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Special Itô maps and an L^2 Hodge theory for one forms on path spaces

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1. Introduction

Let M be a smooth compact Riemannian manifold. For a point x_0 of M let $C_{x_0}M$ denote the space of continuous paths $\sigma : [0,T] \to M$ with $\sigma(0) = x_0$, for some fixed $T > 0$. Then $C_{x_0}M$ has a natural C^∞ Banach manifold structure, as observed by J. Eells, with tangent spaces $T_\sigma C_{x_0}M$ which can be identified with the spaces of continuous maps $v\colon [0,T] \to TM$ over σ such that $v(0) = 0$, each $\sigma \in C_{x_0}M$. The Riemannian structure of M induces a Finsler norm $\| \|_\sigma^\infty$ on each $T_\sigma C_{x_0}M$ with

$$\|v\|_\sigma^\infty = \sup\{|v(t)|_{\sigma(t)} : 0 \le t \le T\}$$

so that $T_\sigma C_{x_0}M$, $\| - \|_\sigma^\infty$ is a Banach space. We can then form the dual spaces $T_\sigma^* C_{x_0}M = \mathbb{L}\,(T_\sigma C_{x_0}M; \mathbb{R})$ to obtain the cotangent bundle $T^*C_{x_0}M$ whose sections are 1-forms on $C_{x_0}M$. To obtain q-vectors, $0 \le q < \infty$ take the exterior product $\wedge^q T_\sigma C_{x_0}M$ completed by the greatest cross norm [**Mic78**] so that the space of continuous linear maps $\mathbb{L}\,(\wedge^q T_\sigma C_{x_0}M; \mathbb{R})$ is naturally isomorphic to the space of alternating q-linear maps

$$\alpha : \qquad T_\sigma C_{x_0}M \times \cdots \times T_\sigma C_{x_0}M \longrightarrow \mathbb{R}$$

(and also to the corresponding completion $\wedge^q T_\sigma^* C_{x_0}M$). Let Ω^q be the space $\Gamma \wedge^q T^* C_{x_0}M$ of sections of the corresponding bundle. These are

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the q-forms. If $C^r\Omega^q$ refers to the C^r q-forms, $0 \le r \le \infty$, then exterior differentiation gives a map

$$d: \qquad C^r\Omega^q \longrightarrow C^{r-1}\Omega^{q+1}, \qquad\qquad r \ge 1.$$

This is given by the formula: if V^j, $j = 1$ to $q+1$, are C^1 vector fields, then for $\phi \in C^1\Omega^q$

(1)
$$\begin{aligned}
&d\phi\,(V^1 \wedge \cdots \wedge V^{q+1})\\
&= \textstyle\sum_{i=1}^{q+1}(-1)^{i+1} L_{V^i}\left[\phi\left(V^1 \wedge \cdots \wedge \widehat{V^i} \wedge \cdots \wedge V^{q+1}\right)\right]\\
&+ \sum_{1\le i<j\le q+1} (-1)^{i+j}\phi\left([V^i, V^j] \wedge V^1 \wedge \ldots \widehat{V^i} \wedge \ldots \widehat{V^j} \cdots \wedge V^{q+1}\right)
\end{aligned}$$

where $[V^i, V^j]$ is the Lie bracket and $\widehat{V^j}$ means omission of the vector field V^j , e.g. see [**Lan62**].

For each $r \ge 1$ there are the deRham cohomology groups $H^q_{deRham(r)}(C_{x_0}M)$. If we were using spaces of Hölder continuous paths, as in [**BFT69**] we would have smooth partitions of unity and the deRham groups would be equal to the singular cohomology groups and so trivial for $q \ge 0$ since based path spaces are contractible. An as yet unpublished result of C. J. Atkin carries this over to continuous paths, even though $C_{x_0}M$ does not admit smooth partitions of unity. In any case since our primary interest is in the differential analysis associated with the Brownian motion measure μ_{x_0} on $C_{x_0}M$, which could equally well be considered on Hölder paths of any exponent smaller than a half, we could use Hölder rather than continuous paths and it is really only for notational convenience that we do not: the resulting manifold would admit C_∞ partitions of unity [**FT72**]. Independently of the existence of partitions of unity: contractibility need not imply triviality of the deRham cohomology group when some restriction is put on the spaces of forms. For example if $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x$ then df determines a non-trivial class in the first bounded deRham group of $\mathbb{R}$. In finite dimensions the L^2 cohomology of a cover $\tilde{M}$ of a compact manifold M gives important topological invariants of M even when $\tilde{M}$ is contractible, eg see [**Ati76**]; note also [**BP98**].

