** Let \( (\Omega, \mathcal{F}, \mathcal{F}_t, P) \) be a filtered probability space and \( \mathcal{G} \) a subfiltration of \( \mathcal{F} \) with the property that for all \( s \geq 0 \),
\[
\mathbb{E}\{A|\mathcal{G}_s\} = \mathbb{E}\{\mathbb{E}\{A|\mathcal{F}_s\}|\mathcal{G}\}, \quad \forall A \in \mathcal{F}, \quad (4.10)
\]
where \( \mathcal{G} = \vee_s \mathcal{G}_s \). Then

(i) \( (\mathbb{E}\{M_t|\mathcal{G}\}, t \geq 0) \) is a \( \mathcal{G}_s \)-martingale whenever \( (M_t : t \geq 0) \) is an \( \mathcal{F}_s \)-martingale;

(ii) For all \( \mathcal{G} \)-measurable and integrable \( H \)
\[
\mathbb{E}\{H|\mathcal{F}_s\} = \mathbb{E}\{H|\mathcal{G}_s\};
\]

(iii) \( \mathbb{E}\{\mathbb{E}\{A|\mathcal{F}_s\}|\mathcal{G}\} = \mathbb{E}\{\mathbb{E}\{A|\mathcal{G}\}|\mathcal{F}_s\}, \quad \forall A \in \mathcal{F}. \)

**Proof.** For (i) set \( N_t = \mathbb{E}\{M_t|\mathcal{G}\}, 0 \leq t < \infty \). By (4.10), \( (N_t) \) is \( \mathcal{G}_t \) measurable. For \( s \leq t \) suppose that \( f \) is \( \mathcal{G}_s \)-measurable and bounded. Then \( \mathbb{E}(N_t f) = \mathbb{E}(M_t f) = \mathbb{E}(M_s f) = \mathbb{E}(N_s f) \). For (ii), let \( H \) and \( F \) be bounded measurable functions with \( \mathcal{G} \)-measurable and \( \mathcal{F}_s \)-measurable representations. Then
\[
\mathbb{E}\{H|F\} = \mathbb{E}\{H|\mathbb{E}\{F|\mathcal{G}\}\} = \mathbb{E}\{H|\mathbb{E}\{F|\mathcal{G}_s\}\} = \mathbb{E}\{H|\mathbb{E}\{F|\mathcal{G}\}\}
\]
using (4.10). Thus \( \mathbb{E}\{H|\mathcal{F}_s\} = \mathbb{E}\{H|\mathcal{G}_s\} \) as required. Part (iii) follows from (ii) on taking \( H = \mathbb{E}\{A|\mathcal{G}\} \) and using equation (4.3.2). \( \Box \)

Part (ii) of the following proposition says that the filtration \( \mathcal{F}^{p(u_0)}_s \) is immersed in the terminology of Tsirelson [71].

**Proposition 4.3.3** (i) For fixed \( t > 0 \) let \( f \) be a bounded \( \mathcal{F}^{u_0}_t \)-measurable function. Then
\[
\mathbb{E}\{f|\mathcal{F}^{p(u_0)}_t\} = \mathbb{E}\{f|\mathcal{F}^{p(u_0)}_t\}.
\]

(ii) All \( \mathcal{F}^{p(u_0)}_s \) martingales are \( \mathcal{F}^{u_0}_s \) martingales. In fact if \( f = G \circ p \) for \( G \) an integrable functional on \( C(M^+) \) with respect to \( P^A \), we have
\[
\mathbb{E}^{u_0}\{f|\mathcal{F}^{p(u_0)}_t\} = \mathbb{E}^{u_0}\{f|\mathcal{F}^{u_0}_t\}.
\]
Proof. (i) Write \( f = F(u_s) : 0 \leq s \leq t \) for \( F \) a bounded measurable function on \( C(N^+) \). Let \( G \) be bounded measurable functions of \( \{ p(u_s) : 0 \leq s \leq t \} \) and \( g^1, \ldots, g^k \) bounded Borel functions on \( M \), with \( h^1, \ldots, h^k \) positive real numbers. By the Markov property of \( u \). and of \( p(u) \),

\[
E(F(u_s) : 0 \leq s \leq t) G P^A \left( g^1 P^A \left( g^2 P^A \left( g^k \right) \right) \right) (p(u_t)).
\]

Therefore,

\[
E \{ F(u_s) : 0 \leq s \leq t \} | \mathcal{F}^{p(u)} = E \{ F(u_s) : 0 \leq s \leq t \} | \mathcal{F}^{p(u)}
\]
as required.

Part (ii) is immediate from (i) by Lemma 4.3.2.

As in §2.1 set \( E_x = \text{Image} \sigma^A_x \) with \( h_u : E_{p(u)} \to T_u N \) the horizontal lift defined by (2.4), although now we have no constant rank assumption and no smoothness of \( \eta \). Also let \( E^B_u = \text{Image} \sigma^B_u \). For an \( \mathcal{F}^{x_0} \)-predictable \( E^* \)-valued process \( \phi_t = \phi_t(\sigma), \) \( 0 \leq t < \zeta^A \) along \( \{ \sigma_t : 0 \leq t < \zeta^A \} \), let \( (p^*(\phi_t) : 0 \leq t < \zeta^B) \) be the pull back restricted to be an \( (E^B)^* \)-valued process along \( (u_t : 0 \leq t < \zeta^B) \) defined by

\[
p^*(\phi_t)(u) = \phi_t(p(u)) \circ T_{u_t} p : E^B_{u_t} \to R.
\]

Since \( \phi_t \) has a predictable extension \( \tilde{\phi}_t \) so does \( p^*(\phi_t) \) and so the latter is predictable. Moreover \( p^*(\phi_t) \sigma^B(p^*(\phi_t)) = \tilde{\phi}_t \sigma^A(\tilde{\phi}_t) \) by Lemma 2.1.1 showing \( \tilde{\phi} \) is in \( L^2_A \) if and only if \( p^*(\phi) \) is in \( L^2_B \). For such \( \phi \) we have the following intertwining:

**Proposition 4.3.4** Let \( \phi \) be a predictable \( L^2_A \)-valued process.

1. For \( \mathbb{P}^{\mathbb{B}}_{u_0} \) almost surely all sample paths, \( M_t^{A,\phi} \circ p = M_t^{B, p^*(\phi)} \) for \( t < \zeta^B \).

2. If \( \alpha \in L^2_B \) with \( \alpha_t \circ h_t = 0 \) almost surely, then \( \{ M_t^\alpha, M_t^{df \circ T_p} \} = 0 \) and \( E^{B, u_0} \{ M_t^\alpha | \mathcal{F}^{p(u)} \} = 0 \) for all \( C^1 \) functions \( f \) on \( M \).

**Proof.** For \( \phi = df \), (1) follows from \( p^*(df)_u = d(f \circ p)_u \) as in the proof of Proposition 4.3.1. For general \( \phi \), taking a predictable extension if necessary, write \( \phi_t(x) = \sum_{j=1}^m g^j_t(x)(df^j)_{x_t} \) for smooth functions \( f^j : M \to R \) and real valued predictable \( \{ g^j_t : 0 \leq t < \zeta^A \} \). Therefore

\[
M_t^{A,\phi} \circ p = \sum_{j=1}^m \int_0^t g^j_s(p(u_t)) \, dM_t^{B, p^*(df^j)} = M_t^{B, p^*(\phi)}
\]
for all $t < \zeta^B$, giving (1). For (2) let $F : N \to \mathbb{R}$ be a smooth measurable function with respect to $\mathcal{F}^{p(u)}$. Then $F = f(p(u))$ for some measurable function $f : M \to \mathbb{R}$.

$$E^{B,u_0}(M_t^\alpha f(p(u))) = \frac{1}{2}E^{B,u_0}(M_t^\alpha, M_t^{df\circ Tp}) = \frac{1}{2}E^{B,u_0} \int_0^t \sigma^B(\alpha_s, df \circ Tp(u_s))ds.$$

If $\alpha_t h_{us} = 0$ almost surely for all $t$, we apply (2.4) to see

$$\sigma^B(\alpha_s, df \circ Tp(u_s)) = \alpha_s\sigma^B_{us}(T^*_p df) = \alpha_s h_{us}\sigma^A_{p(u_s)} \cdot df = 0$$

and thus $E^{B,u_0}(M_t^\alpha f(p(u))) = 0$ giving (2).

For $\alpha \in L_2^B$ define $\beta_s \equiv E^{B,u_0}\{\alpha_s \circ h_{us}|p(u_s) = x\}, 0 \leq s < \zeta$ to be the unique, up to equivalence, element of $L_2^A$ such that

$$E^{B,u_0}\left(\alpha_s \circ h_{us} \sigma^A(\phi_s(p(u)))\right) = E^{A,p(u_0)}\left(\beta_s \sigma^A(\phi_s)\right). \quad (4.11)$$

for any $\phi \in L_2^A$. To see such an element exists and is unique recall that

$$\alpha_s \circ h_{us} \sigma^A_{p(u_s)} = \alpha_s\sigma^B_{us}(t_{us}, p)_s^*$$

which is an $\mathcal{F}^{p(u)}$-predictable process with values in $E_{p(u_s)} \subset T_{p(u_s)}M$ at each time $s$, and by Proposition 4.3.3, (4.11) is equivalent to

$$\beta_s(p(u_s))\sigma^A_{p(u_s)} = E^{B,u_0}\left\{\alpha_s\sigma^B_{us}(T_{us}, p)_s^*|\mathcal{F}^{p(u)}\right\} \quad (4.12)$$

in the sense of Elworthy-LeJan-Li [27]. The predictable projection theorem and the results of [27] shows that there is a unique, up to indistinguishability, $\mathcal{F}^{p(u)}_s$-predictable $TM$ version of $\{\gamma_t : 0 \leq t < \zeta\}$ say, over $\{p(u_t) : 0 \leq t < \zeta\}$, of the right hand side of (4.12). By applying the uniqueness part of this projection theorem to $\{\phi_s(\gamma_s) : 0 \leq s < \zeta\}$ when $\phi$ is $\mathcal{F}^{p(u)}_s$-predictable, $T^*M$-valued over $p(u)$ and $\phi_t$ vanishes on $E_{p(u_s)}$ for all $0 \leq t < \rho$ with probability 1, we see $\gamma_t \in E_{p(u_s)}$ for all $0 \leq t < \zeta$ almost surely. Now set $\beta_s(p(u)) = [\sigma^A_{p(u_s)}]^{-1}\gamma_s$ in $E_{p(u_s)}^\ast$.

**Proposition 4.3.5** For any $\alpha$ in $L_2^B$ we have

$$E^{B,u_0}\left\{M_t^\alpha | p(u) = x\right\} = \int_0^T E^{B,u_0}\left\{\alpha_s \circ h_{us} | p(u) = x\right\} d\{x_s\}.$$
4.4. A FAMILY OF MARKOVIAN KERNELS

Proof. Set $N_t = \mathbb{E}\{M^t \mid \mathcal{F}^{p(u_0)}\}$ and write $N_t(p(u))$ for $\{N_t\}$ a $\mathcal{F}^{x_0}$-measurable function. By Proposition 4.3.3, $(N_t)$ is an $\mathcal{F}^{(u_0)}$-martingale and we see $(\tilde{N}_t)$ is an $\mathcal{F}_t^x$ martingale. Take $g \in C_c^\infty$ then by Proposition 4.3.4,

$$\langle \tilde{N}, M^A, dg \rangle_t \circ p(u) = \mathbb{E}^{B,u_0} \left\{ \langle M^t, M^d(gp) \rangle_t \mid \mathcal{F}^{p(u_0)} \right\}(u)$$

$$= \mathbb{E}^{B,u_0} \left\{ \sigma^B_{u_t} (\alpha_t, (T_{u_t}p)^*(dg)) \mid \mathcal{F}^{p(u_0)}_t \right\}$$

$$= \mathbb{E}^{B,u_0} \left\{ \alpha_t \circ h_{u_t} \sigma^A_{p(u)}(dg) \mid \mathcal{F}^{p(u_0)}_t \right\}$$

by equation (2.4). By Proposition 4.1.1 and the definition above of the conditional expectation, $\tilde{N}_t(p(u_0)) = M^{A,\beta}$ for $\beta \circ p(u_0) = \mathbb{E}^{B,u_0} \left\{ \alpha_t \circ h_{u_t} \sigma^A_{p(u)}(dg) \mid \mathcal{F}^{p(u_0)}_t \right\}$ and so

$$\tilde{N}_t(x) = \int_0^t \mathbb{E} \{ \alpha_s \circ h_{u_s} | p(u) = x \} \, ds$$

as required. □

4.4 A family of Markovian kernels

For a probability measure $\mu_0$ on $N^+$ let the measures $\mu_t$ on $N^+$ be the flow of $u_t$ under $P_{B_{\mu_0}}$ and set $\nu_t = p_*(\mu_t)$ on $M^+$. Let $\eta_{\mu_0}$ be the law of $u. \mapsto (p(u), u_0)$ on $C(M^+) \times N^+$ under $P_{B_{\mu_0}}$ so

$$\eta_{\mu_0}(A, \Gamma) = \int_{y \in M^+} P_y^A(A) \rho^y_{\mu_0}(\Gamma) \, \nu(dy), \quad A \in \mathcal{B}(M^+), \Gamma \in \mathcal{B}(N^+)$$

where $\rho^y_{\mu_0}$ arises from a disintegration of $\mu_0$

$$\mu_0(\Gamma) = \int_{y \in M^+} \rho^y_{\mu_0}(\Gamma) \, \nu(dy), \quad \Gamma \in \mathcal{B}(N^+).$$

For a measurable $f : N^+ \to \mathbb{R}$, integrable with respect to $\mu_t$ set

$$\pi_t^{\mu_0,\sigma} f(v) = \mathbb{E}^{B_{\mu_0}} \{ f(u_t) | p(u_0) = \sigma, u_0 = v \}. \quad (4.13)$$

It is defined for $\eta_{\mu_0}$ almost all $(\sigma, v)$ in $C(M^+) \times N^+$. In particular for $P_{\mu_0}^A$-almost all $\sigma$ it is defined for $\rho^\sigma_{\mu_0}(0)$-almost all $v \in N^+$. We could use the convention that

$$\pi_t^{\mu_0,\sigma} f(v) = 0$$
if \( p(v) \neq \sigma(0) \). With this convention, if we define \( \theta_t \sigma(s) = \sigma(t + s) \) we see that for \( P^A_{\nu_0} \)-almost all \( \sigma \) the map \( y \mapsto \pi^\mu_{\nu_0, \sigma} f(y) \) is defined for \( \mu_0 \)-almost all \( y \) in \( N^+ \).

Further for \( u_0 \in N \) and \( f : N^+ \rightarrow \mathbb{R} \) bounded measurable define

\[
\pi_t f(u_0) : C_{p(u_0)}^1 M^+ \rightarrow \mathbb{R},
\]

\( P^A_{p(u_0)} \)-almost surely, by

\[
\pi_t f(u_0)(\sigma) = \mathbb{E}\left\{ f(u_t) | p(u) = \sigma \right\} = \pi^\delta_{u_0, \sigma}_t f(u_0). \tag{4.14}
\]

This can be extended, as in [27], to the case of predictable process in vector bundles over \( N \), and to define

\[
\pi_t (\alpha \circ h_u)(u_0) : C_{p(u_0)}^1 M^+ \rightarrow \mathbb{R}
\]

as \( \mathbb{E}^{\mathbb{F}, u_0}\{ \alpha s_h_u \mid p(u) = x. \} \), defined above.

### 4.5 The filtering equation

**Theorem 4.5.1**

(1) If \( f \) is \( C^2 \) \( N \), or more generally if \( f \) is \( C^2 \) with \( B f \) and \( \sigma^B(df, df) \circ h \) bounded, then

\[
\pi_t f(u_0) = f(u_0) + \int_0^t \pi_s(B f)(u_0) ds + \int_0^t \pi_s(df \circ h_u)(u_0) d\{x_s\}. \tag{4.15}
\]

In particular \( \{\pi_t f(u_0) : t \geq 0\} \) is a continuous \( \mathbb{F}_s \) semi-martingale.

(2) For bounded measurable \( f : M^+ \rightarrow \mathbb{R} \) and \( P^A_{\nu_0} \) almost all \( \sigma \) in \( C(M^+) \), for each \( s, t \geq 0 \)

\[
S^\mu_{t+s, \sigma} f(v) = \pi^\mu_{t, \sigma} \pi^\sigma_{s, \mu} f(v) \tag{4.16}
\]

for \( \rho_{\mu_0}^{\sigma(0)} \) almost all \( v \) in \( N^+ \).

(3) Moreover there exists a family of probability measures \( Q^\nu_{\mu_0, \sigma} \) on \( C(N^+) \) define for \( \eta_{u_0} \)-almost surely all \( (\sigma, v) \) such that if \( F : C(N^+) \rightarrow \mathbb{R} \) is of the form

\[
F(u.) = f_1(u_{t_1}) \ldots f_n(u_{t_n})
\]
4.5. THE FILTERING EQUATION

some $0 \leq t_1 < t_2 < \ldots < t_n$ and bounded measurable $f_j : N^+ \to \mathbb{R}$, $j = 1, 2, \ldots, n$ then

$$
\int_{u \in \mathcal{C}(N^+)} F(u)Q_{\mu_0,\sigma}(du) = S_{t_1}^{\mu_0,\sigma}(f_1 S_{t_2-t_1}^{\mu_1,\sigma}(f_2 \ldots S_{t_n-t_{n-1}}^{\mu_n,\sigma}(f_n)(v))
= E_{\mu_0}^B \{ F(u) | p(u) = \sigma_0 \}
$$

$\eta_{\mu_0}$-almost surely in $(\sigma, v)$.

Proof. (1). By definition of $M_{df}$ we have

$$
f(u_t) = f(u_0) + \int_0^t Bf(u_s)ds + M_{df}^t
$$

so

$$
\pi_t f(u_0) = f(u_0) + \int_0^t \pi_s Bf(u_0)ds + E \{ M_{df}^t, B | p(u) = x \} \tag{4.17}
$$

and part (1) follows from Proposition 4.3.5.

(2). We observed above that the right hand side of (4.16) is well defined for $P_{\mu_0}$ almost all $\sigma$. The equation then follows from the Markov property.

(3). The existence of regular conditional probabilities in our situation implies the existence of the probabilities $Q_{\mu_0,\sigma}$ as required, together with a standard use of the Markov property. $\square$

Remark 4.5.2 A description of the $Q_{\mu_0,\sigma}$ is given in the next section, in the case where $A$ is cohesive.

Recall we have the decomposition $F_u = H_u + V T_u N$ for each $u \in N$, and $F = \sqcup F_u$. If $\ell \in F_u^*$ there is a corresponding decomposition

$$
\ell = \ell^H + \ell^V \in F_u^*,
$$

where $\ell^H$ vanishes on $V T_u N$ and $\ell^V$ on $H_u$. For $\ell \in T_u^* N$ write $\ell^V = (\ell|F_u)^V$ and $\ell^H = (\ell|F_u)^H$.

Corollary 4.5.3 Suppose $A$ is cohesive. If $f$ is $C^3_\mathcal{C} N$ then there is the Stratonovitch equation

$$
\pi_t f(u_0)(x.) = f(u_0) + \int_0^t \pi_s (B^V f)(u_0)ds + \int_0^t \pi_s (df_{u_0} \circ h_{u_0}) \circ dx_s.
$$

(4.18)
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Proof. We use (4.17). By Proposition 4.3.5,

\[ \mathbb{E}\{ M_t^{df} \mid p(u) = x \} = \mathbb{E}\{ M_t^{df_H} \mid p(u) = x \}. \]

Note that

\[ M_t^{df_H} = \int_0^t (df_H)_{u_s} \circ du_s - \int_0^t \delta^B(df_H)(u_s)ds \]

by Lemma 4.1.2. Furthermore

\[ \delta^B(df_H) = \delta^B(df_H) + \delta^A(df_H) = \delta^A(df_H) = \delta^A_H(df) = \mathcal{A}(f) \]

since \( df^H \) vanishes on vertical vectors and \( df = df^H + df^V \) while \( df^V \) vanishes on horizontal vectors, so \( \delta^A(df^V) = 0 \). This gives

\[ \pi_t f(u_0)(x) = f(u_0) + \int_0^t \pi_s(B^V f)(u_0)(x)ds + \mathbb{E}\left\{ \int_0^t (df^H)_{u_s} \circ du_s \mid p(u) = x \right\} \]

Finally (4.18) follows since \( df^H = p^*(df \circ h_u) = df \circ h_u \circ T_u p \) and \( T_u p \circ du_t = \circ dx_t \). \( \square \)

4.6 Approximations

Assume now that the law of \( u_t \) under \( P_{u_0}^A \) is given by

\[ P_t^A(u_0, A) = \int_A P_{u_0}^A(u_0, v)dv, \quad A \in \mathcal{B}(M) \]

for \( P_{u_0}^A(u_0, v) \) a smooth density with respect to some fixed, smooth, strictly positive measure on \( M \) to which ‘\( dv \)’ refers. This is the case if \( A \) is hypoelliptic.

Consider the conditional probability

\[ q_t^{u_0,b}(V) = P_{u_0}^B\{ u_t \in V \mid p(u_t) = b \}, \quad V \in \mathcal{B}(N) \]

defined for \( p_t^A(u_0, -) \) almost sure all \( b \) in \( M \). There is the disintegration of \( p_t^B(u_0, -) \)

\[ p_t^B(u_0, V) = \int_{b \in M} q_t^{u_0,b}(V)p_t^A(p(u_0), db) \]

and the formula

\[ \mu_t^{u_0,b}(V) = \lim_{\epsilon \downarrow 0} (p_t^A(p(u_0), b))^{-1} \int_V p_{t-\epsilon}^B(u_0, dv)p_t^A(p(v), b). \]
4.7. KRYLOV-VERETENNIKOV EXPANSION

Take a nested sequence \( \{ \Pi^l \}_{l=1}^{\infty} \) of partitions of \([0, t] \)

\[
\Pi^l = \{ 0 = t_0^l < t_1^l < \cdots < t_k^l = t \},
\]
say, with union dense in \([0, t] \). For any continuous bounded \( f : N^+ \to \mathbb{R} \) there is the following approximation scheme to complete \( \pi_t f(u_0) \):

**Proposition 4.6.1**

\[
\pi_t f(u_0)(\sigma) = \lim_{l \to \infty} \int_I^{(u_0, \sigma(t_1^l))} q_{t_1^l}^{v_1, \sigma(t_1^l)}(dv_1)q_{t_2^l-t_1^l}^{v_1, \sigma(t_1^l)}(dv_2) \cdots q_{t_k^l-t_{k-1}^l}^{v_{k-1}, \sigma(t_{k-1}^l)}(dv_{k-1})f(v_{k-1}) = \lim_{l \to \infty} E_{u_0}^G \{ f(u_t) \mid p(u_{t_j}^l) = \sigma(t_j^l), \quad 1 \leq j \leq k \}.
\]

**Proof.** The two versions of the right hand sides are equal before taking limits. For \( l = 1, 2, \ldots, \), set

\[
S_l f(\sigma) = E_{u_0}^G \{ f(u_t) \mid p(u_{t_j}^l) = \sigma(t_j^l), \quad 1 \leq j \leq k \}.
\]

It is defined for \( P_{A_x} \)-almost all \( \sigma \) in \( C(M^+) \), where \( x_0 = p(u_0) \). Let \( Q^l \) be the \( \sigma \)-algebra on \( C(M^+) \) generated by \( \sigma \mapsto (\sigma(t_1^l), \ldots, \sigma(t_{k}^l)) \). Directly from the definitions we see

\[
\pi_t f = E_{u_0} [\pi_t f(u_0) \mid Q^l],
\]

and so \( \{ S_l f \}_{l=1}^{\infty} \) is an \( Q^l \)-martingale. It is bounded and so converges \( P_{A_x} \)-almost surely. Since \( \bigvee_l Q^l \) is the Borel \( \sigma \)-algebra the limit is \( \pi_t f(u_0) \) as required. \( \square \)

4.7 Krylov-Veretennikov Expansion

Suppose \( A = \sum_{j=1}^{m} L_{X_j} + L_A \) for smooth vector fields \( \{ X_j \}_{j=1}^{m} \) and \( A \). We will now take \( \{ x_t : 0 \leq t < \zeta \} \) to be the solution to the stochastic differential equation

\[
dx_t = X(x_t) \circ dB_t + A(x_t)dt, \quad (4.19)
\]

with \( x_0 \) given, for a Brownian motion \( B \) on \( \mathbb{R}^m \), rather than the canonical process. Here \( X(x) : \mathbb{R}^m \to T_x M \) is the map given by

\[
X(x)(a^1, \ldots, a^m) = \sum_{j=1}^{m} a^j X^j(x), \quad x \in M.
\]
CHAPTER 4. PROJECTIONABLE DIFFUSION PROCESSES

Let \( \{P_t : t \geq 0\} \) be the sub-Markovian semi-group generated by \( B \). Let \( f \in C_0^\infty \). Assume \( P_t f \in C_0^\infty \).