In finite dimensions the L^2 theory has especial significance because of its relationship with Hodge theory and the associated geometric analysis. In infinite dimensions L. Gross set the goal of obtaining an analogous Hodge theory at the time of his pioneering work on infinite dimensional potential theory [**Gro67**] in the late 60's. In his work he demonstrated the importance of the Cameron-Martin space H in potential analysis on Wiener space. In particular he showed that H-differentiability was the natural concept in such analysis; a fact which became even more apparent later, especially with the advent of Malliavin calculus. For related analysis on infinite dimensional manifolds such as $C_{x_0}M$ the 'admissible directions' for differentiability have to be subspaces of the tangent spaces and the 'Bismut tangent spaces', subspaces H^1_σ of $T_\sigma C_{x_0}M$, revealed their importance in the work of Jones-Léandre [**JL91**], and later in the integration by parts theory of Driver [**Dri92**] and subsequent surge of activity. They are defined by the parallel translation $//_t(\sigma) : T_{x_0}M \to T_{\sigma(t)}M$ of the Levi-Civita connection and consist of those $v \in T_\sigma C_{x_0}M$ such that $v_t = //_t(\sigma)h_t$ for $h_\cdot \in L^{2,1}_0(T_{x_0}M)$. To have a satisfying L^2 theory of differential forms on $C_{x_0}M$ the obvious choice would be to consider 'H-forms' i.e. for 1-forms these would be ϕ with $\phi_\sigma \in (H^1_\sigma)^*$, $\sigma \in C_{x_0}M$, and this agrees with the natural H-derivative df for $f : C_{x_0}M \to \mathbb{R}$. For L^2 q-forms the obvious choice would be ϕ with $\phi_\sigma \in \wedge^q(H^1_\sigma)^*$, using here the Hilbert space completion for the exterior product. An L^2-deRham theory would come from the complex of spaces of L^2 sections

$$\cdots \xrightarrow{\bar{d}} L^2\Gamma \wedge^q (H^1_\sigma)^* \xrightarrow{\bar{d}} L^2\Gamma \wedge^{q+1} (H^1_\sigma)^* \xrightarrow{\bar{d}} \cdots \tag{2}$$

where $\bar{d}$ would be a closed operator obtained by closure from the usual exterior derivative (1). From this would come the deRham-Hodge-Kodaira Laplacians $\bar{d}\bar{d}^* + \bar{d}^*\bar{d}$ and an associated Hodge decomposition. However the brackets $[V^i, V^j]$ of sections of $H^1_\cdot$ are not in general sections of $H^1_\cdot$, and formula (1) for d does not make sense for ϕ_σ defined only on $\wedge^q H^1_\sigma$, each σ, e.g. see [**CM96**], [**Dri99**]. The project fails at the stage of the definition of exterior differentiation. Ways to circumvent this problem were found, for paths and loop spaces, by Léandre [**Léa96**] [**Léa97**] [**Léa98b**] who gave analytical deRham groups and showed that they agree with the singular cohomology of the spaces. See also [**Léa98a**]. However these were not L^2 cohomology theories and did not include a Hodge theory. For flat Wiener space the problem with brackets does not exist (the $H^1_\cdot$ bundle is integrable) and a full L^2 theory was carried out by Shigekawa [**Shi86**], including the proof of triviality of the groups, see also [**Mit91**]. For Wiener manifolds see [**Pie82**]. For paths on a compact Lie group G with bi-invariant metric, and corresponding loop groups, there is an alternative, natural, H-differentiability structure with the H_σ modified by using the flat left and right connection instead of the Levi-Civita connection usually used. With this Fang&Franchi [**FF97a**], [**FF97b**], carried through a construction of the complex (2) and obtained a Hodge decomposition for L^2 forms. For a recent, and general, survey see [**Léa99**].

Our proposal is to replace the Hilbert spaces $\wedge^q H^1_\sigma$ in (1) by other Hilbert spaces $\mathcal{H}^q_\sigma, q = 2, 3, \ldots$, continuously included in $\wedge^q T_\sigma C_{x_0} M$, though keeping the exterior derivative a closure of the one defined by (1). Here we describe the situation for $q = 2$ which enables us to construct an analogue of the de Rham-Hodge-Kodaira Laplacian on L^2 sections of the Bismut tangent spaces and a Hodge decomposition of the space of L^2 1-forms. At the time of writing this the situation for higher order forms is not so clear and the discussion of them, and some details of the construction here, are left to a more comprehensive article.

The success of Fang and Franchi for path and loop groups was due to a large extent to the fact that the Itô map (i.e. solution map) of the right or left invariant stochastic differential equations for Brownian motion on their groups is particularly well behaved, in particular its structure sends the Cameron-Martin space to the Bismut type tangent space. The basis for the analysis here is the fact that the Itô map of gradient systems is almost as good: the 'almost' being made into precision by 'filtering out redundant noise' [**EY93**] [**ELL96**] [**ELL99**]. Indeed this is used for $q \geq 2$ to define the spaces $\mathcal{H}^q_\sigma$, though it turns out that they depend only on the Riemannian structure of M, not on the embedding used to obtain the gradient stochastic differential equation. The good properties of the Itô maps have been used for analysis on path and loop spaces, particularly by Aida, see [**AE95**] [**Aid96**]: Theorem 2.2 consolidates these and should be of independent interest. The result that the Itô map can be used to continuously pull back elements of $L^2\Gamma(\mathcal{H}^{1^*}_\cdot)$, i.e. L^2 H-forms to L^2 H-forms on Wiener space seems rather surprising.

We should also mention the work done on 'submanifolds' of Wiener space and in particular on the submanifolds which give a model for the based loop space of a Riemannian manifold. For this see [**AVB90**] [**Kus91**] [**Kus92**] [**VB93**]. For a detailed analysis of some analogous properties of the stochastic development Ito map see [**Li99**].

As usual in this subject all formulae have to be taken with the convention that equality only holds for all paths outside some set of measure zero.

2. The Itô map for gradient Brownian dynamical systems

A. Let $j : M \to \mathbb{R}^m$ be an isometric embedding. The existence of such a j is guaranteed by Nash's theorem. Let $X : M \times \mathbb{R}^m \to TM$ be the induced gradient system, so $X(x) : \mathbb{R}^m \to T_xM$ is the orthogonal projection, or equivalently $X(x)(e)$ is the gradient of $x \mapsto \langle j(x), e\rangle_{\mathbb{R}^m}$, for $e \in \mathbb{R}^m$. Take the canonical Brownian motion $B_t(\omega) = \omega(t)$, $0 \le t \le T$, for $\omega \in \Omega = C_0(\mathbb{R}^m)$ with Wiener measure $\mathbb{P}$. The solutions to the stochastic differential equation

$$dx_t = X(x_t) \circ dB_t \tag{3}$$

on M are well known to be Brownian motions on M, [**Elw82**] [**Elw88**] [**RW87**]. Let $\xi_t(x, \omega)$: $0 \le t \le T, x \in M, \omega \in \Omega$ denote its solution flow and

$$\mathcal{I} : C_0(\mathbb{R}^m) \to C_{x_0}(M),$$

$\mathcal{I}(\omega)_t = \xi_t(x_0, \omega)$ its Itô map. Then $\mathcal{I}$ maps $\mathbb{P}$ to the Brownian measure μ_{x_0} on $C_{x_0}M$. The flow is C^∞ in x with random derivative $T_{x_0}\xi_t : T_{x_0}M \to T_{x_t}M$ at x_0. The Itô map is smooth in the sense of Malliavin, as are all such Itô maps, e.g. see [**IW89**] and [**Mal97**], with H-derivatives continuous linear maps

$$T_\omega \mathcal{I} : H \to T_{x.(\omega)} C_{x_0} M, \qquad \text{almost all } \omega \in \Omega.$$

Here $x.(\omega) := \xi.(x_0, \omega)$ and H is the Cameron-Martin space $L_0^{2,1}(\mathbb{R}^m)$ of $C_0(\mathbb{R}^m)$. From Bismut [**Bis81a**] there is the formula

$$T\mathcal{I}(h)_t = T_{x_0}\xi_t \int_0^t (T_{x_0}\xi_s)^{-1} X(x_s)(\dot{h}_s) ds \tag{4}$$

for $h \in H$.