As in the proof of Theorem 4.5.1, from

\[
P_{t-s}f(u_s) = P_t f(u_0) + \int_0^s d(P_{t-r}f)_{u_r} d\{u_r\}, \quad 0 \leq s \leq t
\]

we obtain

\[
\pi_s P_{t-s}(f)(u_0)(x) = P_t f(u_0) + \int_0^s \mathbb{E} \{d(P_{t-r}f)_{u_r} \circ h_{u_r} \mid p(u_s) = x\} d\{x_r\}
\]

so that \( \pi_s P_{t-s}(f)(u_0), 0 \leq s \leq t \), is a continuous \( F_s^{x_0} \) semi-martingale. Therefore

\[
\pi_t f(u_0) - P_t f(u_0) = \int_0^t d_s(\pi_s P_{t-s}(f)(u_s))
\]

\[
= \int_0^t \mathbb{E} \{d(P_{t-r}f) \circ h_{u_r} \mid p(u_s) = x\} X(x_r) dB_r
\]

\[
= \int_0^t S_r[\{d(P_{t-r}f) \circ h_- \circ X^k(p(-))\}](u_0) dB_r^k
\]

giving a ‘Clark-Ocone’ formula for \( \pi_t f(u_0) \). Iterating this procedure formally,

\[
\pi_t f(u_0) = P_t f(u_0) + \int_0^t S_r[\{d(P_{t-r}f) \circ h_- \circ X(p(-))\}](u_0) dB_r
\]

\[
+ \int_0^t \int_0^r \pi_s[\{d(P_{t-r}f) \circ h_- \circ X^k(p(-))\} h_- \circ X^j(p(-)) dB_s^j dB_r^k
\]

we obtain the Wiener chaos expansion of \( \pi_t f(u_0)(x) \).

4.8 Conditional Laws

It will be convenient to extend the notation of section 4.3. For \( 0 \leq l < r < \infty \) let \( C(l, r; N^+) \) and \( C(l, r; M^+) \) be respectively the space of continuous paths \( u : [l, r] \to N^+ \) and \( x : [l, r] \to M^+ \) which remain at \( \Delta \) from the time of explosion; and \( C_{u_0}(l, r; N^+) \) and \( C_{x_0}(l, r; M^+) \) the paths from \( u_0 \in N^+ \) and \( x_0 \in M^+ \) respectively, Let \( \{P_{u_0}^{(l,r),B}\} \) and \( \{P_{x_0}^{(l,r),A}\} \) be the associated diffusion measures.
The conditional law of \( \{u_s : l \leq s \leq r\} \) given \( \{p(u_s) : l \leq s \leq r\} \) will be given by probability kernels \( \sigma \mapsto Q^{l,r}_{\sigma,u_0} \) defined \( P^{(l,r):A} \) almost surely from \( C_{p(u_0)}(l; r; M^+) \) to \( C^p_{u_0}(l, r; N^+) \) for each \( u_0 \in N \), where \( C^p_{u_0}(l, r; N^+) \) is the subspace of \( C_{u_0}(l, r; N^+) \) whose paths satisfy Assumption S. The defining property is that for integrable \( f : C_{u_0}(l, r; N^+) \to \mathbb{R} \)

\[
E \{ f(u.) \mid p(u_s) = \sigma_s, l \leq s \leq r \} = \int_{y \in C_{u_0}(l, r; N^+)} f(y) dQ^{l,r}_{\sigma,u_0}(y). \tag{4.20}
\]

To obtain the conditional law take the decomposition \( B = A^H + B^V \) of Proposition 2.3.5. Represent the diffusion corresponding to \( A \) by a stochastic differential equation

\[
dx'_t = X(x'_t) \circ dB_t + X^0(x'_t) dt. \tag{4.21}
\]

Take a connection \( \nabla^V \) on \( VTN \) and let

\[
(\nabla^V) \quad dz_t = V(z_t) dW_t + V^0(z_t) dt \tag{4.22}
\]

be an Itô equation whose solutions are \( B^V \)-diffusions. Here \( (W_t) \) is the canonical Brownian motion on \( \mathbb{R}^m \) for some \( m \), independent of \( (B) \), the map \( V : M \times \mathbb{R}^m \to TM \) takes values in \( \ker [Tp] \), and \( V \) and \( V^0 \) are locally Lipschitz. For such a representation of \( B^V \) diffusions see the Appendix B. Let \( \tilde{X} : N \times \mathbb{R}^m \to H \) and \( \tilde{X}^0 : N \to H \) be the horizontal lifts of \( X \) and \( X^0 \) respectively using Proposition 2.1.2. The solution to

\[
(\nabla^V) \quad dy_t = \tilde{X}(y_t) \circ dB_t + \tilde{X}^0(y_t) dt + V(y_t) dW_t + V^0(y_t) dt,
\]

\[
y_t = u_0, \quad u_0 \in N, \quad l \leq t \leq r.
\]

has law \( P^{(l,r):B}_{u_0} \). Noting that \( \tilde{X}(u) = h_u X(p(u)) \) for \( u \in M \),

\[
(\nabla^V) \quad dy_t = h_{y_t} \circ dx'_t + V(y_t) dW_t + V^0(y_t) dt,
\]

\[
y_t = u_0, \quad l \leq t \leq r,
\]

where \( x'_t = p(y_t) \) so that \( (x'_t) \) is a solution to (4.21) starting from \( p(u_0) \) at time \( l \). Without changing the law of \( y \) we can replace \( x' \) by the canonical process \( x \). Then

**Theorem 4.8.1** Consider the solution \( (y_t) \) as a process defined on the probability space \( C_{p(u_0)}(l, r; M^+) \times C_0 \mathbb{R}^m \) with product measure,

\[
y : [l, r] \times C_{p(u_0)}(l, r; M^+) \times C_0 \mathbb{R}^m \to N^+,
\]
and define $Q^{l,r}_{\sigma,u_0}$ to be the law of $y(\sigma, -) : C_0 \mathbb{R}^m \rightarrow C_{u_0}(l, r; N^+)$. For bounded measurable $f : C_{u_0}(l, r; N^+)$,

$$
E \{ f(u) \mid p(u_s) = \sigma_s, l \leq s \leq r \} = \int_{y \in C_{u_0}(l, r; N^+)} f(y) dQ^{l,r}_{\sigma,u_0}(y).
$$

**Proof.** Take a measurable function $\alpha : C_{p(u_0)}(l, r; M^+) \rightarrow \mathbb{R}$. Then

$$
E^{P_{u_0}} \left( \alpha(p(u)) \int_{y \in C_{u_0}(l, r; N^+)} f(y) dQ^{l,r}_{\sigma,u_0}(y) \right)
$$

$$
= E^{P_{p(u_0)}} \left( \alpha(x) \int_{y \in C_{u_0}(l, r; N^+)} f(y) dQ^{l,r}_{x,u_0}(y) \right)
$$

$$
= E^{P_{p(u_0)}} \left( \alpha(x) \int_{C_0 \mathbb{R}^m} (\alpha(x)f(y(x, \omega))) dP(\omega) \right)
$$

$$
= \int_{C_{p(u_0)}(l, r; M^+) \times C_0 \mathbb{R}^m} (\alpha(x)f(y(x, \omega))) dP^{A}_{p(u_0)} dP(\omega)
$$

$$
= E f(u) \alpha(p(u)),
$$

as required. \qed

Note that Theorem 4.8.1 is equivalent to the statement that $\omega \mapsto Q^{l,r}_{\sigma,u_0}$, $\omega \in C_{u_0}(l, r; N^+)$, is a regular conditional probability of $P^{(l,r),E}_{u_0}$ given $p$.

**Remark 4.8.2** Let $(\xi^l_t(\cdot, \cdot), 1 \leq t < \infty)$ be a measurable flow for (4.21) and $(\eta^l_t(\sigma, \cdot), 0 \leq t < \infty)$ one for (4.23) with $x'$ replaced by $\sigma \in C_{p(u_0)}(l, r; M^+)$. For $\omega \in \Omega$, the underlying probability space for the Brownian motion $\mathcal{B}$, define $Q^{l,r}_{\omega}$, from the space of bounded measurable functions on $N^+$ to itself, by

$$
Q^{l,r}_{\omega}(f)(u_0) = E f \left( \eta^l_t(\xi^l_t(p(u_0), \omega), u_0) \right).
$$

A direct calculation shows that

$$
Q^{l,r}_{\omega} Q^{r,s}_{\omega} = Q^{l,s}_{\omega}
$$

for $0 \leq l \leq r \leq s < \infty$. Thus their adjoints on a suitable dual space would form an evolution.

More generally, letting $\text{Borel}(X)$ stand for the Borel $\sigma$-algebra of a topological space $X$: 

4.8. CONDITIONAL LAWS

Proposition 4.8.3 Let $\varphi$ be a measurable map from $C_{x_0}(l, r; M^+)$ to some measure space, and let

$$P_{x_0}^{(l, r), \varphi} : C_{x_0}(l, r; M^+) \times \text{Borel}(C_{x_0}(l, r; M^+)) \to [0, 1]$$

be a regular conditional probability for $P_{x_0}^{(l, r)}$ given $\varphi$. For $u_0$ with $p(u_0) = x_0$ set

$$Q_{u_0}^{l, r, \varphi, p}(\omega, A) = \int_{C_{x_0}(l, r; M^+)} Q_{\sigma, u_0}^{l, r}(A) P_{x_0}^{(l, r), \varphi}(p(\omega), d\sigma)$$

for $\omega \in C_{u_0}(l, r; N^+)$ and $A \in \text{Borel}(C_{u_0}(l, r; N^+))$. Then $Q_{u_0}^{l, r, \varphi, p}$ is a regular conditional probability of $P_{u_0}^{(l, r), B}$ given $\varphi \circ p$.

Proof. By definition

$$Q_{u_0}^{l, r, \varphi, p}(\omega, A) = \mathbb{E}^{(l, r), A, x_0} \{ Q_{p(-), u_0}^{l, r}(A) \varphi \} p(\omega) = \mathbb{E}^{(l, r), A, x_0} \{ E^{(l, r), B, u_0} \chi_A | \varphi = - \} | \varphi \} p(\omega) = \mathbb{E}^{(l, r), B, u_0} \{ \chi_A | \varphi \circ p \} (\omega).$$

\[ \square \]

Corollary 4.8.4 For $\varphi$ as in Theorem 4.8.3 suppose that the canonical process on $M^+$ with law $P_{x_0}^{(0, T), \varphi}(\sigma, -)$ is a semi-martingale for almost all $\sigma$, in its own filtration $\mathcal{F}_t$, $0 \leq t \leq T$, for $P_{x_0}^{(0, T), A}$ almost all $\sigma$. Then the solution $y(\sigma, -)$ to the equation

$$\left( \nabla V \right) dy_t = h_{yt} \circ d\sigma_t + V(y_t) dW_t + V^0(y_t) dt, \quad (4.24)$$

$$y_t = u_0, \quad 0 \leq t \leq T$$

where $\sigma_t, 0 \leq t \leq T$ is run with law $P_{x_0}^{(0, T), \varphi}(\sigma, -)$, is a version of the $B$-diffusion from $u_0$ conditioned by $\varphi \circ p$.

Proof. That the law of the solution is as required follows from the discussion at the beginning of this section together with Proposition 4.8.3 and Fubini’s theorem.

\[ \square \]

Conditions under which conditioned processes are semi-martingales are discussed by Baudoin [2]. In particular bridge processes derived from elliptic diffusions are, so we obtain the following version of Carverhill’s result [12]:
**Corollary 4.8.5** Suppose $A$ is elliptic and let $b_t : 0 \leq t \leq T$ be a version of the $A$-bridge going from $x_0$ to $z$ in time $T$, some $z \in M$. Then the solutions to
\[
(\nabla V) \quad dy_t = h(y_t)dW_t + V(y_t)dt, \quad (4.25)
y_0 = u_0, \quad 0 \leq t \leq T
\]
give a version of the $B$ diffusion from $u_0$ conditioned on $p(u_T) = z$.

### 4.9 Equivariant case: skew product decomposition

In the equivariant case, when $N$ is the total space $P$ of a principal bundle $\pi : P \to M$ as in §5, a version of Theorem 4.8.1 is given in [25] which reflects the additional structure. In particular the following is proved there:

**Proposition 4.9.1** Let $B$ be an equivariant diffusion operator on $P$ which induces a cohesive diffusion operator $A$ on $M$. Let $\{y_t : 0 \leq t < \zeta\}$ be a $B$-diffusion on $P^*$. Then
\[
y_t = \tilde{x}_t \cdot g_{\tilde{x}}^t,
\]
where

(i) $\{\tilde{x}_t : 0 \leq t < \zeta\}$ is the horizontal lift of $p(y_t)$, starting at $y_0$, using the semi-connection induced by $B$

(ii) $\{g_{\sigma}^t : 0 \leq t < \zeta(\sigma)\}$ is a diffusion independent of $\{p(y_t) : 0 \leq t < \zeta\}$ on $G$ starting at the identity with time dependent generator $\mathcal{L}_t^\sigma$ given by
\[
\mathcal{L}_t^\sigma f(g) = \sum_{i,j} \alpha^{ij}(\sigma(t) \cdot g) L_{A_i^*} L_{A_j^*} f(g) + \sum \beta^k(\sigma(t)g) L_{A_k^*} f(g),
\]
for any $\sigma \in GP^+$, $0 \leq t < \zeta(\sigma)$, where $A_1^*, \ldots, A_k^*$ are the left invariant vector fields on $G$ corresponding to a basis of $g$ and the $\alpha^{ij}$ and $\beta^k$ are the coefficients for $B^V$ as in Theorem 3.2.1.

Note that for each $t$ the operator $\mathcal{L}_t^\sigma$ is conjugate to the restriction of $B^V$ to the fibre through $\sigma(t)$ by the map
\[
\begin{align*}
g &\mapsto p^{-1}(p(\sigma(t))) \\
g &\mapsto \sigma(t)g.
\end{align*}
\]
It is a right invariant operator.
Remark 4.9.2 Note that by the equivariance of $\mathcal{L}^\sigma$ there will be no explosion of the process $(g^\sigma)$ before that of $\sigma$. Consequently Assumption S of §4.3 holds automatically.

Below we give the equivariant version of Proposition 4.8.1. We shall use the notation of §4.8. However we replace the one point compactification $P^+$ of $P$ by $\tilde{P} = P \cup \Delta$ with the smallest topology agreeing with that of $P$ and such that $\pi : \tilde{P} \to M^+$ is continuous. Also let $G^+$ be the one point compactification $G \cup \Delta$ of $G$ with group multiplication and action of $G$ extended so that

$$u \cdot \Delta = \Delta, \Delta \cdot g = g \cdot \Delta = \Delta, \quad \forall u \in \tilde{P}, g \in \tilde{G}.$$

For $0 \leq l < r < \infty$ if $y \in \mathcal{C}(l, r; \tilde{P})$, we write $l_y = l$ and $r_y = r$. Let $\mathcal{C}(\ast, \ast; \tilde{P})$ be the union of such spaces $\mathcal{C}(l, r; P)$. It has the standard additive structure under concatenation: if $y$ and $y'$ are two paths with $r_y = l_{y'}$ and $y(r_y) = y'(l_{y'})$ let $y + y'$ be the corresponding element in $\mathcal{C}(l_y, r_y; \tilde{P})$. The basic $\sigma$-algebra of $\mathcal{C}(\ast, \ast, \tilde{P})$ is defined to be the pull back by $\pi$ of the usual Borel $\sigma$-algebra on $\mathcal{C}(\ast, \ast; M^+)$. Given an equivariant diffusion operator $B$ on $P$ consider the laws $\{P_a^{(l, r), B} : a \in P\}$ as a kernel from $P$ to $\mathcal{C}(l, r; \tilde{P})$. The right action $R_y$ by $g$ in $G^+$ extends to give a right action, also written $R_y$, of $G^+$ on $\mathcal{C}(\ast, \ast, \tilde{P})$. Equivariance of $B$ is equivalent to

$$P_{a}^{(l, r), B} = (R_y)_{a}^{(l, r), B}$$

for all $0 \leq l \leq r$ and $a \in P$. Therefore $\pi_s(P_{a}^{(l, r), B})$ depends only on $\pi(a)$, $l$, $r$ and gives the law of the induced diffusion $\mathcal{A}$ on $M$. We say that such a diffusion $B$ is basic if for all $a \in P$ and $0 \leq l < r < \infty$ the basic $\sigma$-algebra on $\mathcal{C}(l, r; \tilde{P})$ contains all Borel sets up to $P_{a}^{(l, r), B}$ negligible sets, i.e. for all $a \in P$ and Borel subsets $B$ of $\mathcal{C}(l, r; \tilde{P})$ there exists a Borel subset $A$ of $\mathcal{C}(l, r, M^+)$ s.t. $P_{a}^{(l, r), B}(\pi^{-1}(A) \Delta B) = 0$.

For paths in $G$ it is more convenient to consider the space $\tilde{\mathcal{C}}_{id}(l, r; G^+)$ of cadlag paths $\sigma : [l, r] \to G^+$ with $\sigma(l) = \text{id}$ such that $\sigma$ is continuous until it leaves $G$ and stays at $\Delta$ from then on. It has a multiplication

$$\tilde{\mathcal{C}}_{id}(s, t; G^+) \times \tilde{\mathcal{C}}_{id}(t, u; G^+) \longrightarrow \tilde{\mathcal{C}}_{id}(s, u; G^+)$$

$$(g, g') \mapsto g \times g'$$

where $(g \times g')(r) = g(r)$ for $r \in [s, t]$ and $(g \times g')(r) = g(t)g'(r)$ for $r \in [t, u]$.

Given probability measures $Q$, $Q'$ on $\tilde{\mathcal{C}}_{id}(s, t; G^+)$ and $\tilde{\mathcal{C}}_{id}(t, u; G^+)$ respectively this determines a convolution $Q \ast Q'$ of $Q$ with $Q'$ which is a probability measure on $\tilde{\mathcal{C}}_{id}(s, u; G^+)$. 

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4.9. **EQUIVARIANT CASE: SKEW PRODUCT DECOMPOSITION**

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Theorem 4.9.3  Given the laws \( \{ P_a^{l,r} : a \in P, 0 \leq l < r < \infty \} \) of an equivariant diffusion \( B \) over a cohesive \( A \) there exist probability kernels \( \{ P_{a}^{H,l,r} : a \in P \} \) from \( P \) to \( C(l,r; \tilde{P}) \), \( 0 \leq l < r < \infty \) and \( y \mapsto Q_y^{l,r} \), defined \( P \) a.s. from \( C(l,r; \tilde{P}) \) to \( \tilde{C}(l,r; G^+) \) such that

(i) \( \{ P_{a}^{H,l,r} : a \in P \} \) is equivariant, basic and determining a cohesive generator.

(ii) \( y \mapsto Q_y^{l,r} \) satisfies

\[
Q_y^{l_y+y',r_y} = Q_y^{l_y+r_y} \star Q_y^{l_y',r_y'}
\]

for \( P^{l_y,r_y} \otimes P^{l_y',r_y'} \) almost all \( y, y' \) with \( r_y = l_y' \).

(iii) For \( U \) a Borel subset of \( C(l,r; \tilde{P}) \),

\[
P_a^{l,r}(U) = \int_{C(l,r; \tilde{P})} \int_{\tilde{C}(l,r; G^+)} \chi_U(y \cdot g) Q_y^{l,r}(dg) P_{a}^{H,l,r}(dy).
\]

The kernels \( P_{a}^{H,l,r} \) are uniquely determined as are the \( \{ Q_y^{l,r} : y \in C(l,r; \tilde{P}) \} \), \( P_{a}^{H,l,r} \) a.s. in \( y \) for all \( a \) in \( P \). Furthermore \( Q_y^{l,r} \) depends on \( y \) only through its projection \( \pi(y) \) and its initial point \( y_l \).

The proof of this theorem is as that of Theorem 2.5 in [25] (although there the processes are assumed to have no explosion).

Stochastic differential equations can be given for \( (\tilde{x}_t) \) and \( (g_t^x) \) as in §4.8, from which the decomposition can be proved via Itô’s formula; see Theorem 8.2.5 below for details of a special case.

Proposition 4.9.1 extends results for Riemannian submersions by Elworthy-Kendall [24] and related results by Liao [48]. A rich supply of examples of skew-product decomposition of Brownian motions, with a general discussion, is given in Pauwels-Rogers [58].

For a special class of derivative flows, considered as \( GLM \)-valued process as in §3.3 there is a different decomposition by Liao [49], see also Ruffino [65].

### 4.10  Induced processes on vector bundles

In the notation of §3.4 let \( \rho : G \to L(V,V) \) be a \( C^\infty \) representation with \( \Pi^\rho : F \to M \) the associated bundle. A \( B \)-diffusion \( \{ y_t : 0 \leq t < \zeta \} \) on \( P \) determines
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a family of \(\{\psi_t : 0 \leq t < \zeta\}\) of random linear map \(W_t\) from \(F_{x_0} \rightarrow F_{x_t}\), where \(x_t = \pi(y_t)\). By definition,

\[
\psi_t[(y_0, e)] = [(y_t, e)].
\]

Assuming \(\mathcal{A}\) is cohesive we have the parallel translation \(/t : F_{x_0} \rightarrow F_{x_t}\) along \(\{x_t : 0 \leq t < \zeta\}\) determined by our semi-connection. This is given by

\[
/\![y_0, e]] = [(\tilde{x}_t, e)]
\]

where \(\tilde{x}\) is the horizontal lift of \(x\), starting at \(y_0\).

When taken together with Corollary 3.4.8 the following extends results for derivative flows in Elworthy-Yor[29], Li[47], Elworthy-Rosenberg [28], and Elworthy-LeJan-Li[27].

**Theorem 4.10.1** Let \(\rho : G \rightarrow L(V; V)\) be a representation of \(G\) on a Banach space \(V\) and \(\Pi^\rho : F \rightarrow M\) the associated vector bundle. Let \(\{y_t : 0 \leq t < \zeta\}\) be a \(B\)-diffusion for an equivariant diffusion operator \(B\) over a cohesive diffusion operator \(\mathcal{A}\). Set \(x_t = p(y_t)\) and let \(\Psi_t : F_{x_0} \rightarrow F_{x_t}, 0 \leq t < \zeta\) be the induced transformations on \(F\). Then the local conditional expectation \(\{\bar{\Psi}_t : 0 \leq t < \zeta\}\), for \(\bar{\Psi}_t = \mathbb{E}\{\Psi_t|\sigma\{x_s : 0 \leq s < \zeta\}\}\) exists and is the solution of the covariant equation along \(\{x_t : 0 \leq t < \zeta\}\):

\[
\frac{D}{\partial t} \bar{\Psi}_t = \Lambda^\rho \circ \bar{\Psi}_t
\]

with \(\Psi_0\) the identity map, \(\Lambda^\rho : F \rightarrow F\) given by \(\lambda^\rho\) in Theorem 3.4.1 and where \(\frac{D}{\partial t}\) refers to the semi-connection determined by \(\mathcal{B}\).