Remark: All the following results remain true when the gradient SDE (3) is replaced by an SDE of the form (3) with smooth coefficients whose LeJan-Watanabe connection, in the sense of [**ELL99**], is the Levi Civita connection of our Riemannian manifold (and consequently whose solutions are Brownian motions on M). The only exception is the reference to the shape operator in §3B below. The canonical stochastic differential equation for compact Riemannian symmetric spaces gives a class of such stochastic differential equations, see [**ELL99**] section 1.4, with compact Lie groups giving specific examples.

B. Formula (4) is derived from the covariant stochastic differential equation along $x.$ for $v_t = T\mathcal{I}(h)_t$ using the Levi-Civita connection

$$Dv_t = \nabla X_{v_t}(\circ dB_t) + X(x_t)(\dot{h}_t)dt, \tag{5}$$

where $\frac{D}{\partial t} = //\frac{d}{dt}//^{-1}$ and D is the corresponding stochastic differential. Clearly $v.$ does not lie in the Bismut tangent spaces in general. In fact ∇X is determined by the shape operator of the embedding:

$$A : TM \times \nu M \to TM$$

where νM is the normal bundle of M in $\mathbb{R}^m$. We have

$$\nabla_v X(e) = A(v, K_x e), \qquad v \in T_x M \tag{6}$$

where $K_x : \mathbb{R}^m \to \nu_x M$ is the orthogonal projection, $x \in M$. We can think of $\nu_x M$ as $Ker(x)$. For $v.$ to be a Bismut tangent vector for all h would require $A \equiv 0$ so that M would be isometric to an open set of $\mathbb{R}^n$.

Equation (5) has the Itô form

$$Dv_t = \nabla X_{v_t}(dB_t) - \frac{1}{2} Ric^{\#} v_t dt + X(x_t)(\dot{h}_t)dt \tag{7}$$

where $Ric^{\#} : TM \to TM$ corresponds to the Ricci tensor. From (6) this is driven only by the 'redundant noise' in the kernel of $X(x_t)$. The technique of [**EY93**] shows that if $\mathcal{F}^{x_0}$ denotes the σ-algebra generated by $x_s : 0 \le s \le T$ and

$$\bar{v}_t := \overline{T\mathcal{I}(h)}_t \;:=\; \mathbb{E}\{v_t \,|\mathcal{F}^{x_0}\} \tag{8}$$

then

$$\frac{D}{\partial t}\bar{v}_t = -\frac{1}{2} Ric^{\#}(\bar{v}_t) + X(x_t)(\dot{h}_t). \tag{9}$$

This is described in greater generality in [**ELL99**]. We will rewrite (9) as

$$\frac{\mathbb{D}}{\partial t}\bar{v}_t = X(x_t)(\dot{h}_t) \tag{10}$$

where

$$\frac{\mathbb{D}}{\partial t} := \frac{D}{\partial t} + \frac{1}{2} Ric^{\#}(V_t). \tag{11}$$

If $W_t : T_{x_0}M \to T_{x_t}M$ is the Dohrn-Guerra, or 'damped' parallel transport defined by

$$\frac{\mathbb{D}}{\partial t}(W_t(v_0)) = 0$$

then $\frac{\mathbb{D}}{\partial t} = W_t \frac{d}{dt} W_t^{-1}$. It appears to be of basic importance e.g. see [**Mey82**], [**Nel85**], [**Léa93**], [**CF95**], [**Nor95**], [**Mal97**], [**Fan98**], [**ELL99**].

To take this into account we will change the interior product of the Bismut tangent spaces H^1_σ and take

$$< u^1, u^2 >_\sigma := \int_0^T \left\langle \frac{\mathbb{D}}{\partial s} u^1_s,\ \frac{\mathbb{D}}{\partial s} u^2_s \right\rangle ds. \tag{12}$$

Let $\mathcal{H}^1_\sigma$ denote H^1_σ with this inner product: it consists of those tangent vectors $v.$ above $\sigma.$ such that $\int_0^T \left| \frac{\mathbb{D}}{\partial t} v_t \right|^2_{\sigma(t)} dt < \infty$. Since $X(x_t)$ is surjective we see that $h \mapsto \overline{T_\omega \mathcal{I}}(h.)$ maps H onto $\mathcal{H}^1_{x.(\omega)}$ for each ω, and

$$\left|\left|\overline{T\mathcal{I}(h)}\right|\right|_{x.(\omega)} = \sqrt{\int_0^T \left| X(x_t(\omega))\dot{h}_t \right|^2 dt}. \tag{13}$$