**Proof.** From above and Proposition 4.9.1 we have

\[
\Psi_t[(y_0, e)] = [(\tilde{x}_t \circ g_{g_t}, e)] = [(\tilde{x}_t, \rho(g_t)^{-1} e)]
\]

and so \(\bar{\Psi}_t[(y_0, e)] = [(y_0, \rho(g_t)^{-1} e)]\). Now from the right invariance of \(G^r\), for fixed path \(\sigma\) and time \(t\), we can apply Baxendale’s integrability theorem for the right action

\[
G \times L(V; V) \rightarrow L(V; V)
\]

\[(g, T) \mapsto \rho(g)^{-1} \circ T\]
to see $E|\rho(g^\sigma_t)^{-1}|_L(V;V) < \infty$ for each $\sigma$, $t$ and we have $E(\sigma)_t \in L(V;V)$ given by

$$E(\sigma)_t e = E\rho(g^\sigma_t)^{-1}e.$$ 

By considering $(1 + E|\rho(g^\sigma_t)^{-1}|_L)^{-1}\psi_t$ for $\sigma = \cdot$. We see the local conditional expectation $\bar{\Psi}_t$ exists in $L(F_{x_0};F_{x_t})$ and $\bar{\Psi}_t[(y_0, e)] = [(\tilde{x}_t, E(x.)_t e)]$.

The computation in Theorem 3.4.1 shows that

$$\frac{d}{dt} \bar{\Psi}_t[(y_0, e)] = \frac{d}{dt} [(y_0, E(x.)_t e)] = [(y_0, \lambda^\rho(\tilde{x}_t) E(x.)_t e)]$$

giving

$$D\frac{d}{dt} \bar{\Psi}_t[(y_0, e)] = [(\tilde{x}_t, \lambda^\rho(\tilde{x}_t) E(x.)_t e)] = \Lambda^\rho(x.) \bar{\Psi}_t[(y_0, e)]$$

as required. $\square$

**Remark 4.10.2** Theorem 4.10.1 could also be used to identify the generator of the operator induced on sections of $F^*$, reproving Theorem 3.4.1, since if $\phi \in \gamma F^*$ then $E\phi \circ \bar{\Psi}_t \chi_{t<\cdot} = E\phi \circ \bar{\Psi}_t \chi_{t<\cdot}$ if the expectations exist, by Corollary 3.3.5 of [27]. The extra information in Theorem 4.10.1 is the existence of the conditional expectation. Baxendales’ integrability theorem used for this applies in sufficiently generality to give corresponding results for infinite dimensional $G$, for example in the situation arising in chapter 8 below.
Chapter 5

Filtering with non-Markovian Observations

So far we have considered smooth maps $p : N \rightarrow M$ with a diffusion process $u$ on $N$ mapping to a diffusion process $x = p(u)$ on $M$. From the point of view of filtering we have considered $u$ as the signal and $x$ as the observation process. However the standard set up for filtering does not assume Markovianity of the observation process. Classically we have a signal $z$, a diffusion process on $\mathbb{R}^d$ or a more general space, and an observation process $x$ on some $\mathbb{R}^n$ given by an SDE of the form

$$dx_t = a(t, x_t, z_t)dt + b(t, x_t, z_t)dB_t$$

(5.1)

where $B$ is a Brownian motion independent of the signal. To fit this into our discussion we will need to assume that the noise coefficient of the observation SDE does not depend on the signal other than through the observations, as well as the usual cohesiveness assumptions. We can take $N = \mathbb{R}^d \times \mathbb{R}^n$ and $M = \mathbb{R}^n$ with $p$ the projection and $u_t = (z_t, x_t)$. To reduce to our Markovian case we can use the standard technique of applying the Girsanov-Maruyama theorem. Here we first carry this out in the general context of diffusions with basic symbols, as discussed in Section 2.4 and then show how it fits in with the classical situation. For simplicity we shall assume that the signal is a time homogeneous diffusion, and that the coefficients in the observation SDE are also independent of time. The state spaces are taken to be smooth manifolds and the standard non-degeneracy assumptions on the observation process somewhat relaxed.

For other discussions about filtering with processes which have values in a manifold see [18], [59], and [32].
CHAPTER 5. FILTERING WITH NON-MARKOVIAN OBSERVATIONS

5.1 Signals with Projectible Symbol

Using the notation and terminology of Section 2.4 suppose that our diffusion operator \( \mathcal{B} \) on \( N \) is conservative and descends cohesively over \( p : N \to M \) so that for a horizontal vector field \( b^H \) on \( N \) the diffusion operator \( \tilde{\mathcal{B}} := \mathcal{B} - b^H \) lies over some cohesive \( \mathcal{A} \). Choose such an \( \mathcal{A} \) so that \( \tilde{\mathcal{B}} \), and so \( \mathcal{A} \), is also conservative: *we assume that this is possible*. Also choose a locally bounded one-form \( b^\# \) on \( N \) with \( 2\sigma^\mathcal{B}(b^\#) = b^H \). This is possible since \( b^H \) is horizontal, and we can, and will, choose \( b^\# \) to vanish on vertical tangent vectors and satisfy

\[
 b^\#_y(b^H(y)) = 2\sigma^\mathcal{B}_y(b^\#, b^\#) = |b^H(y)|^2_y \quad y \in N \tag{5.2}
\]

where \( |b^H(y)|_y \) refers to the Riemannian metric on the horizontal tangent space induced by \( 2\sigma^\mathcal{A}^H \). This can be achieved by first choosing some smooth \( \tilde{b} : N \to T^*M \) such that, in the notation of equation (2.10), \( \sigma^\mathcal{A}_p(y)(\tilde{b}(y)) = b(y) \) for \( y \in N \); and then taking \( b^\# \) to be the pull back of \( \tilde{b} \) by \( p \):

\[
 b^\#_y(v) = \tilde{b}(y)(T_y p(v)) \quad y \in N
\]

Now set

\[
 Z_t = \exp\{-M_t^\alpha - \frac{1}{2} \langle M^\alpha \rangle_t\}
\]

for \( \alpha(u) = b^\#_u \) where \( u \in C([0,T]; N) \), our canonical probability space furnished with measures \( \mathcal{P} := \mathcal{P}^\mathcal{B} \) and \( \tilde{\mathcal{P}} := \mathcal{P}^\tilde{\mathcal{B}} \) and corresponding expectation operators \( \mathbb{E} \) and \( \tilde{\mathbb{E}} \).

Here and below we are using the notation of proposition 4.1.1 with \( M^\alpha \) etc referring to taking martingale parts with respect to \( \mathcal{P} \) while \( \tilde{M}^\alpha \) and \( \int_0^t \alpha_s d\{y_s\} \) are with respect to \( \tilde{\mathcal{P}} \).

From the Girsanov-Maruyama-Cameron-Martin theorem (see the Appendix, Section 9.1), we know that \( Z_t \) is a martingale under \( \mathcal{P} \) and the two measures are equivalent with

\[
 \frac{d\tilde{\mathcal{P}}_y}{d\mathcal{P}_{y_0}} = Z_T.
\]

Suppose \( f : N \to \mathbb{R} \) is bounded and measurable. We wish to find \( \pi_t(f) : N \to \mathbb{R}, 0 \leq t \leq T \) where

\[
 \pi_t(f)(y_0) = \mathbb{E}_{y_0}\{ f(u_t) | p(u_s), 0 \leq s \leq t \}.
\]
5.1. SIGNALS WITH PROJECTIBLE SYMBOL

Following the approach due to Zakai, consider the unnormalised filtering process \( \hat{\pi}_t(f) : N \to \mathbb{R} \) given by
\[
\hat{\pi}_t(f)(u_0) = \hat{E}_{u_0}\{ f(u_t)Z_t^{-1} \mid p(u_s), 0 \leq s \leq t \}.
\]

For completeness we state and prove the Kallianpur-Striebel formula, a version of Bayes’ formula:

**Lemma 5.1.1**
\[
\pi_t(f)(u_0) = \frac{\hat{\pi}_t(f)(u_0)}{\hat{\pi}_t(1)(u_0)} \quad \mathbb{P}_{u_0} \text{ - a.s.}
\]

**Proof.** Set \( x_0 = p(u_0) \). Let \( g : C_{u_0}([0,T]; N) \to \mathcal{F}^{u_0}_t \)-measurable. Then
\[
\mathbb{E}_{u_0}\{ f(u_t)g(u) \} = \hat{E}\{ \frac{1}{Z_t} f(u_t)g(u) \}
\]
\[
= \hat{E}\{ \mathbb{E}\{ \frac{1}{Z_t} f(u_t) \mid \mathcal{F}^{u_0}_t \} g(u) \}
\]
\[
= \mathbb{E}\{ Z_t \hat{E}\{ \frac{1}{Z_t} f(u_t) \mid \mathcal{F}^{u_0}_t \} g(u) \}. \quad (5.3)
\]

Thus
\[
\pi_t(f)(u_0) = \mathbb{E}\{ Z_t \mid \mathcal{F}^{u_0}_t \} \hat{\pi}_t(f)(u_0).
\]
Taking \( f \) constant shows that \( \mathbb{E}\{ Z_t \mid \mathcal{F}^{u_0}_t \} \hat{\pi}_t(1)(u_0) = 1 \) and the result follows.

\[\square\]

We can now go on to obtain the analogue of the Duncan-Mortensen-Zakai (DMZ) equation for the unnormalized filtering process, using the results of Section 4.8 on conditional laws:

**Theorem 5.1.2** For any \( C^2 \) function \( f : N \to \mathbb{R} \), under \( \hat{P} \),
\[
\hat{\pi}_t f(u_0) = f(u_0) + \int_0^t \hat{\pi}_s(\mathbb{B}f)(u_0) \, ds + \int_0^t \hat{\pi}_s(fb^\#(\cdot)h_{\cdot})(u_0) d\{ x_s \} + \int_0^t \hat{\pi}_s(df - h_{\cdot})(u_0) d\{ x_s \}; \quad (5.4)
\]
\[
\hat{\pi}_t f(u_0) = f(u_0) + \int_0^t \hat{\pi}_s(\mathbb{B}f)(u_0) \, ds + \int_0^t \hat{\pi}_s(fb)(u_0), d\{ x_s \} , x_s + \int_0^t \hat{\pi}_s(df - h_{\cdot})(u_0) d\{ x_s \}. \quad (5.5)
\]
where \( x_s = p(u_s), 0 \leq s \leq \infty \) is the projection to \( M \) of the canonical process from \( u_0 \) on \( N \), and \( h \) the horizontal lift map for the induced semi-connection.

Using an alternative notation:

\[
\hat{\pi}_t f = \hat{\pi}_0 f + M^\#_t (f \circ h u) \cdot A + M^\#_t (d f \circ h u) \cdot A + \int_0^t \hat{\pi}_s (B f) ds.
\] (5.6)

**Proof.** Since we are working with \( \hat{P} \) we will write \( M^\# \) for \( M^\#_t \cdot \hat{P} \), etc. Also \( Z_t^{-1} \) satisfies:

\[
dZ_t^{-1} = Z_t^{-1} dM^\#_t
\]

while

\[
df(u_t) = dM^f_t + \tilde{B}(f)(u_t) dt
\]

giving

\[
d(Z_t^{-1} f(u_t)) = Z_t^{-1} dM^f_t + \tilde{B}(f)(u_t) dt + f(u_t) Z_t^{-1} dM^b\#_t + Z_t^{-1} + df_u(b^H(u_t)) dt
\]

since \( dM^f_t dM^b\#_t = \sigma^E(df_u, b\#) = df_u(b^H(u_t)) \). Thus

\[
d(Z_t^{-1} f(u_t)) = Z_t^{-1} dM^f_t + \tilde{B}(f)(u_t) dt + f(u_t) Z_t^{-1} dM^b\#_t + Z_t^{-1}.
\]

We can now take conditional expectations using proposition 4.3.5 since \( B - L_{bH} \) is over the cohesive operator \( A \) to complete the proof. \( \square \)

**Lemma 5.1.3** There are the following formulae for angle brackets:

\[
d(\hat{\pi}(1))_t = \langle \hat{\pi}_t(b), \hat{\pi}_t(b) \rangle_{z_t} dt
\] (5.7)

\[
d(\hat{\pi}(1), \hat{\pi}(f))_t = \langle \hat{\pi}_t(fb), \hat{\pi}_t(b) \rangle_{x_t}^E dt + \hat{\pi}_t(df \circ h_u) \circ \hat{\pi}_t(b(u)) dt
\] (5.8)

**Proof.** From the previous theorem

\[
\langle \hat{\pi}(1), \hat{\pi}(f) \rangle dt = (dM^\#_t (fb^\# \circ h), A) + dM^\#_t (df \circ h u, A) dM^\#_t (b^\# \circ h), A
\]

\[
= 2\sigma^A(\hat{\pi}_t (fb^\# \circ h), \hat{\pi}_t (b^\# \circ h)) dt + 2\sigma^A(\hat{\pi}_t(df \circ h u), \hat{\pi}_t(b^\# \circ h)) dt
\]

\[
= \langle \hat{\pi}_t (fb), \hat{\pi}_t (b) \rangle_{x_t} dt + \hat{\pi}_t(df \circ h_u) \circ \hat{\pi}_t(b(u)) dt
\]
since for any one form $\phi$ on $M$ we have:

$$\sigma^A(\phi, \tilde{\pi}_t(b^\# \circ h)) = \tilde{\pi}_t((\phi|_E, b^\# \circ h)|^{E^*})$$

$$= \frac{1}{2}\tilde{\pi}_t(\phi(b))$$

$$= \frac{1}{2}\phi(\tilde{\pi}_t(b)).$$

This gives the second formula, from which comes the first.

We can now give a version of Kushner’s formula in our context:

**Theorem 5.1.4** In terms of the probability measure $\tilde{\mathcal{P}}$

$$\pi_t f = \pi_0 f + \int_0^t \pi_s B(f) ds + \int_0^t \pi_s(df \circ h_u) \left[ d\{x_s\} - \pi_s(b(u)) ds \right]$$

$$+ \int_0^t \langle \pi_s(bf) - \pi_s(f) \pi_s(b), d\{x_s\} - \pi_s(b) \rangle.$$  \hspace{1cm} (5.9)

**Proof.** From the definition and then Ito’s formula:

$$d\pi_t(f) = d\left(\frac{\tilde{\pi}_t(f)}{\tilde{\pi}_t(1)}\right)$$

$$= \frac{d\tilde{\pi}_t(f)}{\tilde{\pi}_t(1)} - \frac{\tilde{\pi}_t(f)d\tilde{\pi}_t(1)}{(\tilde{\pi}_t(1))^2} - \frac{d\tilde{\pi}_t(f)d\tilde{\pi}_t(1)}{(\tilde{\pi}_t(1))^3}$$

$$+ \frac{\tilde{\pi}_t(f)d\tilde{\pi}_t(1)d\tilde{\pi}_t(1)}{(\tilde{\pi}_t(1))^3}.$$ 

Now substitute in the second formula of Theorem 5.1.2 and use the previous lemma.

Note that $\tilde{\pi}_t(f)$, $b$, and $\tilde{\mathcal{P}}$, depend on the choice of $A$. We would like to have a version of formula 5.9 which is independent of such choices. First note that if $B - b^H_1$ is over $A_1$, and $B - b^H_2$ is over $A_2$, then the difference of the two vector fields on $N$ descends to a vector field on $M$: if $g : M \to \mathbb{R}$ is smooth and $\tilde{g} = g \circ p : N \to \mathbb{R}$ then

$$(b^H_2 - b^H_1)\tilde{g} = (B - b^H_1)\tilde{f} - (B - b^H_1)\tilde{g} = (A_1 - A_2)g.$$
Therefore if we set \( b_0(z) = T_y p(b^H_2(y) - b^H_1(y)) \) for \( p(y) = z, z \in M \) then \( A_1 = A_2 + L b_0 \), and by Remark 4.1.4
\[
d\{x_s\}^{A_2} = d\{x_s\}^{A_1} + b_0 ds
\] (5.10)
From this we see immediately that the symbols \( d\{x_s\} - \pi_s(b) ds \), and \( \pi_s(f(b) - \pi_s(f) \pi_s(b) \) in formula 5.9 are in fact independent of the choice we made of \( \mathcal{A} \). To relate to now classical concepts we next discuss the first of these in more detail.

### 5.2 Innovations and innovations processes

Keeping the notation above, for \( \alpha \in L^2_\mathcal{A} \), so \( \alpha_t \in T^*_\mathcal{X}_t M \) for \( 0 \leq t < \infty \), define a real valued process \( I^\alpha_t : 0 \leq t < \infty \), the \( \alpha \)-innovations process by
\[
I^\alpha_t = \int_0^t \alpha_s \left( d\{x_s\}^\mathcal{A} - \pi_s b(u_s) ds \right)
\] (5.11)
A generalisation of a standard result about innovations processes is:

**Proposition 5.2.1** The process \( I^\alpha \) is independent of the choice of \( \mathcal{A} \). Under \( P_{B, \mathcal{M}_0} \) it is an \( \mathcal{F}_{x_0} \) martingale.

**Proof.** The observations just made show it is independent of the choice of \( \mathcal{A} \). It is clearly also adapted to \( \mathcal{F}_{x_0} \). To prove the martingale property note first that by Proposition 4.3.4 and formula (5.10)
\[
\int_0^t \alpha_s d\{x_s\}^\mathcal{A} = \int_0^t p^*(\alpha_s) d\{u_s\}^{B - L b^H} = \int_0^t \alpha_s b(u_s) ds.
\]
From this we see that if \( 0 < r < t \) and \( Z \in \sigma\{x_s : 0 \leq s \leq r\} \) then
\[
E_{\mathcal{X}Z} \left\{ \int_r^t \alpha_s \left( d\{x_s\}^\mathcal{A} - \pi_s b(u_s) \right) ds \right\} = 0
\]
giving the required result.
5.2. INNOVATIONS AND INNOVATIONS PROCESSES

If we fix a metric connection, $\Gamma$, on $E$, as described in Example 4.1.6 we can take the canonical Brownian motion, $B^\Gamma_{-A}$ say, on $E_{x_0}$ determined by $A$ and $\Gamma$. Then, by equation (4.8), we can write $d\{x_s\}^A - \pi_s(b(u))ds = \sqrt{\pi}dB^\Gamma_{-A} - \pi_s(b(u))ds$. In terms of the the $\mathbb{P}$ Brownian motion, $B_{-A}$, on $E_{x_0}$, which is the martingale part under $\mathbb{P}$ of the $\Gamma$- stochastic anti-development of $x$, we can define an $E_{x_0}$-valued process, $z^\Gamma_t : 0 \leq t < \infty$, by

$$z^\Gamma_t = B^\Gamma_t + \int_0^t (\sqrt{\pi})^{-1}(b(u_s) - \pi_s(b(u)))ds.$$  \hspace{1cm} (5.12)

A candidate for the innovations process of our signal -observation system is the stochastic development, $\nu^\Gamma_{-A}$ say, of $z^\Gamma_t$. This can be defined by using the canonical sde on the orthonormal frame bundle of $E$, namely

$$d\tilde{\nu}_t = X(\tilde{\nu}_t)(\tilde{\nu}_0)^{-1} \circ dz_t$$

for a fixed frame $\nu_0$ for $E_{x_0}$. Here

$$X(\mu)(e) = h^\Gamma_{\mu}(\mu(e)).$$

for $\mu : \mathbb{R}^p \to E_m$ a frame in at some point $m \in M$, and $e \in \mathbb{R}^p$, for $p$ the fibre dimension of $E$. The process $\nu^\Gamma$ is then the projection of $\tilde{\nu}$ on $M$. For example see [22]. It will satisfy the Stratonovich equation

$$d\nu_t^\Gamma = \sqrt{\pi} \circ dz_t$$ \hspace{1cm} (5.13)

where the parallel translation is now along the paths of $\nu^\Gamma$. Let $\Theta : C_0(M) \to C_0(M)$ be the map given by $\Theta(\sigma)_t = \nu^\Gamma_t(\sigma)_t$, treating $z^\Gamma_t$ as defined on $C_0(M)$. Let $D = D^\Gamma : C_0(T_{x_0}M) \to C_{x_0}$ be the stochastic development using $\Gamma$ with inverse $D^{-1}$. We will continue to assume that there is no explosion so that these maps are well defined. For example,

$$z(x.) = D^{-1}\Theta(x.).$$

We define a semi-martingale, on $M$ to be a $\Gamma$-martingale if it is the stochastic development using $\Gamma$ of a local martingale, see the Appendix, Section9.3.

**Theorem 5.2.2** For each metric connection $\Gamma$ on $E$ the innovations process $\nu^\Gamma$ is a $\Gamma$-martingale. If $\Gamma$ is chosen so that the $A$-diffusion process is a $\Gamma$-martingale.
under $P^A$ then for $\alpha : [0, \tau) \times C_{x_0} M \to T^* M$ which is predictable and lives over $x$, provided the integrals exist,

$$I^\alpha \circ \Theta(x.) = (\Gamma) \int_0^\alpha \alpha(\nu^\Gamma(x.)) d\nu^\Gamma(x.) - \int_0^\alpha \alpha(x.) \bar{b}(x.) ds$$

(5.14)

where $\bar{b}(-)_s : C_{x_0} \to TM$ is the conditional expectation,

$$\bar{b}_s = E\{b(u_s) \mid p(u) = x.\},$$

and has $\bar{b}(x.)_s \in T_{x_s}$ almost surely for all $s$.

**Proof.** The fact that $\nu^\Gamma$ is a $\Gamma$-martingale is immediate from the definition and Proposition 5.2.1. To prove the claimed identity note that our extra assumption on $\Gamma$ implies that $\parallel s^{-1} d\{x_s\}^A = d(D^{-1}(x.)_s$. Therefore

$$I^\alpha(x.) = \int_0^\alpha \alpha_s(x.) \parallel_s dD^{-1}(x.)_s - \int_0^\alpha \alpha(x.) \bar{b}(x.)_s ds$$

(5.15)

while by definition

$$(\Gamma) \int_0^\alpha \alpha_s d\nu^\Gamma_s(x.) = \int_0^\alpha \alpha_s(\nu^\Gamma_s(x.)) \parallel_s^{\nu^\Gamma_s(x.)} d(D^{-1}(\nu^\Gamma_s(x.)))_s$$

(5.16)

where the superscript on the parallel translation symbol indicates that it is along the paths $\nu^\Gamma(x.)$. Our identity follows. \qed

**Remark 5.2.3** (1) For $\Gamma$ such that the $A$-process is a $\Gamma$ martingale we can easily see that $\Theta$ has an adapted inverse. Indeed its inverse is defined almost surely by

$$\Theta^{-1} = D \circ Mart^A \circ D^{-1}$$

where $Mart^A$ denotes the operation of taking the martingale part under the probability measure $P^A$.

(2) If we are given a connection $\Gamma$ on $E$ we could make our choice of $A$ so that its diffusion process gives a $\Gamma$ martingale. This specifies $A$ uniquely and might be more natural sometimes, for example in the classical case with $M = R^n$.

(3) The results and earlier discussion still hold if $\Gamma$ is not a metric connection. However then $B^{\Gamma,A}$ cannot be expected to be a Brownian motion. The connection could even be on $TM$ rather than on $E$ in which case $B^{\Gamma,A}$ will be a local martingale in $T_{x_0} M$. This will be a natural procedure when $N = R^n$, using the standard flat connection.
5.3 Classical Filtering

For an example of the situation treated above consider a signal process \((z_t, 0 \leq t \leq T)\) on \(\mathbb{R}^d\) satisfying an SDE
\[
dz_t = V(z_t, x_t) dW_t + \beta(z_t, x_t) dt
\]
(5.17)
with \((x_t, 0 \leq t \leq T)\), the observation process, taking values in \(\mathbb{R}^n\) and satisfying:
\[
dx_t = X^{(1)}(x_t) dB_t + X^{(2)}(x_t) dW_t + b(z_t, x_t) dt.
\]
(5.18)
Here \(B\) and \(W\) are independent Brownian motions of dimension \(q\) and \(p\) respectively. We then take \(N = \mathbb{R}^d \times \mathbb{R}^n\) and \(M = \mathbb{R}^n\), with \(p : N \to M\) the projection.