Let $\overline{T\mathcal{I}}()_\sigma : H \to \mathcal{H}^1_\sigma$ denote the map

$$h \mapsto \mathbb{E}\left\{ T\mathcal{I}.(h.) \,| x.(\omega) = \sigma \right\} \tag{14}$$

defined for μ_{x_0} almost all $\sigma \in C_{x_0}M$.

When $h : C_0(\mathbb{R}^m) \to H$ gives an adapted process we see, e.g. from [**ELL99**] that

$$\overline{TI(h)}_\sigma \ := \ \mathbb{E}\left\{ T\mathcal{I}.(h.) \,| x. = \sigma \right\} = \overline{T\mathcal{I}}(\bar{h}.(\sigma))_\sigma$$

where $\bar{h}(\sigma)_t = \mathbb{E}\left\{ h_t \,| x.(\omega) = \sigma \right\}$. For non-adapted $h : C_0(\mathbb{R}^m) \to H$ see Theorem 2.2 below.

C. Let ϕ be a C^1 1-form on $C_{x_0}M$ which is bounded together with $d\phi$, using the Finsler structure defined above. For example ϕ could be cylindrical and C^∞. Then there is the pull back $\mathcal{I}^*(\phi) : C_0(\mathbb{R}^m) \to H^*$, the H-form on $C_0(\mathbb{R}^m)$ given by

$$\mathcal{I}^*(\phi)_\omega(h) = \phi\left(T_\omega \mathcal{I}(h) \right), \qquad h \in H. \tag{15}$$

Also $\mathcal{I}^*(\phi) \in D(\bar{d})$, for $\bar{d}$ the closure of the exterior derivative on H-forms on Wiener space and

$$\bar{d}\left(\mathcal{I}^*(\phi) \right) = \mathcal{I}^*\left(d\phi \right), \tag{16}$$

for $\mathcal{I}^*$ the pull back

$$\mathcal{I}^*(\psi)_\omega(h^1 \wedge h^2) = \psi\left(T_\omega \mathcal{I}(h^1) \wedge T_\omega \mathcal{I}(h^2) \right),$$

when $\psi \in \Omega^2$ and $h^1, h^2 \in H$, so $\mathcal{I}^*(\psi) : C_0(\mathbb{R}^m) \to (H \wedge H)^*$, c.f. [**Mal97**] and [**FF97b**]. This can be proved by approximation.

We will show that $\mathcal{I}^*(\phi)$ can be defined when ϕ is only an H-form on $C_{x_0}M$ although the right hand side of (15) is not, classically, defined in this case.

D. For each element h of a Hilbert space H, let $h^\#$ denote the dual element in H^*, and conversely, so $(h^\#)^\# = h$. For Hilbert spaces H, H' and measure spaces $(\Omega, \mathcal{F}, \mathbb{P})$, $(\Omega', \mathcal{F}', \mathbb{P}')$, a linear map

$$S : L^2\left(\Omega', \mathcal{F}', \mathbb{P}'; H'^*\right) \to L^2\left(\Omega, \mathcal{F}, \mathbb{P}; H^*\right)$$

will be said to be the co-joint of a linear map

$$T : L^2\left(\Omega, \mathcal{F}, \mathbb{P}; H\right) \to L^2\left(\Omega', \mathcal{F}', \mathbb{P}'; H'\right)$$

if

$$h \mapsto \left[S(h(\cdot)^\#)(\cdot)\right]^\#$$

is the usual Hilbert space adjoint T^* of T, i.e. if

$$\int S(\phi)_\omega h(\omega) d\mathbb{P}(\omega) = \int \phi_\omega \left(T(h)(\omega)\right) d\mathbb{P}'(\omega)$$

all $\phi \in L^2(\Omega'; H')$ and $h \in L^2(\Omega; H)$.

A linear map T of Hilbert spaces is a *Hilbert submersion* if TT^* is the identity map.