We set \(u_t = (z_t, x_t)\) so that
\[
B f(z, x) = \frac{1}{2} D_{1,1} f(V^i(z, x), V^j(z, x)) + D_1 f(\beta(z, x))
+ \frac{1}{2} D_{2,2} f(X^{(1)}, X^{(2)}, x) + \frac{1}{2} D_{2} f(X^{(2)}, X^{(1)}, x)
+ D_2 f(z, x) (b(z, x)) + D_{1,2} f(z, x) (V^i(z, x), X^{(1)}(z, x))
\]
using the repeated summation convention where \(i\) goes from 1 to \(p\) and \(j\) from 1 to \(q\), with the \(V^j\) referring to the components of \(V\) and similarly for \(X^{(1)}\) and \(X^{(2)}\). Also \(D_{l,m}\) refers to the second partial Frechet derivative, mixed if \(l \neq m\), etc.

The filtering problem would be to find \(E\{g(z_t) | x_s : 0 \leq s \leq t\}\) for suitable \(g : \mathbb{R}^d \to \mathbb{R}\). This would fit in with the discussion above by defining \(f : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}\) by \(f(z, x) = g(z)\). Note that we have allowed feedback from the signal to the observation; usually only the special case where \(V\) and \(\beta\) are independent of \(x\) is considered. Also we have allowed the noise driving the signal to also affect the observations (“correlated noise”). This can give a non-trivial connection, in which case the terms involving horizontal derivatives of \(f\) will not vanish even for \(f\) independent of \(x\). This vanishing would occur otherwise (i.e. for uncorrelated noise) so that in that case the formula in Theorem 5.1.2 reduces to the usual DMZ equation, for example as in [56] or [57].

Our basic assumptions are smoothness of the coefficients, non-explosion (for simplicity of exposition), and the cohesiveness of our observation process. By the latter we mean that for all \(x \in \mathbb{R}^n\) and \(z \in \mathbb{R}^d\) the image of the map \((e^1, e^2) \mapsto X^1(e^1) + X^2(e^2)\) from \(\mathbb{R}^q \times \mathbb{R}^p\) to \(\mathbb{R}^n\) contains \(b(z, x)\) and has dimension independent of \(x\). Some bounds are needed on \(b\) to ensure the existence of its conditional expectations.
To carry out the procedure for the signal and observation given above we must first identify the horizontal lift operator determined by \( B \). For this for each \( x \in M \) let \( Y_x : \mathbb{R}^n \rightarrow \mathbb{R}^{p+q} \) be the inverse of the restriction of the map \((e^1, e^2) \mapsto X^1(x)(e^1) + X^2(x)(e^2)\), from \( \mathbb{R}^q \times \mathbb{R}^p \) to \( \mathbb{R}^n \), to the orthogonal complement of its kernel. Then from Lemma 2.2.1 we see that the horizontal lift \( h_u : \mathbb{R}^n \rightarrow \mathbb{R}^d \times \mathbb{R}^n \) is given by

\[
h_u(v) = (V(z, x) \circ Y_x(v), v) \quad u = (z, x) \in \mathbb{R}^d \times \mathbb{R}^n. \tag{5.20}
\]

A natural choice of \( A \) is

\[
A(f)(x) = \frac{1}{2} D_2^2 f \left( X^{(1)i}(x), X^{(1)i}(x) \right) + \frac{1}{2} D_2^2 f \left( X^{(2)j}(x), X^{(2)j}(x) \right).
\]

Having done that the ‘b’ of our general discussion is just the drift \( b : \mathbb{R}^d \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) of our observation’s stochastic differential equation. Moreover for suitable \( T^*M \)-valued processes \( \alpha \), we have the \( \alpha \)-innovations process

\[
I_{\alpha}^t = \int_0^t \alpha(x_s) \left( X^{(1)}(x_s) dB_s + X^{(2)}(x_s) dW_s \right) + \int_0^t \alpha(x_s) \left( b(z_s, x_s) - \bar{b}(x_s) \right) ds,
\]

where \( \bar{b}(\sigma) = \mathbb{E}^\sigma \{ b(z_s, x_s) \mid x_r = \sigma_r \quad 0 \leq r \leq s \} \).

From Theorem 5.9, Kushner’s formula, given smooth \( g : \mathbb{R}^d \rightarrow \mathbb{R} \), one has

\[
\nu_t = g(z_0) + \int_0^t \left( \tau_s \frac{1}{2} D_2^2 \left( g(-)(V^i(-), V^i(-)) \right) + \tau_s D_1 g(-)(\beta(-)) \right) ds
\]

\[
+ I_{\tau}^t \left( g(\bar{b}_s) - g, \bar{b}_s \right)_{\tau_s}.
\]

This can be compared, for example, with the formula given in the remark on page 85 of [57], following the proof of Proposition 2.2.5 there. Alternatively see [56].

Using the standard flat connection of \( R^n \) we get the innovations process given by

\[
\nu_t = x_0 + \int_0^t \left( X^{(1)}(x_s) dB_s + X^{(2)}(x_s) dW_s \right) + \int_0^t \left( b(z_s, x_s) - \bar{b}(x_s) \right) ds.
\]
5.4. Examples

Consider the stochastic partial differential equation on $L^2([0,1];\mathbb{R}^p)$:

$$du_t(x) = \Delta u_t(x) + \sum_{i=1}^{m} \Phi_i(x, u_t(x))dB^i_t$$

where $(B^i_t)$ are independent Brownian motions. For $p > 1$ it can be considered as a system of equations. One natural question is to find the law of $u_t$ given that of $u_s(x_0), 0 \leq s \leq t$ for some given point $x_0$, or to find the conditional law of $u_t$ given $u_s(x_0), 0 \leq s \leq t$. Here we indicate briefly how the approach we have been following may sometimes be applied to this or similar problems. For simplicity we take $p = 1$, so our “observations” process is one dimensional; $M = \mathbb{R}$.

Let $y_t = u_t(x_0)$. It satisfies:

$$dy_t = (\Delta u_t)(x_0)dt + \sum_{i=1}^{m} \Phi_i(x_0, y_t)dB^i_t.$$

Because of the drift term we cannot expect this to be Markovian so we will have to remove the term $(\Delta u_t)(x_0)dt$ by a Girsanov transformation.

Let $(e_i)$ be the standard orthonormal base of $\mathbb{R}^m$. Define

$$\Phi : L^2([0,1];\mathbb{R}) \times \mathbb{R}^m \to L^2([0,1];\mathbb{R})$$

and

$$\tilde{\Phi} : \mathbb{R} \to \mathcal{L}(\mathbb{R}^m;\mathbb{R})$$

by

$$\Phi(u)(e)(x) = \sum_{i=1}^{m} \Phi_i(x, u(x))\langle e, e_i \rangle$$

and

$$\tilde{\Phi}(z)(e) := \sum_{i=1}^{m} \Phi_i(x_0, z)\langle e, e_i \rangle,$$

respectively. Consider $T_z\mathbb{R}$, identified with $\mathbb{R}$ and furnished with the metric induced by $\Phi(z)$:

$$\langle v_1, v_2 \rangle_z = \frac{v_1v_2}{\sum_{i=1}^{m}(\Phi_i(z))^2}.$$
To have cohesivity and to be able to apply the Girsanov-Maruyama-Cameron-Martin theorem this must be well defined, i.e. the denominator must never vanish, and it must determine a non-explosive Brownian motion. If these conditions hold, we still have to be sure that the Girsanov transformed S.P.D.E has solutions existing for all time and that we can apply the martingale method approach used in the proof of 9.1.3. Alternatively we can try to apply one of the standard tests to show that the local martingale which arises is a true martingale. First we apply Lemma 2.2.1 to obtain the horizontal lift map. For this we need the dual map $\tilde{\Phi}^*(z) : \mathbb{R} \to \mathbb{R}^m$ is given by:

$$\Phi^*(z)(1) = \frac{1}{\sum_{i=1}^m (\Phi_i(x_0, z))^2} \sum_{j=1}^m \Phi_j(x_0, z)e_j.$$ 

Then from equation (2.8) the horizontal lift $h_u : T_{u(x_0)}\mathbb{R} \to L^2([0, 1]; \mathbb{R})$ at a function $u$ is given by

$$h_u(1)(x) = \Phi(x, u(x)) \circ \tilde{\Phi}^*(u(x_0)).$$

In particular a natural choice of drift $b^h$ to remove by the Girsanov-Maruyama-Cameron-Martin theorem, namely $b^h(u) = h_u(\Delta u(x_0))$, is given by

$$b^h(u)(x) = \frac{\sum_{j=1}^m \Phi_j(x_0, u(x_0))\Phi_j(x_0, u(x))}{\sum_{k=1}^n (\Phi_k(x_0, u(x_0)))^2} \Delta u(x_0). \quad (5.21)$$

Making the change of probability to $\tilde{P}$ we see that our SPDE becomes

$$du_t(x) = \Delta u_t(x) - \frac{\sum_{j=1}^m \Phi_j(x_0, u_t(x_0))\Phi_j(x_0, u_t(x))}{\sum_{k=1}^n (\Phi_k(x_0, u_t(x_0)))^2} \Delta u_t(x_0) + \sum_{i=1}^m \Phi_i(x, u_t(x))d\tilde{B}_t^i$$

for new, independent Brownian motions $\tilde{B}^1, ... \tilde{B}^m$ and has the decomposition

$$du_t(x) = \left[ \frac{\sum_{j=1}^m \Phi_j(x_0, u_t(x_0))\Phi_j(x_0, u_t(x))}{\sum_{k=1}^n (\Phi_k(x_0, u_t(x_0)))^2} \Phi_i(x_0, u_t(x_0)) \right] d\tilde{B}_t^i$$

$$+ \left[ \frac{\sum_{j=1}^m \Phi_j(x_0, u_t(x_0))\Phi_j(x_0, u_t(x))}{\sum_{k=1}^n (\Phi_k(x_0, u_t(x_0)))^2} \Delta u_t(x_0) \right] dt$$

$$+ \sum_{i=1}^m \left( \Phi_i(x, u(x)) - \frac{\sum_{j=1}^m \Phi_j(x_0, u_t(x_0))\Phi_j(x_0, u_t(x))}{\sum_{k=1}^n (\Phi_k(x_0, u_t(x_0)))^2} \Phi_i(x_0, u_t(x_0)) \right) d\tilde{B}_t^i.$$
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In this decomposition the term in the first square brackets relates to the horizontal lift of the $\mathcal{A}$-process, while that in the second is the vertical component. They are independent (under $\mathbb{P}$), given $u$ at $x_0$.

We could continue by applying the Kallianpur-Striebel formula, Lemma 5.1.1 or go directly to our version of Kushner’s formula, Theorem 5.9. In that formula the operator $\mathcal{B}$ will be the infinite dimensional diffusion operator on $L^2([0, 1]; \mathbb{R})$ which is the generator of the solution of our SPDE, so there are extra analytical problems. However there are cases where the situation is fairly straightforward. For example:

1. $\Phi_i(z, u) = \phi_i(z)$, where the vector $\{\phi_1(z), \ldots, \phi_m(z)\}$ never vanishes for any $z$. In this case $y_t$ is basically Gaussian.

2. $\Phi(z, u) = u$ with one dimensional noise $B_t$, in which case the solution of the SPDE is $u_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} e^{B_t - \frac{t^2}{2}}$. 

Chapter 6

The Commutation Property

In certain cases the filtering is in a sense trivial: the process decomposes into the observable and an independent process. From the geometric point of view this means the commutation of the vertical operator $B^V$ and the horizontal operator $A^H$. See Theorem 6.2.8 below.

For $p$ a Riemannian submersion (defined in Chapter 7 below) with totally geodesic fibres and $B$ the Laplacian, Berard-Bergery & Bourguignon [7] show that $A^H$ and $B^V$ commute. Their proof is based on the result of R.Hermann [37]

**Theorem 6.0.1** [R.Hermann] A Riemannian submersion $p : N \to M$ has totally geodesic fibres iff the Laplace-Beltrami operator of $N$ commutes with all Lie derivations by horizontal lifts of vector fields on $M$.

From this, and the Hörmander form representation of $A^H$, it follows immediately that $A^H$ with $B^V$ will commute in their situation. In this section we consider some extensions of this and their consequences.

First, for $p : N \to M$ with a diffusion operator $B$ over a cohesive $A$, as usual, we will say that a vector field on $N$ is basic if it is the horizontal lift of a section of $E$. From our Hörmander form representation of $A^H$ we get the following extension of Berard-Bergery & Bourguignon’s result:

**Theorem 6.0.2** For a diffusion operator $B$ over a cohesive diffusion operator $A$ the following are equivalent:

- [i] $B^V$ commutes with all Lie derivations by smooth basic vector fields of $N$;
- [ii] the operators $B$, $B^V$, and $A^H$ commute (on $C^4$ functions);
• [iii] the operator $B^V$ commutes with the horizontal lifts of the vector fields which appear in one Hörmander form representation of $A$.

Proof. It is clear that [i] implies [iii], and [iii] implies [ii]. To show [ii] implies [i] observe that every section of $E$ has the form $\sigma$ since every one form on $M$ can be written as $\sum_{j=1}^{m} \lambda^j df_j$ for $\lambda^j : M \to R$ and $f_j : M \to R$ and some integer $m$. By definition of the connection this shows that every basic vector field on $N$ has the form $\sum_{j=1}^{m} \lambda^j p^j \sigma_A^H (p^* df_j)$. It will therefore suffice to show that if [ii] holds then $B^V$ commutes with Lie differentiation by $\lambda^j p^j \sigma_A^H (p^* df)$ for all smooth $\lambda, f : M \to R$.

For this assume [ii] holds and take a smooth $g : N \to R$. By definition of the symbol and Remark 1.4.5:

$$2B^V dg \left( \lambda \sigma_A^H (p^* df) \right) = 2 \lambda \frac{\partial}{\partial \lambda} B^V dg \left( \sigma_A^H (p^* df) \right)$$

$$= \lambda \frac{\partial}{\partial \lambda} B^V \left( \sigma_A^H (p^* df) \right)$$

$$= \lambda \frac{\partial}{\partial \lambda} \left( \sigma_A^H (p^* df) \right)$$

$$= 2d(B^V g) \sigma_A^H \left( \lambda \frac{\partial}{\partial \lambda} p^* df \right)$$

as required. \hfill \square

For the special case of an equivariant diffusion on a principal bundle as considered in Chapter 3 we can obtain a working criterion for commutativity: see also Example 6.2.12.

**Corollary 6.0.3** In the notation of Theorem 3.2.1 commutativity of $B^V$ and $A^H$ holds if and only if both $\alpha$ and $\beta$ are constant along all horizontal curves. This holds if and only if $A^H (\alpha^i,j) = 0$ and $A^H (\beta^k) = 0$ for all $i, j, k$.

Proof. First note that each vector field $A^*_k$ commutes with all basic vector fields. Indeed if $V$ is basic it is equivariant and so

$$(R_{\exp t A^*_k})_*(V) = V \quad t > 0.$$  

Differentiating in $t$ at $t = 0$ gives the required commutativity. Thus the operators $\mathcal{L}_{A^*_k}$ are invariant under flows of basic vector fields and so for $B^V$ to commute with basic vector fields the coefficients $\alpha$ and $\beta$ must be constant along their flows. By
the theorem this gives the first result since any horizontal curve can be considered as an integral curve of a (possible time dependent) basic vector field.

Clearly, from the Hörmander form of \( A^H \), if this holds both \( \alpha \) and \( \beta \) are \( A^H \)-harmonic. The converse holds since from above \( A^H \) commutes with all of the vertical vector fields \( L^*_A \).

The Corollary is applied to derivative flows in Example 6.2.12 of Section 6.2 below.

Hermann proved that a Riemannian submersion with totally geodesic fibres has the natural structure of a fibre bundle with group the isometry group of a typical fibre.

**Theorem 6.0.4 (Hermann)** If \( N \) is a complete Riemannian manifold and \( \phi : N \to M \) is a \( C^\infty \) Riemannian submersion then \( \phi \) is a locally trivial fibre space. If in addition the fibres of \( \phi \) are totally geodesic submanifolds of \( N \), \( \phi \) is a fibre bundle with structure group the Lie group of isometries of the fibre.

An analogous result given the hypothesis of theorem 6.0.2 together with some completeness and hypoellipticity conditions is proved in Theorem 6.2.8 below.

Before that we consider when the associated semi-groups commute.

### 6.1 Commutativity of Diffusion Semigroups

It is well known that in general the commutativity of two diffusion generators (on \( C^4 \) functions) does not imply that of their associated semi-groups. One reference is [61] page 273 where an example they ascribe to Nelson is given. Here is a minor modification of that construction:

Cut \( \mathbb{R}^2 \) along the positive \( x \)-axis. Take a copy \( A \), say, of \( (0, \infty) \times (-\infty, 0] \) and glue it along the cut to the upper part of the cut plane, identifying \( (0, \infty) \times \{0\} \) in \( A \) with the positive \( x \)-axis. Similarly glue a copy \( B \), of \( (0, \infty) \times (0, \infty) \) along the cut to the lower part of the cut plane. This gives a version of the plane but with two copies of the upper and lower quadrants, and with the origin missing. On this we have naturally defined vector fields \( X^1 \) given by \( \frac{\partial}{\partial x} \) and \( X^2 \) given by \( \frac{\partial}{\partial y} \). These certainly commute. However their associated semi-groups do not, as can be seen by starting at the point \((-1, -1)\) moving along the \( X^1 \)-trajectory for time 2 and then along the \( X^2 \) trajectory for the same amount of time. We end up at the point \((1, 1)\) of copy \( B \). However if we had changed the order of the vector fields we would be at \((1, 1)\) of copy \( A \). A more geometrically satisfying construction would
be, as Nelson, to use the double covering of the punctured plane as state space with similarly behaved vector fields. Here is an easy positive result:

**Proposition 6.1.1** Let $A_1$ and $A_2$ be diffusion operators with associated semi-groups $\{P_t^1\}_{t>0}$ and $\{P_t^2\}_{t>0}$ acting as strongly continuous semi-groups on a Banach space $E$ of functions which contains the $C^2$ functions with compact support. Let $G_1$ and $G_2$ be the corresponding generators, (closed extensions of the restrictions of $A_1$ and $A_2$ to the space of $C^2$ functions with compact support). Assume there is a core $C_2$ for $G_2$ consisting of bounded $C^\infty$ functions such that for $f \in C_2$:

[i] For all $t > 0$ the function $P_t^1 f$ is $C^4$.

[ii] $A_2 A_1 P_t^1 f$ is uniformly bounded in $t \in (0, 1)$ and in space, and it converges pointwise to $A_2 A_1 P_t^1 f$ as $t \to 0+$.

[iii] $A_2 P_t^1 f$ is uniformly bounded in $t \in (0, 1)$ and in space.

Then commutativity of $P_t^1$ with $P_s^2$, $0 \leq s, 0 \leq t$ follows from commutativity of $A_1$ with $A_2$ on $C^2$ functions. Moreover if this holds the semi-group $\{P_{t}^{A_1+A_2}\}_{t>0}$ associated to $A_1 + A_2$ satisfies

$$P_{t}^{A_1+A_2} = P_t^1 P_t^2.$$  

**Proof.** Let $f : M \to \mathbb{R}$ be in $C_2$.

We show first that

$$A_2 P_t^1 f = P_t^1 A_2 f$$  \hspace{1cm} (6.1)

For this set $V_t = A_2 P_t^1 f$. Then, by hypothesis [ii],

$$\frac{\partial}{\partial t} V_t = A_2 A_1 P_t^1 f = A_1 V_t$$  \hspace{1cm} (6.2)

by commutativity. By assumption [ii] we know $V_s$ is bounded uniformly in $s \in [0, t]$ for any $t > 0$. However there is a unique $C^2$ and uniformly bounded solution, $P_1^1 V_0$, to any diffusion equations such as (6.2) with given smooth bounded initial condition $V_0$ (as is easily seen by the standard use of Itô’s formula applied to $V_{t-s}$ acting on a diffusion process with generator $A_1$). This gives

$$A_2 P_t^1 f = P_t^1 V_0 = P_t^1 A_2 f.$$
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as required. Now suppose \( f \in \text{Dom}(G_2) \). By assumption there is a sequence \( \{f_n\}_n \) of functions in \( C_2 \) converging in \( G_2 \)-graph norm to \( f \). Then \( P^1_t A_2 f_n \to P^1_t G_2 f \) and \( P^1_t f_n \to P^1_t f \). Equation (6.1) therefore shows that \( P^1_t f \in \text{Dom}(G_2) \) and we have

\[
G_2 P^1_t \supset P^1_t G_2.
\]  

(6.3)

Next, for \( f \in \text{Dom}(G_2) \), and our fixed \( t > 0 \) set \( W_s = P^1_t P^2_s f \). Since the convergence of \( \frac{1}{\epsilon} \{ P^2_{s+\epsilon} f - P^2_{s} f \} \) to \( G_2 P^2_s f \) is in \( E \) we see, using equation (6.3),

\[
\frac{\partial}{\partial s} W_s = P^1_t G_2 P^2_s f = G_2 P^1_t P^2_s f = G_2 W_s
\]

since \( P^2_s f \in \text{Dom}(G_2) \). In particular \( W_s \in \text{Dom}(G_2) \).

Although now it is not clear that \( W \) is \( C^2 \) we see from this that \( \frac{\partial}{\partial u} P^2_s W_{s-u} = 0 \) for \( 0 < u < s \), giving

\[
P^1_t P^2_s f = P^2_s W_s = P^2_s W_0 = P^2_s P^1_t f
\]

for \( 0 \leq s \leq t \). For \( s > t \) it is now only necessary to use the semigroup property of \( P^2 \), to commute with \( P^1_t \) portion by portion.

Finally since \( P^2_s f \in \text{Dom}(G_2) \) the above gives

\[
\frac{\partial}{\partial t} P^1_t P^2_s f = A_1 P^1_t P^2_s f + P^1_t A_2 P^2_s f = (A_1 + A_2) P^1_t P^2_s f
\]

and we can repeat the second argument showing uniqueness of solutions of the diffusion equation to obtain \( P^1_t^{A_1 + A_2} f = P^1_t P^2_t f \). \( \square \)

**Remark 6.1.2** Condition [i] does not always hold. A simple example is when the state space is \( \mathbb{R}^2 - \{(0,1)\} \) and the operator is \( \frac{\partial^2}{\partial x^2} \). The standard positive result for degenerate operators on \( \mathbb{R}^n \) is due to Oleinik, [54].

6.2 Consequences for the Horizontal Flow

For our standard set up of \( p : N \to M \) with diffusion operator \( \mathcal{B} \) over a cohesive \( \mathcal{A} \), let \( P^V \) and \( P^H \) denote the semi-groups generated by the vertical and horizontal components of \( B \), and let \( p^V_t (u, -) \), \( t \leq 0 \), \( u \in N \), be the transition probabilities of \( P^V \). If we set \( N_x = p^{-1}(x) \) for \( x \in M \) then \( p^V_t (u, -) \) will be a probability measure
on $N_{p(u)}^+$, the union of $N_{p(u)}$ with $\Delta$. For and For $P_{x_0}$-almost all $\sigma \in C_{x_0}M^+$ for each $x_0 \in M$ there are measurable maps

$$\|\sigma\| : N_{x_0}^+ \rightarrow N_{x_0}^+$$

such that for each $u \in N_{x_0}$ the process $(t, \sigma) \mapsto \|\sigma\| (u)$ is an $A^H$-diffusion and is over $\sigma$. These can be obtained, for example, by taking a stochastic differential equation, as equation (4.21),

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt$$

for our $A$-diffusion. Let $Y_x : E_x \rightarrow R^m$ be the adjoint (and right inverse) of $X(x)$, each $x \in M$. Then consider the SDE on $N$

$$dy_t = \tilde{X}(y_t)Y(\sigma_t) \circ d\sigma_t$$

and let $(t, \sigma) \mapsto \|\sigma\|$ be the restriction of its flow to $N_{x_0}$, augmented by mapping the coffin state, $\Delta$, to itself. This SDE is canonical since it can be rewritten as

$$dy_t = h_{y_t} \circ d\sigma_t$$

for $h$ the horizontal lift map of Proposition 2.1.2.

We will often need to assume that the lifetime of this diffusion is the same as that of its projection on $M$:

**Definition 6.2.1** The semi-connection induced by $B$ is said to be *stochastically complete* if

$$C_{u_0}^p M^+ := \{\sigma : [0, \infty) \rightarrow M^+ : \lim_{t \rightarrow \zeta} p(u_t) = \Delta \text{ when } \zeta(u) < \infty\}$$

has full $P_{u_0}^{A-u}$ measure for each $u_0 \in N$ or equivalently if the lifetimes satisfy

$$\zeta(u) = \zeta(p(u))$$

for $P_{u_0}^{A-u}$-almost all paths $u$.

The semi-connection is said to be *strongly stochastically complete* if also we can choose a version of $\|\sigma\| : N_{\sigma(0)} \rightarrow N_{\sigma(t)}$ which is a smooth diffeomorphism whenever $\sigma(0)$ is a regular value of $p$ and $t < \zeta(\sigma)$.

Note that strong stochastic completeness of the connection will hold whenever the fibres of $p$ are compact by the basic properties of the domains of local flows of SDE, [43], [21]. This also holds if the stochastic horizontal differential equation is strongly $p$-complete in the sense of Li [47] for $p = \dim(N) - \dim(M)$. 
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**Proposition 6.2.2** Suppose the semi-groups $P^V$ and $P^H$ commute and stochastic completeness of the connection holds. Then the horizontal flow preserves the vertical transition probabilities in the sense that for all positive $s$ and $0 < t < \zeta(\sigma)$,

\[
\left(\begin{array}{c}
\llbracket t \rrbracket \\
\end{array}\right)P_s^V(u_0, -) = P_s^V\left(\begin{array}{c}
\llbracket t \rrbracket (u_0, -)
\end{array}\right)
\]

(6.4)

for all $u_0 \in N_\sigma$ for $P^A$-almost all $\sigma$. Equivalently for any bounded measurable $h : N \to \mathbb{R}$ we have $P^A$-almost surely;

\[
P_s^V(h \circ \llbracket t \rrbracket)(u_0) = P_s^Vh(\llbracket t \rrbracket(u_0))
\]

(6.5)

**Proof.** It suffices to show that given any finite sequence $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k < t$, bounded measurable $f_j : M \to \mathbb{R}$, $j = 1, \ldots, k$ and bounded measurable $h : N \to \mathbb{R}$, if $u_0 \in N_{x_0}$ then

\[
E_{x_0}\left\{f_1(\sigma_{t_1}) \cdots f_k(\sigma_{t_k})1_{t<\zeta(\sigma)}P_s^V(h \circ \llbracket t \rrbracket(u_0))\right\}
\]

(6.6)

where $\chi_Z$ denotes the indicator function of a set $Z$. To see this set $\tilde{f}_j = f_j \circ p : N \to \mathbb{R}$. Then the left hand side of (6.6) is

\[
E_{x_0}\left\{\tilde{f}_1(\llbracket t_1 \rrbracket(u_0)) \cdots \tilde{f}_k(\llbracket t_k \rrbracket(u_0))\chi_{t<\zeta(\sigma)}P_s^V(h \circ \llbracket t \rrbracket(u_0))\right\}
\]

(6.6)

which reduces to the right hand side of (6.6).

\[\square\]

**Remark 6.2.3** Assuming strong stochastic completeness of our semi-connection let \{z_t : 0 \leq t < \zeta(p(u))\} be a semi-martingale in $N$ with $p(z_t) = x_t := p(u) : 0 \leq t < \zeta(p(u))$. If $x_0$ is a regular value of $p$ we have the Stratonovich equation:

\[
d\llbracket t \rrbracket^{-1}z_t = T\llbracket t \rrbracket^{-1} \circ T\llbracket t \rrbracket^{-1}(h_{z_t} \circ dx_t)
\]

(6.7)

where $\llbracket t \rrbracket$ refers to $\llbracket \cdot \rrbracket^x$. To see this, for example set $b_t = \llbracket t \rrbracket^{-1}z_t$ and observe that

\[
dz_t = d(\llbracket t \rrbracket b_t) = T\llbracket t \rrbracket \circ db_t + h_{b_t} \circ dx_t.
\]
Now assume that our induced semi-connection is strongly stochastically complete. For a regular value \( x_0 \) of \( p \) and \( u_0 \in N_{x_0} \), define a process \( \alpha_{u_0} : [0, \infty) \times C_{u_0} N^+ \to N^+ \) by

\[
\alpha_{u_0}^{u_0}(u) = \alpha_t(u) = (\|p(u)\|^{-1}u_t)
\]

(6.8)

if \( u \in C_{u_0} N \) with \( t < \zeta(u) \) and define \( \alpha_{u_0}^{u_0}(u) = \triangle \) if \( t \geq \zeta(u) \). Note that \( \alpha_t \) may not go out to infinity in \( N_{x_0} \) as \( t \) increases to its extinction time.

Also define

\[
//^*(B^V)(f) = B^V(f \circ //_s) \circ //^{-1}_t
\]

to obtain a random time dependent diffusion operator \( //^*(B^V) \) on each fibre over a regular value of \( p \).

**Lemma 6.2.4** In the notation of equation (4.23) we have the Itô equation for \( \alpha_t := \alpha_{u_0}^{u_0} \):

\[
\nabla^V d\alpha_t = T//^{-1}_t V(//_t \alpha_t) dW_t + T//^{-1}_t V^0(//_t \alpha_t) dt.
\]

(6.9)

In particular for \( f : N \to \mathbb{R} \) in \( C^2 \)

\[
M_t^{df,\alpha} := f(\alpha_t) - \int_0^t //^*(B^V)(f)(\alpha_s) ds
\]

(6.10)

is a local martingale.

**Proof.** Formula (6.9) is immediate from equations (4.23) and (6.7). That \( M_t^{df,\alpha} \) is a local martingale follows immediately using the properties of pull-backs under diffeomorphisms of Lie derivatives when \( V \) is \( C^1 \), and by going to local co-ordinates otherwise.

**Lemma 6.2.5** At all points above regular values of \( p \) we have:

\[
\frac{d}{ds} \mathbb{E}\{ //^*(B^V) \}_{s=0} = [A^H, B^V]
\]

**Proof.** This is an exercise in the use of Ito’s formula. For example write \( A \) in the Hörmander form

\[
A = \frac{1}{2} \sum_{j=1}^m L_{X^j} L_{X^j} + L_0^0_X
\]
so that \( \tilde{\mathbf{v}}_s \) is the flow of the SDE

\[
dz_s = \sum_{j=1}^{m} \tilde{X}^j(z_s) dB^j + \tilde{X}^0(z_s)
\]

using the horizontal lifts of the vector fields \( X^j \). From the Ito formula in lemma 9B Chapter VII of [21] we have

\[
\frac{d}{ds} \mathbb{E}\{ \tilde{\mathbf{v}}_s^*(B^V) \}|_{s=0} = \frac{1}{2} \sum_{j=1}^{m} \frac{d^2}{ds^2} (\tilde{\mathbf{v}}^*_j(B^V))|_{s=0} + \frac{d}{ds} (\tilde{\mathbf{v}}^*_0(B^V)|_{s=0}
\]

where \( \tilde{\mathbf{v}}^*_i \) is the flow of the vector field \( \tilde{X}^j \). Since

\[
\frac{d}{ds} (\tilde{\mathbf{v}}^*_j(B^V)) = [\mathbf{L}_{\tilde{X}^j}, (\tilde{\mathbf{v}}^*_j)^* B^V]
\]

we have the result.

\[
\Box
\]

**Definition 6.2.6** For a regular value \( x_0 \) of \( p \). We say \( B^V \) is stochastically holonomy invariant at \( x_0 \) if on \( N_{x_0} \) we have \( \tilde{\mathbf{v}}^*_s(B^V) = B^V \) for all \( 0 \leq t < \zeta \) with probability one. If this holds for all regular values \( x_0 \) then we say \( B^V \) is stochastically holonomy invariant. Similarly we say \( B^V \) is holonomy invariant at \( x_0 \) if the corresponding result holds for parallel translation along any piecewise \( C^1 \) curve starting at \( x_0 \) in \( M \), and is holonomy invariant if this holds for all regular values \( x_0 \).

**Remark 6.2.7** 1. If the \( A \)-diffusion on \( M \) is represented by a stochastic differential equation we can lift that equation to \( N \) and obtain a local flow \( \eta \) where \( \zeta : 0 \leq t < \zeta \) gives its explosion times; so that with probability one \( \eta \) is defined and smooth on the open set \( \{ y \in N : t \leq \zeta(y) \} \), see [43] or [21]. We can say that \( B^V \) is invariant under the horizontal flow if for all \( C^2 \) functions \( f : N \to \mathbb{R} \) we have

\[
B^V(f) \circ \eta_t = B^V(f \circ \eta_t)
\]

on \( \{ y \in N : t \leq \zeta(y) \} \), almost surely, for all \( t > 0 \). This does not require strong stochastic completeness of the semi-connection, nor do we have to restrict attention to fibres over regular values. On the other hand if it holds, and given such strong stochastic completeness, if \( x_0 \) is a regular value it follows that \( N_{x_0} \) lies in \( \{ y \in N : t \leq \zeta(y) \} \) for all \( t < \zeta(x_0) \) and that we have stochastic holonomy invariance at \( x_0 \).
2. Assume completeness of the semi-connection. If $\mathcal{A}$ satisfies the standard Hörmander condition, or more generally if the space $\mathcal{D}^0(x_0)$, as in Section 2.6 is all of $M$, then holonomy invariance at $x_0$ implies holonomy invariance. This follows since concatenation of paths gives composition of the corresponding parallel translations and the conditions imply that any two points can be joined by a smooth path with derivatives in $E$. Moreover by Theorem 2.6.1 every point is a regular value and so given also strong stochastic completeness of the connection from the theorem below we see that holonomy invariance of $B^V$ at one point implies it is invariant under the horizontal flow induced by any SDE on $M$ which gives one point motions with generator $\mathcal{A}$. The same holds for stochastic holonomy invariance: see Theorem 6.2.8 below.

**Theorem 6.2.8** Suppose the induced semi-connection is complete and strongly stochastically complete, and $x_0$ is a regular value of $p$. Then the following are equivalent:

[i$_{x_0}$] For all $u_0 \in N_{x_0}$ and for any $\mathcal{F}^\alpha$-stopping time $\tau$ with $\tau(\alpha(u)) < \zeta(p(u))$, the process $\{\alpha_t : 0 \leq t < \tau\}$ is independent of $\mathcal{F}^{x_0}$;

[ii$_{x_0}$] $B^V$ is stochastically holonomy invariant at $x_0$;

[iii$_{x_0}$] $B^V$ is holonomy invariant at $x_0$;

[iv$_{x_0}$] $B^V$ and $A^H$ commute at all points of $\overline{\mathcal{D}^0(x_0)}$;

[v$_{x_0}$] $P^V$ and $P^H$ commute at all points of $\overline{\mathcal{D}^0(x_0)}$.

If the above hold at some regular value $x_0$ they hold for all elements in $\mathcal{D}^0(x_0)$. Moreover $\alpha^{x_0}$ is a Markov process on $N_{x_0}$ with generator $B^V$.

**Proof.** We will show that [i$_{x_0}$] is equivalent to [ii$_{x_0}$] which implies [iv$_{x_0}$]. Then [iv] implies [iii$_y$] for all $y \in \mathcal{D}^0(x_0)$ which implies [v]. Finally we show [v] implies [ii$_y$] for all $y \in \mathcal{D}^0(x_0)$.

Assume [i$_{x_0}$] holds. Let $f : N_{x_0} \to R$ be smooth with compact support. Then the local martingale $M^{f,\alpha}$ given by formula (6.10) is a martingale and from equation (6.9) we see that

$$\mathbb{E}\{M^{f,\alpha} | \mathcal{F}^{x_0}\} = f(u_0).$$
Therefore for $\mathbb{P}^{x_0}$-almost all $\sigma$ in $C_{x_0}M$

$$
\mathbb{E}\{f(\alpha_t)\} = \mathbb{E}\{f(\alpha_t)|p(u) = \sigma\} = f(u_0) + \int_0^t \mathbb{E}\{(\mathbb{I}_s^*\mathbb{B}^V)(f)(\alpha_s)\}ds.
$$

(6.11)

Also, in the notation of equation (6.9), with the obvious notation for the filtrations generated by our processes, we have

$$
F_{\alpha_t} \subset F_{W_t} \land F_{x_0} \quad \text{and} \quad F_{W_t} \subset F_{\alpha_t} \land F_{x_0},
$$

so our assumption implies that $F_{W_t} = F_{\alpha_t}$, for all positive $t$, after stopping $W$ at the explosion time of $\alpha$. From this, and equation (6.9) we see that if we set

$$
\bar{M}_{\alpha, t} = \mathbb{E}\{M_{\alpha, t}|F_{\alpha_t}\}
$$

we obtain a martingale with respect to $F_{\alpha_t}$ and

$$
\mathbb{I}_s^*\mathbb{B}^V = \mathbb{E}\{\mathbb{I}_s^*\mathbb{B}^V\}. \quad \text{Thus by the usual martingale characterisation of Markov processes we see that $\alpha$ is Markov with (possibly time dependent) generator $\mathbb{I}_s^*\mathbb{B}^V$ at time $s$. However equation (6.11) then implies, for example by [62] Proposition(2.2), Chapter VII, that the generator is given by $\mathbb{I}_s^*\mathbb{B}^V$ for arbitrary $\sigma$ in a set of full measure in $C_{x_0}M$. Thus [i] implies the stochastic holonomy invariance [ii].

Conversely if [ii] holds, equation (6.10) gives

$$
f(\alpha_t) = M_{\alpha, t} + \int_0^t \mathbb{I}_s^*\mathbb{B}^V(f)(\alpha_s)ds.
$$

(6.12)

Then $M_{\alpha, t}$ is an $F_{\alpha_t}$-martingale and again we see that $\alpha$ is Markov, with generator $\mathbb{B}^V$. It is therefore independent of $x$, giving [i]. Moreover, in an obvious notation, if $0 \leq s \leq t$, by the flow property of parallel translations, on $N_{x_0}$,

$$
\mathbb{B}^V = \mathbb{I}_s^*\mathbb{B}^V = \mathbb{I}_s^*(\mathbb{I}_t^*\mathbb{B}^V),
$$

and so, almost surely, at all points of $N_{x_0}$ we have

$$
(\mathbb{I}_t^*)^*\mathbb{B}^V = (\mathbb{I}_s^*)^{-1}\mathbb{B}^V = \mathbb{B}^V.
$$

Since $(\mathbb{I}_t^*)^*\mathbb{B}^V$ has the same law as $(\mathbb{I}_s^*)\mathbb{B}^V$ and is independent of $F_{x_0}$ this shows that [iii] holds for $p_{x_0}(x_0,-)$-almost all $y \in M$ for all $s > 0$.

On the other hand [iii] implies that $\mathbb{B}^V$ and $A$ commute on $N_y$ by Lemma 6.2.5. Thus by continuity of $[\mathbb{B}^V, A]^H$ and the support theorem we see that [iv] implies [iv].
Furthermore as in Theorem 6.0.2 we see that $B^V$ commutes with basic vector fields at all points over $\mathcal{D}^0(x_0)$. From this the holonomy invariance [iii] holds for all $y \in \mathcal{D}^0(x_0)$.

Now assume [iii] and so by Remark 6.2.7(2.) we have [iii] for all $y \in \mathcal{D}^0(x_0)$. Since $\|\sigma\| (u_0)$ stays above $\mathcal{D}^0(x_0)$ for any suitable piecewise smooth $\sigma$ we find the solution to the martingale problem of $B^V$ for any point $u_0$ of $N_{x_0}$ is holonomy invariant at $u_0$, i.e. along piecewise smooth curves $\sigma$ in $M$ starting at $x_0$.

$$P_t^V (f \circ \|\sigma\|)(u_0) = P_t^V (f)(\|\sigma\| u_0).$$

By Wong-Zakai approximations we see that stochastic holonomy invariance of $P^B^V$ holds over $x_0$ and hence on taking expectations we get [v]. As observed we also get [v] for all $y \in \mathcal{D}^0(x_0)$ and hence by continuity for all $y \in \mathcal{D}^0(x_0)$. Thus [iii] implies [v].

Finally assuming [v] we can apply Proposition 6.2.2, observing that the proof still holds since it only involves points in $\mathcal{D}^0(x_0)$. Differentiating equation (6.5) in $s$ at $s = 0$ gives the stochastic holonomy invariance [ii] for all $y \in \mathcal{D}^0(x_0)$.

**Remark 6.2.9** From the proof and Theorem 6.0.2 we see that the stochastic completeness of the connection is not needed to ensure that [iv] and [iii] are equivalent.

We can now go further than our Theorem 2.6.1 in extending Hermann’s result, Theorem 6.0.1. For this we will need some extra hypoellipticity conditions to deal with the case of non-compact fibres. Take a Hörmander form $A$ corresponding to a smooth factorisation

$$\sigma^A_x = X(x)X(x)^*$$

with $X(x) \in L(R^m : T_x M$ for $x \in M$. Let $H$ denote the usual Cameron-Martin space of finite energy paths $H = L^2(\mathbb{R}, [0, 1]; R^m)$. For $h \in H$ and $x \in M$ let $\phi^h_t(x), 0 \leq t \leq 1$ be the solution at time $t \in [0, 1]$ to the ordinary differential equation

$$\dot{z}(t) = X(z(t))(\dot{h})$$

with $\phi^h_0(x) = x$. In particular we assume such a solution exists up to time $t = 1$. For each $x \in M$ this gives a smooth mapping $\phi^h_x(x) : H \rightarrow M$, namely $h \mapsto \phi^h_x(x)$. Let $C^{h,x} : E_x \rightarrow E_x$ be the deterministic Malliavin covariance operator, see [9], given by

$$C^{h,x} = T_h \phi^x_x(T_h \phi^x_x)^*.$$
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Then \( \phi_1^{-1}(x) \) is a submersion in a neighbourhood of \( h \) if and only if \( C^{h,x} \) is non-degenerate. It is shown in [9] that this condition is independent of the choice of Hörmander form for \( A \), and follows from the standard Hörmander condition that \( X^1, \ldots, X^m \) and their iterated Lie brackets span \( T_x M \) when evaluated at the point \( x \). A more intrinsic formulation of it can be made in terms of the manifold of \( E \)-horizontal paths of finite energy, as described in [52].

**Theorem 6.2.10** Consider a smooth map \( p : N \rightarrow M \) with diffusion operator \( B \) on \( N \) over a cohesive diffusion operator \( A \). Suppose that the connection induced by \( B \) is complete. Also assume that \( D^0(x) \) is dense in \( M \) for all \( x \in M \) and that either the fibres of \( p \) are compact or that the solutions to equation (6.13) exist up to time 1 and there exists \( h_0 \in H \) and \( x_0 \in M \) such that \( C^{h_0,x_0} \) is non-degenerate. Then \( p : N \rightarrow M \) is a locally trivial bundle.

If also \( B \) and \( A^H \) commute we can take \( N_{x_0} \), the fibre over \( x_0 \), to be the model fibre and choose the local trivialisations

\[
\tau : U \times N_{x_0} \rightarrow p^{-1}(U)
\]

to satisfy

\[
\tau(x,-)^*(B^V|N_x) = B^V|N_{x_0}.
\]

**Proof.** The local triviality given compactness of the fibres is a special case of Corollary 2.6.3 so we will only consider the other case.

For this set \( y = \phi_1^{h_0}(x_0) \). Our assumption on the covariance operator together with the smoothness of \( h \mapsto \phi_1^h(x_0) \) implies by the inverse function theorem that there is a neighbourhood \( U_y \) of \( y \) in \( M \) and a smooth immersion \( s : U_y \rightarrow H \) with \( s(y) = h_0 \) and \( \phi_1^{s(x)}(x_0) = x \) for \( x \in U_y \).

We know from Theorem 2.6.1 that \( p \) is a submersion so all its fibres are submanifolds of \( N \). Define \( \tau_{U_y} : U_y \times N_{x_0} \rightarrow p^{-1}(U_y) \) by using the parallel translation along the curves \( \phi_t^{s(x)} : 0 \leq t \leq 1 \) that is:

\[
\tau_{U_y}(x,v) = \phi_1^t(v) \quad (x,v) \in (U_y \times N_{x_0}).
\]

(6.14)

For a general point \( x \) of \( M \) we can find an \( x' \in U_y \cap D^0(x) \) and argue as in the proof of Theorem 2.6.1 to obtain open neighbourhoods \( U_x \) of \( x \) in \( M \) and \( U'_{x'} \) of \( x' \) in \( U_{x_0} \) and a fibrewise diffeomorphism of \( p^{-1}(U_{x'}) \) with \( p^{-1}(U_x) \) obtained from parallel translations. This can be composed with a restriction of \( \tau_{U_{x_0}} \) to give a trivialisation near \( x \). This proves local triviality. The rest follows directly from Remark 6.2.9 since our trivialisations came from parallel translations. \( \square \)
Remark 6.2.11 Set

\[ \mathcal{G}(\mathcal{B}^V_{x_0}) = \{ \alpha \in \text{Diff}(N_{x_0}) : \alpha^*(\mathcal{B}^V|N_{x_0}) = \mathcal{B}^V|N_{x_0} \} \].

(6.15)

Then assuming the commutativity in the theorem we can consider \( \mathcal{G}(\mathcal{B}^V_{x_0}) \) as a structure group for our bundle though unless the fibres of \( p \) are compact it is not clear if we have a smooth fibre bundle with this as group in the usual sense, since this requires smoothness into \( \mathcal{G}(\mathcal{B}^V_{x_0}) \) of the transition maps between overlapping trivialisations. See the next section and Michor [51] section 13.

Note that elements of \( \mathcal{G}(\mathcal{B}^V_{x_0}) \) preserve the symbol of \( \mathcal{B}^V \) and so if that symbol has constant rank preserve the inner product induced on the image of \( \sigma\mathcal{B}^V \). In particular if \( \mathcal{B}^V \) is elliptic they are isometries of the Riemannian structure induced on the fibre \( N_{x_0} \). This is the situation arising from Riemannian submersions as in Hermann’s Theorem 6.0.4 and described in detail in Chapter 7 below. The space of isometries of a Riemannian manifold with compact-open topology is well known to form a Lie group, for example see [40]. However there appears to be no detailed proof that the same holds in degenerate cases even when the Hörmander condition holds at each point. When Hörmander’s condition holds the Caratheodory metric on the manifold determines the standard manifold topology, e.g. see [52] Theorem 2.3, which is locally compact, and the group of isometries of a connected locally compact metric space is locally compact in the compact-open topology, see [40], Chapter 1, Theorem 4.7. Thus in this case \( \mathcal{G}(\mathcal{B}^V_{x_0}) \) will be locally compact.

In general preserving the possibly degenerate Riemannian structure determined by its symbol will not be enough to characterise \( \mathcal{G}(\mathcal{B}^V_{x_0}) \). Even in the elliptic case there may be a “drift vector” which needs to be preserved as well and this may lead to \( \mathcal{G}(\mathcal{B}^V_{x_0}) \) being very small. For example if \( N_{x_0} \) is \( \mathbb{R}^2 \) and \( \mathcal{B}^V = \frac{1}{2} \Delta - |x|^2 \frac{\partial}{\partial x^2} \) the group is trivial.

Example 6.2.12 1. As an example consider the situation described in Section 3.3 of the derivative flow of a stochastic differential equation (3.8) on \( M \) acting on the frame bundle \( GLM \) to produce a diffusion operator \( \mathcal{B} \) on \( GLM \). Assume that \( M \) is Riemannian and complete, and that the one point motions are Brownian motions, so that \( \mathcal{A} = \frac{1}{2} \Delta \). Assume also that the connection induced is the Levi-Civita connection. Then if \( \mathcal{B} \) and \( \mathcal{A}^H \) commute, by Corollary co:equ-comm , we see that the co-efficients \( \alpha \) and \( \beta \) of \( \mathcal{B}^V \) described in Theorem 3.3.1 must be constant along horizontal curves. However as pointed out in the proof of Corollary 3.4.8, the restriction of
\[\alpha(u)\] for \(u \in GLM\) to anti-symmetric tensors is essentially (one half of) the curvature operator. It follows that the curvature is parallel, \(\nabla R = 0\). In turn this implies, [40] page 303, that \(M\) is a local symmetric space and so if simply connected, a symmetric space. In Section 7.2 we show how such stochastic differential equations arise on any symmetric space. Also from Example 3.3 we see that the standard gradient SDE for Brownian motion on spheres also give derivative flows with this property.

2. For the apparently weaker property of commutativity for the derivative flow \(T\xi_t\) of our SDE (3.8) acting directly on the tangent bundle \(TM\) recall first that if the generator \(\mathcal{A}\) is cohesive (and even if it just happens that the symbol of \(\mathcal{A}\) has constant rank, see [27]) then for \(v_t = T\xi_t(v_0)\) some \(v_0 \in T_{x_0}M\) we have the covariant SDE

\[
\hat{D}v_t = \hat{\nabla}_{v_t} X dB_t - \frac{1}{2} \hat{\text{Ric}}^# (v_t) dt + \hat{\nabla}_{v_t} \text{Ad}t.
\]  

(6.16)

From this we see that if \(\mathcal{A}\) is cohesive the process \(\alpha_t\) defined by \(\alpha_t = \int_t^1 T\xi_t(v_0)\) satisfies the SDE

\[
d\alpha_t = \int_t^{t^{-1}} \left( \hat{\nabla}_{\hat{\alpha}_{-X}} X dB_t - \frac{1}{2} \hat{\text{Ric}}^# (\hat{\alpha}_{-X}) dt + \hat{\nabla}_{\hat{\alpha}_{-X}} \text{Ad}t \right).
\]

Suppose also that \(A = 0\). We see that \(\alpha_t\) is independent of \(\xi_t(x_0)\) if and only if both \(\hat{\nabla}_- X\) and \(\hat{\text{Ric}}^#\) are holonomy invariant. If \(M\) is Riemannian and the solutions of the SDE are Brownian motions and the induced connection is the Levi-Civita connection we can deduce, as above, using Theorem 6.2.8, that commutativity of the the vertical and horizontal diffusions operators on \(TM\) holds only if \(M\) is locally symmetric.
Chapter 7

Example: Riemannian Submersions & Symmetric Spaces

7.1 Riemannian Submersions

Recall that when $N$ and $M$ are Riemannian manifolds a smooth surjection $p : N \to M$ is a Riemannian submersion if for each $u$ in $N$ the map $T_u p$ is an orthogonal projection onto $T_{p(u)} M$, i.e. restricted to the orthogonal complement of its kernel it is an isometry. Note that if $p : N \to M$ is a submersion and $M$ is Riemannian we can choose a Riemannian structure for $N$ which makes $p$ a Riemannian submersion. If a diffusion operator $B$ on $N$ which has projectible symbol for $p : N \to M$ is also elliptic its symbol induces Riemannian metrics on $N$ and $M$ for which $p$ becomes a Riemannian submersion. A well studied situation is when $p$ is a Riemannian submersion and $B$ is the Laplacian, or $\frac{1}{2} \triangle_N$, on $N$. The basic geometry of Riemannian submersions was set out by O’Neill in [55]; he ascribes the term ‘submersion’ to Alfred Gray. In this section we shall mainly be relating the work of B’rard-Bergery & Bourguignon [7], Hermann, [37], Elworthy & Kendall, [24], and Liao, [48], to the discussion above. The book [33] shows the breadth of geometric structures which can be considered in association with Riemannian submersions.

A simple example of a Riemannian submersion is the map $p : \mathbb{R}^n - \{0\} \to \infty$ given by $p(x) = |x|$. Then, for $n > 1$, Brownian motion on $\mathbb{R}^n - \{0\}$ is mapped to the Bessel process on $(0, \infty)$ with generator $A = \frac{1}{2} \frac{d^2}{dx^2} + \frac{n-1}{2x} \frac{d}{dx}$. Thus in this case $\frac{1}{2} \triangle_N$ is projectible but its projection is not $\frac{1}{2} \triangle_M$. The well known criterion for the latter to hold is that $p$ has minimal fibres as we show below. See also [21], and
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[48]. To examine this in more detail we follow Liao,[48]. Suppose that \( p \) is a Riemannian submersion. The horizontal subbundle on \( N \) is just the orthogonal complement of the vertical bundle. Working locally take an orthonormal family of vector fields \( X^1, \ldots, X^n \) in a neighbourhood of of a given point \( x_0 \) of \( M \). Let \( \tilde{X}^1, \ldots, \tilde{X}^n \) be their horizontal lifts to a neighbourhood of some \( u_0 \) above \( x_0 \), and let \( V^1, \ldots, V^p \) be a locally defined orthonormal family of vertical vector fields around \( u_0 \). Then near \( u_0 \), using the summation convention over \( j = 1, \ldots, n \), \( \alpha = 1, \ldots, p \), we have

\[
\Delta_N = \tilde{X}^j \tilde{X}^j + V^\alpha V^\alpha - \nabla^N_{\tilde{X}^j} \tilde{X}^j - \nabla^V_{V^\alpha} V^\alpha
\]

(7.1)

while

\[
\Delta_M = X^j X^j - \nabla^N_{X^j} X^j.
\]

(7.2)

Here \( \nabla_M, \nabla_N \) refer to the Levi-Civita connections on \( M \) and \( N \), and we are identifying the vector fields with the Lie differentiation in their directions.

Now \( \tilde{X}^j \tilde{X}^j \) lies over \( X^j X^j \) while \( V^\alpha V^\alpha \) is vertical. Also the horizontal component of the sum \( \nabla^V_{V^\alpha} V^\alpha \) at a point \( u \in N \) is the trace of the second fundamental form of the fibre \( N_{p(u)} \) of \( p \) through \( u \), denoted by \( T_{V^\alpha} V^\alpha \) in O’Neill’s notation, while \( \frac{1}{2} \Delta_N \) lies over \( \nabla_{X^j} X^j \) by Lemma 1 of [55].

Thus we see that \( \frac{1}{2} \Delta_N \) is projectible if and only if the trace of the second fundamental form, \( \text{trace} T \), of each fibre \( p^{-1}(x) \) is constant along the fibre in the sense of being the horizontal lift of a fixed tangent vector, \( 2A(x) \in T_x M \). If so \( \frac{1}{2} \Delta_N \) lies over \( \frac{1}{2} \Delta_M - A \). In particular \( A = 0 \), or equivalently \( p \) maps Brownian motion to Brownian motion, if and only if \( p \) has minimal fibres.

In general to relate to the discussion in Section 2.4 we can set \( b^H(u) = -\frac{1}{2} \text{trace} T(u) \), with \( b(u) = T_u p b^H(u) \) in \( T_{p(u)} M \). Let \( \Delta^V \) be the vertical operator on \( N \) which restricts to the Laplacian on each fibre, and let \( \Delta^H \) be the horizontal lift of \( \frac{1}{2} \Delta_M \). Our decomposition in Theorem 2.4.6 becomes

\[
\frac{1}{2} \Delta_N = \left( \frac{1}{2} \Delta^H - \frac{1}{2} \text{trace} T \right) + \frac{1}{2} \Delta^V
\]

(7.3)

since the vertical part of \( \nabla^V_{V^\alpha} V^\alpha \) is just \( \nabla^V_{V^\alpha} V^\alpha \) where \( \nabla^V \) refers to the connection on the vertical bundle which restricts to the Levi-Civita of the fibres, and also the vertical part of \( \tilde{X}^j \tilde{X}^j \) vanishes because by Lemma 2 of [55] the vertical part of \( \tilde{X}^j \tilde{X}^k \) is the vertical part of \( \frac{1}{2} [\tilde{X}^j, \tilde{X}^k] \).
7.2 Riemannian Symmetric Spaces

Let $K$ be a Lie group with bi-invariant metric and let $M$ be a Riemannian manifold with a symmetric space structure given by a triple $(K, G, \sigma)$. This means that there is a smooth left action $K \times M \rightarrow M, (k, x) \mapsto L_k(x)$ of $K$ on $M$ by isometries such that if we fix a point $x_0$ of $M$ and define $p : K \rightarrow M$ by $p(k) = L_k(x_0)$ then $p$ is a Riemannian submersion and a principal bundle with group the subgroup $K_{x_0}$ of $K$ which fixes $x_0$. Write $G$ for $K_{x_0}$. Thus $M$ is diffeomorphic to $K/G$. Moreover if $g$ denotes the Lie algebra of $G$, and $\mathfrak{k}$ that of $K$, (identified with the tangent spaces at the identity to $G$ and $K$ respectively), there is an orthogonal and $\text{ad}_G$-invariant decomposition

$$\mathfrak{k} = \mathfrak{g} + \mathfrak{m}$$

where $\mathfrak{m}$ is a linear subspace of $T_{\text{id}}K$. Further $\sigma$ is an involution on $K$ and $\mathfrak{g}$ and $\mathfrak{m}$ are, respectively, the $+1$ and the $-1$ eigenspaces of the involution on $T_{\text{id}}K$ induced by $\sigma$. See Note 7, page 301, of Kobayashi & Nomizu Volume I, [40], for definitions and basic properties, and Volume II, [41], for a detailed treatment.

We shall also let $\sigma$ denote the involutions induced by $\sigma$ on $\mathfrak{k}$ and on $M$, and by differentiation on $TM$ and $OM$. On $M$ it is an isometry, so it does act on $OM$.

Note that on $T_{x_0}M$ it acts as $v \mapsto -v$.

Since $G$ fixes $x_0$ the derivative of the left action $L_k$ at $x_0$ gives a representation of $G$ by isometries of $T_{x_0}M$. The linear isotropy representation. We shall assume it to be faithful, i.e. injective. As a consequence the action of $K$ on $M$ is effective, so that $K$ can be considered as a sub-group of the diffeomorphism group of $M$, and also the action of $K$ on the frame bundle of $M$ is free, i.e the only element of $K$ which fixes a frame is the identity element. See page 187 and the remark on page 198 of [41] for a discussion of this, and how the condition can be avoided.

Taking a fixed orthonormal frame $u_0 : \mathbb{R}^n \rightarrow T_{x_0}M$, say, at $x_0$, we can consider $G$ as acting by isometries on $\mathbb{R}^n$ by

$$g \cdot e = u_0^{-1}TL_gu_0(e).$$

(7.4)

Let $\rho : G \rightarrow O(n)$ denote this representation. We then have the well known identification of $K$ as a subbundle of the orthonormal frame bundle of $M$:

**Proposition 7.2.1** Let $\Phi : K \rightarrow OM$ be defined by $\Phi(k)(e) = TL_k(u_0e)$ for $e \in \mathbb{R}^n$. Then $\Phi$ is an injective homomorphism of principle bundles. Moreover $\Phi$ is equivariant for the actions of $\sigma$ on $K$ and $OM$.
Proof. To see that \( \Phi \) is a bundle homomorphism it is only necessary to check that \( \Phi \) commutes with the actions of \( G \). For this take \( e \in \mathbb{R}^n \) and \( g \in G \). Then, for \( k \in K \),

\[
\Phi(k \cdot g)(e) = TL_k TL_g u_0(e) = \Phi(k) TL_g u_0(e) = \Phi(k)(g \cdot e)
\]

as required. For the equivariance with respect to \( \sigma \) observe that by definition, \( \sigma(L_k x_0) = L_{\sigma(k)} x_0 \) so that acting on the frame \( \Phi(k) \) we have

\[
\sigma(\Phi(k)) = \sigma(TL_k \circ u_0) = TL_{\sigma(k)} u_0 = \Phi(\sigma(k)).
\]

It is easy to see that \( p : K \to M \) has totally geodesic fibres. We can therefore take \( B = \frac{1}{2} \triangle^K \) to have \( B \) lying over \( \frac{1}{2} \triangle^M \). Moreover in the decomposition of \( B \) the vertical component \( \frac{1}{2} \triangle^V \) restricts to the one half the Laplacian of \( G \) on the fibre \( p^{-1}(x_0) \). The induced connection has horizontal subspace \( m \) at the identity element of \( K \). It is clearly left \( K \)-invariant and so \( H_k = TL_k[m] \) for general \( k \in K \). From the equivariance under the right action of \( G \) it is a principle connection: \( TR_g[H_k] = H_{kg} \). Since \( H_{kg} = TL_k TL_g[m] = TR_k TL_k ad_g[m] \) this holds because of the \( ad_G \)-invariance of \( m \). This is the canonical connection.

The connection on \( K \) extends to one on \( \mathcal{O}M \) as described in Proposition 3.1.3. This is known as the canonical linear connection. Since the connection on \( K \) is invariant under \( \sigma \), by the equivariance of \( \Phi \) so is the canonical linear connection. As in [41] we have:

**Proposition 7.2.2** The canonical linear connection is the Levi-Civita connection.

Proof. It is only necessary to check that its torsion \( T \) vanishes. By left invariance it is enough to do that at the point \( x_0 \). Let \( u, v \in T_{x_0} M \). However by invariance under \( \sigma \) we see

\[
T(u, v) = \sigma T(\sigma(u), \sigma(v)) = -T(-u, -v) = -T(u, v),
\]

as required. \( \square \)
Let $k_t, t \geq 0$ be the canonical Brownian motion on $K$ starting at the identity, $\text{id}$, and let $B_t$ be the Brownian motion on the Euclidean space $\mathfrak{k}$ given by the right flat anti-development:

$$B_t = \int_0^t TR_{k_s}^{-1} d\{k_s\}.$$ 

Define $\xi_t : M \rightarrow M$ by $\xi_t(x) = L_{k_t}x$, for $t \geq 0$, $x \in M$.

**Proposition 7.2.3** The diffeomorphism group valued process $\xi_t, t \geq 0$ is the flow of the sde

$$dx_t = X(x_t) \circ dB_t$$

where

$$X(x)\alpha = \frac{d}{dt} L_{\exp_{t\alpha}x}|_{t=0}$$

**Proof.** Observe that $k_\cdot$ satisfies the right invariant SDE

$$dk_t = TR_{k_t} \circ dB_t$$

which is $p$-related to the given SDE on $M$. \hfill \square

**Remark 7.2.4** The last two propositions relate to the discussion of connections determined by stochastic flows in the next section, and to the discussion about canonical SDE on symmetric spaces in [27]. In [27] it was shown that the connection determined by our SDE is the Levi-Civita connection. In Proposition 8.1.3 below, and in Theorem 3.1 of [25], it is shown that the connection determined by a flow (in this case the canonical linear connection) is the adjoint of that induced by its SDE. this is confirmed in our special case since the adjoint of a Levi-Civita connection is itself.

We can also apply our analysis of the vertical operators and Weitzenböck formulae to our situation, For this it is simplest to assume the symmetric space is irreducible. This means that the restricted linear holonomy group of the canonical connection on $p : K \rightarrow M$ is irreducible i.e. for every $g \in G$ there is a null-homotopic loop based at $x_0$ whose horizontal lift starting at $\text{id} \in K$ ends at the point $g$. The definition in [41] is that $[m, m]$ acts irreducibly on $m$ via the adjoint action, and it is shown there, page 252, that this implies that $g = [m, m]$. As a consequence the linear isotropy representation of $G$ on $T_{x_0}M$ is irreducible, and equivalently so is our representation $\rho$. 

The vertical operators determined by $\mathcal{B}^V$ on the bundles associated to $p$ via our representation $\rho$ and its exterior powers $\wedge^k \rho$ are given in Theorem 3.4.1 by the function $\lambda^{\wedge^k \rho} : K \to \mathcal{L}(\wedge^k \mathbb{R}^n; \wedge^k \mathbb{R}^n)$. By Corollary 3.4.8 and the discussion above they correspond to the Weitzenböck curvatures of the Levi-Civita connection, and so in particular are symmetric. To calculate them using Theorem 3.4.1 first use the fact that $\mathcal{B}^V$ restricts to $\frac{1}{2} \triangle G$ on $p^{-1}(x_0)$ to represent it as $\frac{1}{2} \sum L_{A_j^*} L_{A_j^*}$ for $A_j^*$ as in Section 3.2. The computation in the proof of Corollary 3.4.3 shows that

$$\lambda^{\wedge^k \rho}(u) = -\frac{(n-2)!}{(k-1)!(n-k-1)!} c_{\wedge^k}(u),$$

(7.5)

for

$$c_{\wedge^k}(u) = (d\wedge^k A_i(u) \circ (d\wedge^k A'_i)(u)$$

the Casimir element of our representation $\wedge^k \rho$ of $G$.

If $\wedge^k \rho$ is irreducible then $c_{\wedge^k}(u)$ is constant scalar. As remarked in Corollary 3.4.3 this happens when $G = SO(n)$, given our irreducibility hypothesis on the $\rho$ and then it is just $\frac{1}{2} n(n-1)/n! \ldots (n-k+1)/k!$. Thus for the sphere $S^n(\sqrt{2})$ of radius $\sqrt{2}$, considered as $SO(n+1)/SO(n)$ we have

$$\lambda^{\wedge^k \rho}(u) = -\frac{1}{4} k(n-k).$$

(7.6)
Chapter 8

Example: Stochastic Flows

Before analysing stochastic flows by the methods of the previous paragraphs we
describe some purely geometric constructions which will enable us to identify the
semi-connections which arise in that analysis.

8.1 Semi-connections on the Bundle of Diffeomorphisms

Assume that $M$ is compact. For $r \in \{1, 2, \ldots \}$ and $s > r + \dim M/2$ let $\mathcal{D}^s = \mathcal{D}^s(M)$ be the space of diffeomorphisms of $M$ of Sobolev class $H^s$. See, for
example, Ebin-Marsden [20] and Elworthy [21] for the detailed structure of this
space. Elements of $\mathcal{D}^s$ are then $C^r$ diffeomorphisms. The space is a topological
group under composition, and has a natural Hilbert manifold structure for which
the tangent space $T_\theta \mathcal{D}^s$ at $\theta \in \mathcal{D}^s$ can be identified with the space of $H^s$ maps
$v : M \to TM$ with $v(x) \in T_{\theta(x)}M$, all $x \in M$. In particular $T_{id} \mathcal{D}^s$ can be
identified with the space $H^s \Gamma(TM)$ of $H^s$ vector fields on $M$. For each $h \in \mathcal{D}^s$
the right translation

$$R_h : \mathcal{D}^s \to \mathcal{D}^s$$
$$R_h(f) = f \circ h$$

is $C^\infty$. However the joint map

$$\mathcal{D}^{s+r} \times \mathcal{D}^s \to \mathcal{D}^s$$

(8.1)

is $C^r$ rather than $C^\infty$ for each $r$ in $\{0, 1, 2, \ldots \}$. 

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For \( x_0 \in M \) fixed, define \( \pi : D^s \to M \) by

\[
\pi(\theta) = \theta(x_0).
\]

(8.2)

The fibre \( \pi^{-1}(y) \) at \( y \in M \) is given by: \( \{ \theta \in D^s : \theta(x_0) = y \} \). Set \( D^s_{x_0} := \pi^{-1}(x_0) \). Then the elements of \( D^s_{x_0} \) act on the right as \( C^\infty \) diffeomorphisms of \( D^s \). We can consider this as giving a principal bundle structure to \( \pi : D^s \to M \) with group \( D^s_{x_0} \), although there is the lack of regularity noted in equation (8.1).

A smooth semi-connection on \( \pi : D^s \to M \) consists of a family of linear horizontal lift maps \( h_\theta : E_{\pi(\theta)} \to T_\theta D^s, \theta \in D^s \), which is smooth in the sense that it determines a \( C^\infty \) section of \( \mathcal{L}(\pi^*E; TD^s) \to D^s \). In particular we have

\[
h_\theta(u) : M \to TM
\]

with

\[
h_\theta(u)(y) \in T_{\theta(y)}M,
\]

\( u \in E_{\theta(x_0)}, \theta \in D^s, y \in M \).

We shall relate semi-connections on \( D^s \to M \) to certain reproducing kernel Hilbert spaces. For this let \( E \) be a smooth sub-bundle of \( TM \) and \( \mathcal{H} \) a Hilbert space which consists of smooth section of \( E \) such that the inclusion \( \mathcal{H} \to C^\infty \Gamma E \) is continuous (from which comes the continuity into \( \mathcal{H}^s \Gamma E \) for all \( s > 0 \)). Such a Hilbert space determines and is determined by its reproducing kernel \( k \), a \( C^\infty \) section of the bundle \( \mathcal{L}(E^*; E) \to M \times M \) with fibre \( \mathcal{L}(E^*_x; E_y) \) at \( (x, y) \), see [4]. By definition,

\[
k(x, -) = \rho_x^* : E_x^* \to \mathcal{H}
\]

where \( \rho_x : \mathcal{H} \to E_x \) is the evaluation map at \( x \), and so

\[
k(x, y) = \rho_y \rho_x^* : E_x^* \to E_y.
\]

Assume \( \mathcal{H} \) spans \( E \) in the sense that for each \( x \) in \( M \), \( \rho_x : \mathcal{H} \to E_x \) is surjective. It then induces an inner product \( \langle \cdot, \cdot \rangle^\mathcal{H}_x \) on \( E_x \) for each \( x \) via the isomorphism \( \rho_x \rho_x^* : E_x^* \to E_x \).

Using the metric on \( E \) the reproducing kernel \( k \) induces linear maps

\[
k^\#(x, y) : E_x \to E_y, \quad x, y \in M,
\]

with \( k^\#(x, x) = \text{id} \).
8.1. SEMI-CONNECTIONS ON THE BUNDLE OF Diffeomorphisms

Proposition 8.1.1 A Hilbert space $\mathcal{H}$ of smooth sections of a sub-bundle $E$ of $TM$ which spans $E$ determines a smooth semi-connection $h^H$ on $\pi : D^s \to M$ over $E$ by

$$h^H_\theta(u)(y) = k^#(\theta(x_0), \theta(y))(u), \quad \theta \in D^s, u \in E_{\theta(x_0)}, y \in M, \quad (8.3)$$

for $k^#$ derived from the reproducing kernel of $\mathcal{H}$ as above. In particular the horizontal lift $\tilde{\alpha}$ starting from $\tilde{\alpha}(0) = \text{id}$, of a curve $\alpha : [0, T] \to M$, $\alpha(0) = x_0$ with $\dot{\alpha}(t) \in E_{\alpha(t)}$ for all $t$, is the flow of the non-autonomous ODE on $M$

$$\dot{z}_t = k^#(\alpha(t), z_t)\dot{\alpha}(t). \quad (8.4)$$

The mapping $\mathcal{H} \mapsto (h^H, \langle \cdot, \cdot \rangle^H)$ from such Hilbert spaces to semi-connections over $E$ and Riemannian metrics on $E$ is injective.

Proof. From the definition of $k^#$ we see $h^H_\theta(u)(y)$, as given by (8.3), takes values in $T_{\theta(y)}M$, is linear in $u \in E_{\theta(x_0)}$ into $T_{\theta(y)}D^s$, and is $D^s_{x_0}$-invariant. Moreover,

$$T_{\theta} \pi \circ h^H_\theta(u) = h^H_\theta(u)(x_0) = k^#(\theta(x_0), \theta(x_0))(u) = u$$

for $u \in E_{\theta(x_0)}$ and so $h^H_\theta$ is a ‘lift’.

To see that $h$ is $C^\infty$ as a section of $\mathcal{L}(\pi^*E; TD^s) \to D^s$ note that for each $r \in \{0, 1, 2, \ldots \}$ the composition map

$$T_{id}D^{r+s} \times D^s \to TD^s$$

$$(V, \theta) \mapsto TR_\theta(V)$$

is a $C^r$ vector bundle map over $D^s$, being a partial derivative of the composition $D^{r+s} \times D^s \to D^s$. Therefore it induces a $C^{r-1}$ vector bundle map $Z \mapsto TR_\theta \circ Z$, for $Z : E_{\theta(x_0)} \to \mathcal{H}$ and for $\mathcal{H}$ the trivial $\mathcal{H}$-bundle over $D^s$, by composition

$$\mathcal{L}(\pi^*E; \mathcal{H}) \to \mathcal{L}(\pi^*E; TD^s)$$

$D^s$
On the other hand \( y \mapsto k(y, -) \) can be considered as a \( C^\infty \) section of \( \mathcal{L}(E; H_\gamma) \rightarrow M \) and so \( \theta \mapsto k(\theta(x_0), -) \) as a \( C^\infty \) section of \( \mathcal{L}(\pi^* E; H_\gamma) \). This proves the regularity of \( h \).

That the horizontal lift \( \tilde{\alpha} \) is the flow of (8.4) is immediate. To see that the claimed injectivity holds, given \( \theta \mapsto k(\theta(x_0), -) \) as a \( C^\infty \) section of \( \mathcal{L}(\pi^* E; H_\gamma) \). This proves the regularity of \( h \).

Remark 8.1.2 We cannot expect surjectivity of the map \( \mathcal{H} \rightarrow h^{\mathcal{H}} \) into the space of semi-connections on \( \pi : \mathcal{D}^s \rightarrow M \). Indeed for \( k^\# \) defined by (8.5) to be the reproducing kernel for some Hilbert space of sections of \( E \) we need

1) \( h^{\mathcal{H}}_\theta(u)(y) \in E_{\theta(y)} \), for \( u \in E_{\theta(x_0)}, y \in M \), and a metric \( \langle \cdot, \cdot \rangle \) on \( E \) with respect to which the following holds:

2) for \( x, y \in M \),

\[
  k^\#(x, y) = \left( k^\#(y, x) \right)^\ast,
\]

3) For any finite set \( S \) of points of \( M \) and \( \{\xi_a\} \in E_a, a \in S \)

\[
  \sum \langle k^\#(a, b)\xi_a, \xi_b \rangle \geq 0.
\]

For each frame \( u_0 : \mathbb{R}^n \rightarrow T_{x_0}M \) there is a homomorphism of principal bundles

\[
  \Psi^{u_0} : \mathcal{D}^s \rightarrow \text{GLM} \quad \theta \mapsto T_{x_0}\theta \circ u_0. \tag{8.6}
\]

As with connections such a homeomorphism maps a semi-connection on \( \mathcal{D}^s \) over \( E \) to one on \( \text{GLM} \). The horizontal lift maps are related by

\[
  \begin{array}{ccc}
  T_{\mathcal{D}^s} & \xrightarrow{T_{\mathcal{H}}\Psi^{u_0}} & T_{\Psi^{u_0}(\theta)\text{GLM}} \\
  h_\theta & \downarrow & h_{\Psi^{u_0}(\theta)} \\
  E_{\theta(x_0)} & \xleftarrow{E_{\theta(x_0)}} & \end{array}
\]

and if \( \tilde{\alpha} : [0, T] \rightarrow \mathcal{D}^s \) is a horizontal lift of \( \alpha : [0, T] \rightarrow M \) then

\[
  \Psi^{u_0}(\tilde{\alpha}(t)) = T_{x_0}\tilde{\alpha}(t) \circ u_0, \quad 0 \leq t \leq T
\]

is a horizontal lift of \( \alpha \) to \( \text{GLM} \).
Theorem 8.1.3 Let $h^\mathcal{H}$ be the semi-connection on $\pi : D^s \to M$ over $E$ determined by some $\mathcal{H}$ as in Proposition 8.1.1. Then the semi-connection induced on $GLM$, and so on $TM$, by the homeomorphism $\Psi^{u_0}$ is the adjoint $\hat{\nabla}$ of the metric connection which is projected on $(E, \langle \cdot, \cdot \rangle^\mathcal{H})$ by the evaluation map $(x, e) \mapsto \rho_x(e)$ from $M \times \mathcal{H} \to E$, c.f. (1.1.10) in [27]. In particular every semi-connection on $TM$ with metric adjoint connection arises this way from some, even finite dimensional, choice of $\mathcal{H}$.

Proof. Let $\alpha : [0, T] \to M$ be a $C^1$ curve with $\dot{\alpha}(t) \in E_{\alpha(t)}$ for each $t$. By Proposition 8.1.1 its horizontal lift $\tilde{\alpha}$ to $D^s$ starting from $\theta \in \pi^{-1}(\alpha(0))$ is the solution to

$$\frac{d\tilde{\alpha}}{dt} = k^\#(\tilde{\alpha}(t)(x_0), \tilde{\alpha}(t) - \dot{\alpha}(t)) \quad (8.7)$$

$$\tilde{\alpha}(0) = \theta. \quad (8.8)$$

The horizontal lift to $GLM$ is $t \mapsto T_{x_0} \tilde{\alpha}(t) \circ u_0$ and to $TM$ through $v_0 \in T_{\pi(x_0)}M$, i.e. the parallel translation $\{/i(v_0) : 0 \leq t \leq T\}$ of $v_0$ along $\alpha$, is given by

$$/i(v_0) = T_{x_0} \tilde{\alpha}(t) \circ (T_{x_0} \theta)^{-1}(v_0) = T_{\alpha(0)} \left( \tilde{\alpha}(t) \circ \theta^{-1} \right)(v_0).$$

However this is $T_{\alpha(0)} /i(v_0)$ for $\{/i : 0 \leq t \leq T\}$ the solution flow of

$$\frac{dz_t}{dt} = k^\#(\alpha(t), z(t)) \dot{\alpha}(t)$$

which by Lemma 1.3.4 of [27] is the parallel translation of the adjoint of the associated connection (in [27] $k^\#$ is denoted by $k$).

The fact that all such semi-connections on $TM$ arise from some finite dimensional $\mathcal{H}$ comes from Narasimhan-Ramanan [53] as described in [27], or more directly from Quillen [60].

8.2. Semi-connections Induced by Stochastic Flows

From Baxendale [5] we know that a $C^\infty$ stochastic flow $\{\xi_t : t \geq 0\}$ on $M$, i.e. a Wiener process on $D^\infty := \cap_k D^s$, can be considered as the solution flow of a stochastic differential equation on $M$ driven by a possibly infinite dimensional noise. Its one point motions form a diffusion process on $M$ with generator $\mathcal{A}$, say. The noise comes from the Brownian motion $\{W_t : t \geq 0\}$ on $\mathcal{H}^{*}(TM)$.
determined by a Gaussian measure $\gamma$ on $\mathcal{H}^s \Gamma(TM)$. (In our $C^\infty$ case they lie on $\mathcal{H}^\infty(TM) := \cap_\gamma \mathcal{H}^s \Gamma(TM)$.) We will take $\gamma$ to be mean zero and so we may have a drift $A$ in $\mathcal{H}^\infty(TM)$. The stochastic flow $\{\xi_t : t \geq 0\}$ can then be taken to be the solution of the right invariant stochastic differential equation on $D^s$

$$d\theta_t = TR_{\theta_t} \circ dW_t + TR_{\theta_t}(A)dt \quad (8.9)$$

with $\xi_0$ the identity map $id$. In particular it determines a right invariant generator $B$ on $D^s$.

For fixed $x_0$ in $M$ the one point motion $x_t := \xi_t(x_0)$ solves

$$dx_t = \circ dW_t(x_t) + A(x_t)dt. \quad (8.10)$$

We can write (8.10) as

$$dx_t = \rho x_t \circ dW_t + A(x_t)dt. \quad (8.11)$$

Thus $\pi(\xi_t) = \xi_t(x_0) = x_t$. For a map $\theta$ in $D^s$, the solution $\xi_t \circ \theta$ to (8.9) starting at $\theta$ has $\pi(\xi_t \circ \theta) = \xi_t(\pi(\theta))$, the solution to (8.11) starting from $\pi(\theta)$, and we see that the diffusions are $\pi$-related (c.f. [21]), and $A$ and $B$ are intertwined by $\pi$.

The measure $\gamma$ corresponds to a reproducing kernel Hilbert space, $H_{\gamma}$ say, or equivalently to an abstract Wiener space structure $i : H_{\gamma} \to \mathcal{H}^s \Gamma(TM)$ with $i$ the inclusion (although $i$ may not have dense image). Then

$$\sigma^B_B : (T_y D^s)^* \to T_y D^s$$

is right invariant and determined at $\theta = id$ by the canonical isomorphism $H_{\gamma}^* \simeq H_{\gamma}$ through the usual map $j = i^*$

$$(\mathcal{H}^s \Gamma(TM))^* \xrightarrow{j} H_{\gamma}^* \simeq H_{\gamma} \xleftarrow{i} \mathcal{H}^s \Gamma(TM),$$

i.e.

$$\sigma^B_{id} = i \circ j.$$
The reproducing kernel $k$ of $H_\gamma$ is the covariance of $\gamma$ and:

$$k^\#(x, y)v = \int_{U \in \gamma(E)} \langle U(x), v \rangle U(y) \, d\gamma(U), \quad v \in E_x; \quad x, y \in M.$$  

Analogously to Lemma 2.2.1 we have the commutative diagram

$$
\begin{array}{ccc}
(T_\theta D^s)^* & \xrightarrow{j \circ (TR_\theta)^*} & H_\gamma \\
\downarrow & & \downarrow T \theta \pi \circ i \\
T^*_\theta(x_0) & \xrightarrow{k(\theta(x_0), -)} & H_\gamma \\
\downarrow & & \downarrow \rho_{\theta(x_0)} \\
E^*_{\theta(x_0)} & \xrightarrow{\rho_{\theta(x_0)}} & E_{\theta(x_0)} \hookrightarrow T_{x_0} M
\end{array}
$$

with $\ell_\theta$ uniquely determined under the extra condition

$$\ker \ell_\theta = \ker \rho_{\theta(x_0)}.$$

Writing $K : M \to \mathcal{L}(H_\gamma; H_\gamma)$ for the map giving the projection $K(x)$ of $H_\gamma$ onto $\ker \rho_x$ for each $x$ in $M$ and letting $K^\perp(x)$ be the projection onto $[\ker \rho_x]^\perp$ we have

$$\ell_\theta = K^\perp(\theta(x_0)),$$

(agreeing with the note following Lemma 2.2.1), and so

$$\ell_\theta(U) = k^\#(\theta(x_0), -) U(\theta(x_0)), \quad U \in H_\gamma.$$

Note that the formula

$$K^\perp(y)(U) = k^\#(y, -) U(y)$$

for $U$ in $H_\gamma$ determines an extension $K^\perp(y) : \Gamma E \to H_\gamma$. We then define $K(y)U = U - K^\perp(y)U$. Note that $\rho_y(K(y)U) = 0$ for all $U$ in $\Gamma E$.

The horizontal lift map determined by $B$ as in Proposition 2.1.2 is therefore given by

$$h_\theta : E_{\theta(x_0)} \to T \gamma \pi(T_\theta D_s) \subset T_\theta D^s$$

$$h_\theta(U) = T \gamma \pi \ell_\theta \left[ k^\#(\theta(x_0), -) U \right], \quad (8.12)$$

for $\theta \in D^s$. Consequently

$$h_\theta(U)(y) = k^\#(\theta(x_0), \theta(y))(U). \quad (8.13)$$

Comparing this with formula (8.3) we have
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Proposition 8.2.1 The semi-connection $h$ determined on $\pi : D^s \to M$ by the equivariant diffusion operator $B$ is just that given by the reproducing kernel Hilbert space $H_\gamma$ of the stochastic flow which determines $B$, i.e.

$$h = h^{H_\gamma}.$$  

The horizontal lift $\{\tilde{x}_t : t \geq 0\}$ of the one point motion $\{x_t : t \geq 0\}$ with $\tilde{x}_0 = \text{id}$ is the solution to

$$d\tilde{x}_t = k^\#(\tilde{x}_t(x_0), \tilde{x}_t - ) \circ dx_t;$$  

which in a more revealing notation is:

$$d\tilde{x}_t = TR_{\tilde{x}_t}\left(K^\perp(\tilde{x}_t(x_0)) \circ dW_t\right) + TR_{\tilde{x}_t}\left(K^\perp(\tilde{x}_t(x_0))A\right).$$  

Equivalently $\{\tilde{x}_t : t \geq 0\}$ can be considered as the solution flow of the non-autonomous stochastic differential equation on $M$

$$dy_t = k^\#(x_t, y_t) \circ dx_t$$

$i.e.$

$$dy_t = \left(K^\perp(x_t) \circ dW_t\right)(y_t) + K^\perp(x_t)(A)(y_t).$$  

The standard fact that the solution to such equation as (8.16) starting at $x_0$ is just $\{x_t : t \geq 0\}$, i.e. that $\tilde{x}_t(x_0) = x_t$ reflects the fact that $\tilde{x}.$ is a lift of $x$. The lift through $\phi \in D^s_{x_0}$ is just $\{\tilde{x}_t \circ \phi : t \geq 0\}$.

Remark 8.2.2 If our solution flow is that of an SDE

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt$$

for $X(x) : \mathbb{R}^m \to TM$ arising, for example, from Hörmander form representation of $A$ as in §4.7 above the relationships with the notation in this section is as follows: $H_\gamma = \{X(\cdot)e : e \in \mathbb{R}^m\}$ with inner product induced by the surjection $\mathbb{R}^m \to H_\gamma$. If $Y_x = [X(x)|_{\ker X(x)}]^{-1}$ then $k^\#(y, -) : E_y \to H_\gamma$ is

$$k^\#(y, -)u = X(-)Y_y(u), \quad u \in E_y.$$  

Also $K^\perp(y) : \Gamma E \to H_\gamma$ is $K^\perp(y)U = X(-)Y_y(U(y))$. 

...
Remark 8.2.3 The reproducing kernel Hilbert space $H_\gamma$ determines the stochastic flow and so by the injectivity part of Proposition 8.1.1 the semi-connection together with the generator $A$ of the one-point motion determines the flow, or equivalently the operator $B$. This is because the symbol of $A$ again gives the metric on $E$ which together with the semi-connection determines $H_\gamma$ by Proposition 8.1.1. The generator $A$ then determines the drift $A$. A consequence is that the horizontal lift $A^H$ of $A$ to $D^s$ determines the flow (and hence $B$, so $B^V$ really is redundant).

To see this directly note that given any cohesive $A$ on $M$ and $D^s_{\rho_0}$-equivariant $A^H$ on $D^s$ over $A$, with no vertical part, there is at most one vertical $B^V$ such that $A^H + B^V$ is right invariant. This follows from the following lemma

Lemma 8.2.4 Suppose $B^1$ is a diffusion operator on $D^s$ which is vertical and right invariant then $B^1 = 0$.

Proof. By Remark 1.3.2 (i) the image $E_\theta$, say, of $\sigma^B_\theta$ lies in $VT_\theta D^s$ for $\theta \in D^s$ and so if $V \in E_\theta$. On the other hand, by right invariance $E_\theta = TR_\theta(E_{id})$. Therefore if $V \in E_{id}$ then $V(\theta_{x_0}) = 0$ all $\theta \in D^s$ and so $V \equiv 0$. Thus $E_{id} = \{0\}$ and by right invariance, $B^1$ must be given by some vector field $Z$ on $D^s$. But $Z$ must be vertical and right invariant, so again we see $Z \equiv 0$. $\square$

Proposition 3.1.3 applies to the homomorphism $\Psi^{u_0} : D^s \to GL(M)$ of (8.6). From this and Theorem 8.1.3 we see that the semi-connection $\nabla$ on $GLM$ determined by the generator of the derivative flow in $\S 3.3$ is the adjoint $\hat{\nabla}$ of the connection $\tilde{\nabla}$, so giving an alternative proof of Theorem 3.3.1 above. Proposition 3.1.3 also gives a relationship between the curvature and holonomy group of $\nabla$ and those of the connection induced by the flow on $D^s \to M$.

We can summarize our decomposition results as applied to these stochastic flows in the following theorem. The skew product decomposition was already described in [25] for the case of solution flows of SDE of the form (4.19), and in particular with finite dimensional noise: however the difference is essentially that of notation, see Remark 8.2.2 above.

Theorem 8.2.5 Let $\{\xi_t : t \geq 0\}$ be a $C^\infty$ stochastic flow on a compact manifold $M$. Let $A$ be the generator of the one point motion on $M$ and $B$ the generator of the right invariant diffusion on $D^s$ determined by $\{\xi_t : t \geq 0\}$. Assume $A$ is strongly cohesive. Then there is a unique decomposition $B = A^H + B^V$ for $A^H$ a diffusion operator which has no vertical part in the sense of definition 2.3.3
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and $B^V$ a diffusion operator which is along the fibres of $\rho_{x_0}$, both invariant under the right action of $D^s_{x_0}$. The diffusion process $\{\theta_t : t \geq 0\}$ and $\{\phi_t : t \geq 0\}$ corresponding to $A^H$ and $B^V$ respectively can be represented as solutions to

$$d\theta_t = TR_{\theta_t} \left( K^+(\theta_t(x_0)) \circ dW_t \right) + TR_{\theta_t} \left( K^+(\theta_t(x_0))A \right)$$  \hspace{1cm} (8.17)

and

$$d\phi_t = TR_{\phi_t} \left( K(z_0) \circ dW_t \right) + TR_{\phi_t} \left( K(z_0)A \right)$$  \hspace{1cm} (8.18)

for $z_0 = \phi_0(x_0) = \phi_t(x_0)$. There is the corresponding skew-product decomposition of the given stochastic flow

$$\xi_t = \tilde{x}_t g^+_t, 0 \leq t < \infty$$

where $\{\tilde{x}_t : t \geq 0\}$ is the horizontal lift of the one point notion $\{\xi_t(x_0) : t \geq 0\}$ with $\tilde{x}_0 = \text{id}_M$ and for $P_{A^H_{x_0}}$-almost all $\sigma : [0, \infty) \to M$, $\{g_t^\sigma : t \geq 0\}$ is a $D^s_{x_0}$-valued process independent of $\{\tilde{x}_t : t \geq 0\}$ and satisfying

$$dg_t^\sigma = T\tilde{\sigma}_t^{-1} \rho(\tilde{\sigma}_t g_t^\sigma -) \left( K(\sigma_t) \circ dW_t \right) + T\tilde{\sigma}_t^{-1} \rho(\tilde{\sigma}_t g_t^\sigma -) \left( K(\sigma_t)A \right)$$

where $\tilde{\sigma}$ is the horizontal lift of $\sigma$ to $D^s$ with $\tilde{\sigma}_0 = \text{id}_M$.

**Remark 8.2.6** As in [27] we could rewrite the terms such as $K(\sigma_t) \circ dW_t$ and $K^+(\sigma_t) \circ dW_t$ above as Itô differentials which can be written as

$$K(\sigma_t)dW_t = \tilde{\gamma}(\sigma_t)d\beta_t$$

$$K^+(\sigma_t)dW_t = \tilde{\gamma}_l(\sigma_t)d\tilde{B}_t$$

where $\tilde{\gamma}_l(\sigma_t) : \gamma \to \gamma$, $0 \leq t < \infty$, is a family of orthogonal transformations mapping $\ker \rho_{x_0} \to \ker \rho_{\sigma_t}$ defined for $P_{A^H_{x_0}}$-almost all $\sigma : [0, \infty) \to M$ and $\{\beta_t : t \geq 0\}$, $\{\tilde{B}_t : t \geq 0\}$ are independent Brownian motions, ($\beta_t$ could be cylindrical), on $\ker \rho_{x_0}$ and $[\ker \rho_{x_0}]^\perp$ respectively.

**Proof.** Our general result gives the decomposition $B = A^H + B^V$ into horizontal and vertical parts. We have just proved the representation (8.17) for $A^H$. To show
that $B - A^H$ corresponds to (8.18) take an orthonormal base $\{X^j\}$ for $H_\gamma$. Then, on a suitable domain,

$$B = \frac{1}{2} \sum_j L_{X^j} L_{X^j} + L_A$$  \hspace{1cm} (8.19)$$

for $X^j(\theta) = TR_\theta(X^j)$ and $A = TR_\theta(A)$, while, by (8.17),

$$A^H = \frac{1}{2} \sum_j L_{X^j} L_{X^j} + L_B$$  \hspace{1cm} (8.20)$$

for $X^j(\theta) = TR_\theta(K_\perp(\theta(x_0))X^j)$, $B = TR_\theta(K_\perp(\theta(x_0)))A$.

Define vector fields $Z^j, C$ on $D^s$ by

$$Z^j(\phi) = TR_\phi(K(\phi(x_0))X^j), \quad \text{and} \quad C(\phi) = TR_\phi(K(\phi(x_0))A), \quad \phi \in D^s.$$  

Then $A = B + C$ and $X^j = Y^j + Z^j$ each $j$. Moreover

$$\sum_j L_{Y^j} L_{Z^j} + \sum_j L_{Z^j} L_{Y^j} = 0$$

by Lemma 8.2.7 which follows below. This shows that

$$B^V = \frac{1}{2} \sum_j L_{Z^j} L_{Z^j} + L_C.$$  \hspace{1cm} (8.21)$$

Thus the diffusion process from $\phi_0$ corresponding to $B^V$ can be represented by the solution to

$$d\phi_t = TR_{\phi_t}(K(\phi_t(x_0) \circ dW_t)) + TR_{\phi_t}(K(\phi_t(x_0)A)) \, dt.$$  \hspace{1cm} (8.22)$$

If we set $z_t = \rho_{x_0}(\phi_t) = \phi_t(x_0)$. We obtain, via Itô’s formula

$$z_t = \rho_{z_t}(K(z_t) \circ dW_t) + \rho_{z_t}(K(z_t)A),$$

i.e. $dz_t = 0$. Thus $\phi_t(x_0) = z_0$ and (8.18) holds.

The skew product formula is seen to hold by calculating the stochastic differential of $\tilde{x}_t g^2_t$ using (8.15) to see it satisfies the SDE (8.9) for $\{\xi_t : t \geq 0\}$.  \hspace{1cm} $\square$

**Lemma 8.2.7**

$$\sum_j L_{Y^j} L_{Z^j} + L_{Z^j} L_{Y^j} = 0.$$
Proof. Since, for fixed $\theta$, we can choose our basis $\{X^j\}$, such that either $Y^j(\theta) = 0$ or $X^j(\theta) = 0$, and since for $f : D^s \to \mathbb{R}$ we can write

$$df\left(Z^j(\theta)\right) = \left(df \circ TR_\theta\right)\left(K(\theta(x_0))X^j\right)$$

and

$$df(Y^j(\theta)) = \left(df \circ TR_\theta\right)\left(K^\perp(\theta(x_0))X^j\right), \quad \theta \in D^s,$$

it suffices to show that

$$\sum_j \left\{ (dK^\perp)_{\theta(x_0)}\left(Z^j(\theta)(x_0)\right)X^j + (dK)_{\theta(x_0)}\left(Y^j(\theta)(x_0)\right)X^j \right\} = 0, \quad (8.23)$$

for all $\theta \in D^s$.

Now $K^\perp(y)K(y) = 0$ for all $y \in M$. Therefore

$$(dK^\perp)_y(v)K(y) + K^\perp(y)(dK)_y(v) = 0, \quad \forall v \in T_x M, x \in M.$$  

Writing

$$X^j = K(\theta(x_0))X^j + K^\perp(\theta(x_0))X^j$$

this reduces the right hand side of (8.23) to

$$\sum_j \left\{ (dK^\perp)_{\theta(x_0)}\left(Z^j(\theta)(x_0)\right)\left(K^\perp(\theta(x_0))X^j\right) + (dK)_{\theta(x_0)}\left(Y^j(\theta)(x_0)\right)\left(K(\theta(x_0))X^j\right) \right\} = 0$$

with our choice of basis this clearly vanishes, as required. \qed

8.3 Semi-connections on Natural Bundles

Our bundle $\pi : \text{Diff} M \to M$ can be considered as a universal natural bundles over $M$, and a connection on it induces a connection on each natural bundle over $M$. Natural bundles are discussed in Kolar-Michor-Slovak [42]), they include bundles such as jet bundles as well as the standard tensor bundles. For example let $G^*_r$ be the Lie group of $r$-jets of diffeomorphisms $\theta : \mathbb{R}^n \to \mathbb{R}^n$ with $\theta(0) = 0$ for positive integer $r$. An ‘$r$-th order frame’ $u$ at a point $x$ of $M$ is the $r$-jet at 0 of some $\psi : U \to M$ which maps an open set $U$ of $\mathbb{R}^n$ diffeomorphically onto an open subset of $M$ with $0 \in U$ and $\psi(0) = x$. Clearly $G^*_r$ acts on the right of such
jets, by composition. From this we can define the rth order frame bundle \( G^r_nM \) of \( M \) with group \( G^r_n \).

If we fix an rth order frame \( u_0 \) at \( x_0 \) we obtain a homomorphism of principal bundles

\[
\Psi^{u_0} : D^s \to G^r_nM
\]

\[
\theta \mapsto j^r_{x_0}(\theta) \circ u_0
\]
as for \( GLM \) (which is the case \( r = 1 \)) with associated group homomorphism \( D^s \to G_n \) given by \( \theta \mapsto u_0^{-1} \circ j^r_{x_0}(\theta) \circ u_0 \). As for the case \( r = 1 \) there is a diffusion operator induced by the flow on \( G^2_nM \) and we are in the situation of Proposition 8.1.3. The behaviour of the flow induced on \( G^2_nM \) is essentially that of \( j^2_{x_0}(\xi_t) \) and so relevant to the effect on the curvature of sub-manifolds of \( M \) as they are moved by the flow e.g. see Cranston-LeJan [14], Lemaire [46].

Alternatively rather having to choose some \( u_0 \) we see that \( G^r_nM \) is (weakly) associated to \( \pi : D^s \to M \) by taking the action of \( D^s \) on \( (G^r_xM)_{x_0} \) by

\[
(\theta, \alpha) \mapsto j^r_{x_0}(\theta) \circ \alpha.
\]

As a geometrical conclusion we can observe

**Theorem 8.3.1** Any classifying bundle homomorphism

\[
OM \xrightarrow{\Phi} V(n, m - n)
\]

\[
M \xrightarrow{\Phi_0} G(n, m - n)
\]

for the tangent bundle to a compact Riemannian manifold \( M \), (where \( G(n, m - n) \) is the Grassmannian of \( n \)-planes in \( \mathbb{R}^m \) and \( V(n, m - n) \) the corresponding Stiefel manifold) induces not only a metric connection on \( TM \) as the pull back of Narasimhan and Ramanan’s universal connection \( \varpi_U \), but also a connection on \( \Pi : D^s \to M \). The latter induces a connection on each natural bundle over \( M \) to form a consistent family; that induced on the tangent bundle is the adjoint of \( \Phi^*(\varpi_U) \).

The above also holds with smooth stochastic flows replacing classifying bundle homomorphisms, and the resulting map from stochastic flows to connections on \( \pi : D^s \to M \) is injective.

**Proof.** It is only necessary to observe that \( \Phi \) determines and is determined by a surjective vector bundle map \( X : M \times \mathbb{R}^m \to TM \) (e.g. see [27], Appendix 1). This in turn determines a Hilbert space \( \mathcal{H} \) of sections of \( TM \) as in Remark 8.2.2 so we can apply Proposition 8.1.1 and 8.1.3. \( \square \)
Some of the conclusions of Theorem 8.3.1 are explored further in [30].

**Remark 8.3.2** This injectivity result in Proposition 8.3.1 implies that all properties of the flow can, at least theoretically, be obtainable from the induced connection on $D^s$.

**Flows on Non-compact Manifolds**

In general if $M$ is not compact we will not be able to use the Hilbert manifolds $D^s$, or other Banach manifolds without growth conditions on the coefficients of our flow. One possibility could be use the space $\text{Diff} M$ of all smooth diffeomorphisms using the Frölicher-Kriegl differential calculus as in Michor [51]. In order to do any stochastic calculus we would have to localize and use Hilbert manifolds (or possibly rough path theory). The geometric structures would nevertheless be on $\text{Diff} M$. This was essentially what was happening in the compact case. However it is useful to include partial flows of stochastic differential equations which are not strongly complete, see Kunita [43] or Elworthy[21]. For the partial solution flow $\{\xi_t : t < \tau\}$ of an SDE as in Remark 8.2.2 we obtain the decomposition in Theorem 8.2.5 but now only for $\xi_t(x)$ defined for $t < \tau(x, -)$. This can be proved from the compact versions by localization as in Carverhill-Elworthy [13] or Elworthy [21].
Chapter 9

Appendices

9.1 Girsanov-Maruyama-Cameron-Martin Theorem

To apply the Girsanov-Maruyama theorem it is often thought necessary to verify some condition such as Novikov’s condition to ensure that the exponential (local) martingale arising as Radon-Nikodym derivative is a true martingale. In fact for conservative diffusions this is automatic, and we give a proof of this fact here since it is not widely appreciated. The proof is along the lines of that given for elliptic diffusions in [21] but with the uniqueness of the martingale problem replacing the uniqueness of minimal semi-groups used in [21]. See also [[45]]. On the way we relate the expectation of the exponential local martingale to the probability of explosion of the trajectories of the associated diffusion process: a special case of this appeared in [50]. Let \( B \) be a conservative diffusion operator on a smooth manifold \( N \). For fixed \( T > 0 \) and \( y_0 \in N \) let \( \mathbf{P}_{y_0} \) denote the solution to the martingale problem for \( B \) on \( C_{y_0}([0, T]; N^+) \). Using the notation of chapter 4, let \( b \) be a vector field on \( N \) for which there is a \( T^* N \)-valued process \( \alpha \) in \( L^2_{B,loc} \) such that

\[
2 \sigma^B(\alpha_t) = b(y_t) \quad 0 \leq t \leq T
\]

for \( \mathbf{P}_{y_0} \) almost all \( y \in C_{y_0}([0, T]; N^+) \). Set

\[
Z_t = \exp\{M^\alpha_t - \frac{1}{2} \langle M^\alpha \rangle_t\} \quad 0 \leq t \leq T.
\]

This exists by the non-explosion of the diffusion process generated by \( B \), and is a local martingale with \( \mathbf{E}Z_t \leq 1 \).

For bounded measurable \( f : N \to \mathbb{R} \) define \( Q_tf(y_0) = \mathbf{E}_{y_0}^B[Z_t f(y_t)] \) for \( y_0 \in N \). Since the pair \((y, Z)\) is Markovian this determines a semi-group on the
space of bounded measurable functions with corresponding probability measures \( \{ Q_{y_0} \}_{y_0 \in N} \).

**Proposition 9.1.1** The family \( \{ Q_{y_0} \}_{y_0 \in N} \) is a solution to the martingale problem for the operator \( B + b \).

**Proof.** Let \( f : N \to \mathbb{R} \) be \( C^\infty \) with compact support. We must show, for arbitrary \( y_0 \in N \), that

\[
    f(y_t) - f(y_0) - \int_0^t (B + b) f(y_s) \, ds \quad 0 \leq t \leq T
\]

is a local martingale under \( Q_{y_0} \). For this first note that \( Z \) satisfies the usual stochastic equation which in our notation becomes:

\[
    Z_t = 1 + M_t^{Z_{\alpha}}, \quad 0 \leq t \leq T
\]

Now use Ito’s formula and the definition of \( M^{\alpha} \) to see that

\[
    f(y_t) Z_t = f(y_0) + M_t^{Z(df)} + M_t^{Z\alpha} + \int_0^t B f(y_s) Z_s ds + \langle M^{df}, M^{Z\alpha} \rangle_t. \tag{9.1}
\]

Now

\[
    \langle M^{df}, M^{Z\alpha} \rangle_t = 2 \int_0^t df \left( \sigma_y^{B} (Z_s \alpha_s) \right) ds \tag{9.2}
\]

Thus

\[
    f(y_t) Z_t - f(y_0) - \int_0^t B f(y_s) Z_s ds - \int_0^t df (Z_s b(y_s)) ds, \quad 0 \leq t \leq T,
\]

is a local martingale under \( P_{y_0}^{B} \) and so there is a sequence \( \{ \tau_n \} \) of stopping times, increasing to \( T \), such that if \( \phi : C_{y_0}([0, T]; N^+) \to \mathbb{R} \) is \( \mathcal{F}_{y_0}^{B} \)-measurable and bounded then, using the definition of \( Q \) and Fubini’s theorem, if \( 0 \leq r \leq t \leq T \),

\[
    E_{y_0}^Q \left[ \left( f(y_t \land \tau_n) - \int_0^{t \land \tau_n} (B + b)(f)(y_s) ds \right) \phi \right] \\
    = E_{y_0}^B \left[ \left( f(y_{r \land \tau_n}) Z_{t \land \tau_n} - \int_0^{t \land \tau_n} (B + b)(f)(y_s) Z_s ds \right) \phi \right] \\
    = E_{y_0}^B \left[ \left( f(y_{r \land \tau_n}) Z_{r \land \tau_n} - \int_0^{r \land \tau_n} (B + b)(f)(y_s) Z_s ds \right) \phi \right].
\]
9.2. **STOCHASTIC DIFFERENTIAL EQUATIONS FOR DEGENERATE DIFFUSIONS**

giving the required martingale property.

Since \( Q_t(1) = E Z_t \) we immediately obtain the following corollary and a theorem:

**Corollary 9.1.2**. Suppose further that uniqueness of the martingale problem holds for \( \mathcal{B} + b \), e.g. suppose \( b \) is locally Lipschitz \([39]\). Then

\[
E^B_{y_0} Z_t
\]

is the probability that the diffusion process from \( y_0 \) generated by \( \mathcal{B} + b \) has not exploded by time \( t \).

**Theorem 9.1.3** Suppose the diffusion operator \( \mathcal{B} \) and its perturbation \( \mathcal{B} + b \) by a locally Lipschitz vector field \( b \) on \( N \) are both conservative. Assume that \( \mathcal{B} + b \) is cohesive or more generally that there is a locally bounded, measurable one-form \( b^\# \) on \( N \) such that

\[
2\sigma^B_y(b^\#_y) = b(y), \quad y \in N.
\]

Then

\[
\exp \left( M_{t}^{b^\#} - \frac{1}{2} \langle M_{t}^{b^\#} \rangle_t \right), \quad 0 \leq t \leq T
\]

is a martingale under \( P^B \) and for each \( y_0 \in N \) the measures \( P^B_{y_0} \) and \( P^{B+b}_{y_0} \) on \( C_{y_0}([0, T]; N) \) are equivalent with

\[
\frac{dP^{B+b}_{y_0}}{dP^B_{y_0}} = \exp \left( M_T^{b^\#} - \frac{1}{2} \langle M_T^{b^\#} \rangle_T \right).
\]

### 9.2 Stochastic differential equations for degenerate diffusions

Let \( \mathcal{B} \) be a (smooth) diffusion diffusion operator on \( N \). If its symbol \( \sigma^B : T^*N \to TN \) does not have constant rank there may be no smooth, or even \( C^2 \), factorisation

\[
T^*N \xrightarrow{X} \mathbb{R}^m \xrightarrow{X} TN
\]

of \( \sigma^B_x \) into \( X_x X^*(x) \) for \( X : N \times \mathbb{R}^m \to TN \), as usual, for any finite dimensional \( m \). \([\]\). A factorisation with \( X : N \times H \to TN \), for \( H \) a separable Hilbert space, can be found following Stroock and Varadhan, Appendix in \([68]\). , with the property that \( X \) is continuous and each vector field \( X^j \) is \( C^\infty \), where
\[ X^j(x) = X(x)(e^j) \] for an orthonormal basis \((e_j)_{j=1}^\infty\) of \(H\). However it seems unclear if such an \(X\) can be found with each \(x \mapsto X(x)e, e \in H\), smooth. The following is well known:

**Theorem 9.2.1** Let \(\sigma : \mathbb{R}^d \to \mathcal{L}_+(\mathbb{R}^m; \mathbb{R}^m)\) be a \(C^2\) map into the symmetric positive semi-definite \((m \times m)\)-matrices then \(\sqrt{\sigma} : \mathbb{R}^d \to \mathcal{L}_+(\mathbb{R}^m; \mathbb{R}^m)\) is locally Lipschitz.

For a proof see Freidlin [34], page 97 in [67] or Ikeda-Watanabe [39].

**Corollary 9.2.2** For a \(C^2\) diffusion operator \(B\) on \(N\) there is a locally Lipschitz \(X : \mathbb{R}^m \to TN\) with \(\sqrt{\sigma_B} = XX^*\) for some \(m\).

**Proof.** Take a smooth inclusion \(TN \ni x \mapsto \nu^x\) as a sub-bundle (e.g. by embedding \(N\) in \(\mathbb{R}^m\)) and extend \(\sigma_B\) trivially to \(\sigma_B^x : N \to \mathcal{L}_((\mathbb{R}^m)^*; \mathbb{R}^m)\) by

\[
\begin{align*}
  (\mathbb{R}^m)^* \xrightarrow{i^*} T^*_xN & \quad \sigma_B^x \xrightarrow{\pi} T_xN \quad \xrightarrow{\pi}\mathbb{R}^m
\end{align*}
\]

identifying \((\mathbb{R}^m)^*\) with \(\mathbb{R}^m\) and take the square root.

Let \(\nabla\) be a connection on a sub-bundle \(G\) of \(TN\) and let \(X : \mathbb{R}^m \to G\) be a locally Lipschitz bundle map. Let \(A\) be a locally Lipschitz vector field on \(N\). As in Elworthy [21] (p184) we can form the Itô stochastic differential equation on \(N\)

\[
(\nabla) \quad dx_t = X(x_t)dB_t + A(x_t)dt
\]

where \((B_t)\) is a Brownian motion on \(\mathbb{R}^m\). For given \(x_0 \in N\) there will be a unique maximal solution \(\{x_t : 0 \leq t < \zeta^{x_0}\}\) as usual, where by a solution we mean a sample continuous adapted process such that for all \(C^2\) functions \(f : N \to \mathbb{R}\)

\[
f(x_t) = f(x_0) + \int_0^t (df)_{x_s}X(x_s)dB_s + \int_0^t (df)_{x_s}A(x_s)ds
\]

\[
= \int_0^t \sum_{j=1}^m \nabla X^j(x_s)(df|_G)X^j(x_s)ds.
\]

Indeed in a local coordinate \((U, \phi)\) system the equation is represented by

\[
dx^\phi_t = X^\phi_t dB_t + \frac{1}{2} \sum_{j=1}^m \Gamma^\phi_t(x^\phi_t) \left( X^j_t(x^\phi_t) \right) dt + A^\phi_t(x^\phi_t)dt,
\]
where $X_\phi$, $X_i^\phi$, and $A_\phi$ are the local representations of $X$, $X^i$ and $A$, and $\Gamma_\phi$ is the Christoffel symbol.

Note that the generator of the solution process has symbol $\sigma_x = X(x)X(x)^*$, $x \in \mathcal{N}$, and so a Lipschitz factorisation of $\sigma^B$ together with a suitable choice of $A$ will give a diffusion process with generator $\mathcal{B}$.

If in addition we have another generator $G$ on $\mathcal{N}$ given in Hörmander form

$$G = \sum_{k=1}^{p} L_{Y_k}L_{Y_k} + L_{Y_0}$$

for $Y^0, Y^1, \ldots, Y^k$ vector fields of class $C^2$ we can consider an SDE of mixed type

$$(\nabla) \quad dx_t = \sum_{k=1}^{p} Y^k(x_t) \circ d\tilde{B}^k_t + X(x_t)dB_t + (Y^0(x_t) + A(x_t))dt$$

for $\tilde{B}^1, \ldots, \tilde{B}^k$ independent Brownian motions on $\mathbb{R}$ independent of $(B_t)$. For a $C^2$ map $f : \mathcal{N} \to \mathbb{R}$, a solution $\{x_t : 0 \leq t < \zeta\}$ will satisfy

$$f(x_t) = f(x_0) + \int_0^t (df)_{x_s} X(x_s)dB_s + \int_0^t \sum_{k=1}^{n} (df)_{x_s} (X^k(x_s))d\tilde{B}^k_s$$

$$= \int_0^t (B + G)f(x_s)ds, \quad t < \zeta$$

giving the unique solution to the martingale problem for $B + G$. These SDE’s fit into the general frame work of the ‘Itô bundle’ approach of Belopol'skaya-Dalecky [6], see the Appendix of Brzeziak-Elworthy[11]; also see Emery [31](section 6.33, page 85) for a more semi-martingale oriented approach.

9.3 Semi-martingales & $\Gamma$-martingales along a Sub-bundle

Several of the concepts we have defined for diffusions also have versions for semi-martingales, and these are relevant to the discussion of non-Markovian observations in Chapter 5. Only continuous semi-martingales will be considered. Let $S$ denote a sub-bundle of the tangent bundle $TM$ to a smooth manifold $M$. 

**Definition 9.3.1** A semi-martingale $y_s, 0 \leq s < \tau$ is said to be along $S$ if whenever $\phi$ is a $C^2$ one-form on $M$ which annihilates $S$ we have vanishing of the Stratonovich integral of $\phi$ along $y$: \[
abla \int_0^t \phi_s \circ dy_s = 0 \quad 0 < t < \tau.\]

For simplicity take $y_0$ to be a point of $M$.

**Proposition 9.3.2** The following are equivalent:

1. the semi-martingale $y_s$ is along $S$;
2. if $\alpha_s: 0 \leq s < \tau$ is a semi-martingale with values in the annihilator of $S$ in $T^*M$, lying over $y_s$, then \[
abla \int_0^t \alpha_s \circ dy_s = 0 \quad 0 < t < \tau;\]
3. for some, and hence any, connection $\Gamma$ on $S$ the process $y_s$ is the stochastic development of a semi-martingale $y^\Gamma_s, 0 \leq s < \tau$ on the fibre $S_{y_0}$ of $S$ above $y_0$.

If $L$ is a diffusion operator then the associated diffusion processes are all along $S$ if and only if $L$ is along $S$ in the sense of Section 1.3.

**Proof.** Let $\parallel.$ denote the parallel translation along the paths of $y_s$ using $\Gamma$. If (3) holds then 
\[
dy_s = \parallel. \circ dy^\Gamma_s\]
and it is immediate that (2) is true. Also (2) trivially implies (1).

Now suppose that (1) holds. Let $\Gamma$ be a connection on $E$ and $\Gamma^0$ some extension of it to a connection on $TM$, so that the corresponding parallel translation $\parallel^0$ will preserve $S$ and some complementary sub-bundle of $TM$. Let $y^\Gamma_0$ be the stochastic anti-development of $y_s$ using this connection. To show (3) holds it suffices to show that $y^{\Gamma_0}$ takes values in $S_{y_0}$. For this choose a smooth vector bundle map $\Phi : TM \to M \times \mathbb{R}^m$ whose kernel is precisely $S$ and let $\phi : TM \to \mathbb{R}^m$ denote its principal part and $\phi^j, j = 1, \ldots, m$ the components of $\phi$. These are one-forms which annihilate $S$. Then, for each $j$
\[
0 = \int_0^t \phi_s \circ dy_s = \int_0^t \phi_s \parallel^0_s \circ dy^\Gamma_s = 0 < t < \tau.
\]
9.3. **SEMI-MARTINGALES & \( \Gamma \)-MARTINGALES ALONG A SUB-BUNDLE**

By the lemma below we see that \( y_s^{\Gamma_0} \in S_{y_0} \) for each \( s \), almost surely, and the result follows.

Finally suppose that \( y_0 \) is a diffusion process with generator \( \mathcal{L} \). By lemma 4.1.2 we have

\[
M^\alpha_t = \int_0^t \alpha_{y_s} \circ dy_s - \int_0^t (\delta^\mathcal{L} \alpha)(y_s) ds, \quad 0 \leq t < \zeta. \tag{9.4}
\]

for any \( C^2 \) one form \( \alpha \). Suppose \( \alpha \) annihilates \( S \). Then if \( y_0 \) is along \( S \) both the martingale and finite variation parts of \( \int_0^t \alpha_{y_s} \circ dy_s \) vanish and so \( (\delta^\mathcal{L} \alpha)(y_s) = 0 \) almost surely for almost all \( 0 \leq s < \tau \). If this is true for all starting points we see \( \mathcal{L} \) is along \( S \). On the other hand if \( \mathcal{L} \) is along \( S \) and \( \alpha \) annihilates \( S \) we see that \( M^\alpha \) vanishes by its characterisation in Proposition 4.1.1, since \( \sigma^\mathcal{L} \) takes values in \( S \). Thus both the martingale and finite variation parts of \( \int_0^t \alpha_{y_s} \circ dy_s \) vanish, and so the integral itself vanishes and the diffusion processes are along \( S \).

**Lemma 9.3.3** Suppose \( z_0 \) and \( \Lambda_0 \) are semi-martingales with values in a finite dimensional vector space \( V \) and the space of linear maps \( \mathcal{L}(V;W) \) of \( V \) into a finite dimensional vector space \( W \), respectively. Let \( V_0 \) denote the kernel of \( \Lambda_s \) which is assumed non-random and independent of \( s \geq 0 \). Assume

\[
\int_0^\cdot \Lambda_s \circ dz_s = 0.
\]

Then \( z_0 \) lies in \( V_0 \) almost surely.

**Proof.** We can quotient out by \( V_0 \) to assume that \( V_0 = 0 \), so we need to show that \( z_0 \) vanishes. Giving \( W \) an inner product, let \( P_s : W \to \Lambda_s[V] \) be the orthogonal projection. Compose this with the inverse of \( \Lambda_s \) considered as taking values in \( \Lambda_s[V] \), to obtain an \( \mathcal{L}(W;V) \)-valued semi-martingale \( \tilde{\Lambda} \) formed by left inverses of \( \Lambda_s \). By the composition law for Stratonovich integrals

\[
z_t = \int_0^t dz_s = \int_0^t \tilde{\Lambda}_s \Lambda_s \circ dz_s = \int_0^t \tilde{\Lambda}_s \circ d\left( \int_0^s \Lambda_r \circ dz_r \right) = 0 \tag{9.5}
\]

as required.

Let \( \Gamma \) be a connection on \( S \). Note that by the previous proposition any semi-martingale \( y_0 \) which is along \( S \) has a well defined anti-development \( y^\Gamma_0 \), say , which is a semi-martingale in \( S_{y_0} \).
Definition 9.3.4 An $M$-valued semi-martingale is said to be a $\Gamma$-martingale if its anti-development using $\Gamma$ is a local martingale.

Also we can make the following definition of an Itô integral of a differential form, using the analogue of a characterisation by Darling, [16], for the case $S = TM$;

Definition 9.3.5 If $\alpha$ is a predictable process with values in $T^*M$, lying over our semi-martingale $y,$ define its Itô integral, $(\Gamma) \int_0^t \alpha_s dy_s$ along the paths of $y,$ with respect to $\Gamma$ by

\[
(\Gamma) \int_0^t \alpha_s dy_s = \int_0^t \alpha_s // dy^\Gamma.
\]

whenever the (standard) Itô integral on the right hand side exists.

As usual this Itô integral is a local martingale for all suitable integrands $\alpha$, if and only if the process $y$ is a $\Gamma$-martingale.
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