We first note a preliminary result

PROPOSITION 2.1. *The map* $\overline{T\mathcal{I}} : L^2\left(C_0(\mathbb{R}^m), \mathcal{F}^{x_0}, \mathbb{P}|_{\mathcal{F}^{x_0}}; H\right) \to L^2\Gamma\mathcal{H}^1_{\cdot}$ *given by* $\overline{T\mathcal{I}}(h_\cdot)(\sigma) = \overline{T\mathcal{I}}(h_\cdot(\sigma))(\sigma)$ *is*

$$h \mapsto W_\cdot \int_0^\cdot W_s^{-1} X(\sigma(s)) \dot{h}(\sigma)_s ds \tag{17}$$

and is a Hilbert submersion with inverse and adjoint given by

$$v \mapsto \int_0^\cdot Y(x_s(\cdot)) \frac{\mathbb{D}}{\partial s} v_s ds \tag{18}$$

for $Y(x) : T_xM \to \mathbb{R}^m$ *the adjoint of* $X(x)$, $x \in M$. *Its co-joint can be written* $\overline{\mathcal{I}^*(-)} : L^2\Gamma(\mathcal{H}^{1*}_{\cdot}) \to L^2\left(C_0(\mathbb{R}^m), \mathcal{F}^{x_0}; H^*\right)$ *in the sense that it agrees with* $\phi \mapsto \mathbb{E}\{\mathcal{I}^*(\phi)\,|\mathcal{F}^{x_0}\}$ *for* ϕ *a 1-form on* $C_{x_0}M$.

Proof. Note $h \mapsto X(\sigma_\cdot)(\dot{h}_\cdot)$ maps h to the space $L^2T_\sigma C_{x_0}M$ of L^2 'tangent vectors' to $C_{x_0}M$ at σ, and as such is a Hilbert projection with inverse and adjoint $u \mapsto \int_0^{\cdot} Y(\sigma_s)(u_s)ds$ since

$$\begin{aligned}\langle \int_0^{\cdot} Y(\sigma_s)u_s ds, h_\cdot \rangle_H &= \int_0^T \langle Y(\sigma_s)u_s, \dot{h}_s \rangle_{\mathbb{R}^m} ds \\ &= \int_0^T \langle u_s, X(\sigma_s)\dot{h}_s \rangle_{\sigma_s} ds \\ &= \langle u_\cdot, X(\sigma_\cdot)\dot{h}_\cdot \rangle_{L^2 T_\sigma C_{x_0} M}.\end{aligned}$$

Also $u \mapsto W_\cdot \int_0^{\cdot} W_s^{-1} u_s \, ds$ is an isometry of $L^2 T_\sigma C_{x_0} M$ with $\mathcal{H}^1_\sigma$ by definition, and its inverse is $\frac{\mathbb{D}}{\partial s}$. To check the co-joint: if $h \in L^2(C_0(\mathbb{R}^m), \mathcal{F}^{x_0}; H)$ and $\phi \in \Omega^1$ is bounded and continuous then

$$\begin{aligned}\int_{C_0(\mathbb{R}^m)} \overline{\mathcal{I}^*(\phi)}(h) \, d\mathbb{P} &= \int_{C_0(\mathbb{R}^m)} \mathcal{I}^*(\phi)(h) \, d\mathbb{P} \\ &= \int_{C_0(\mathbb{R}^m)} \phi(T\mathcal{I}(h)) \, d\mathbb{P} \\ &= \int_{C_0(\mathbb{R}^m)} \phi\left(\mathbb{E}\left\{T\mathcal{I}(h) \quad | \quad \mathcal{F}^{x_0}\right\}\right) \, d\mathbb{P} \\ &= \int_{C_{x_0}(M)} \phi\left(\overline{T\mathcal{I}}(h)\right) \, d\mu_{x_0}.\end{aligned}$$

Q.E.D.

Our basic result on the nice behaviour of our Itô map is the following:

THEOREM 2.2. *The map* $h \mapsto \mathbb{E}\{T\mathcal{I}(h) \,|\mathcal{F}^{x_0}\}$ *determines a continuous linear map*

$$\overline{T\mathcal{I}(-)} : L^2\left(C_0(\mathbb{R}^m); H\right) \to L^2\Gamma\mathcal{H}^1,$$

which is surjective. The pull back map $\mathcal{I}^*$ *on 1-forms extends to a continuous linear map of H-forms:*

$$\mathcal{I}^* : L^2\Gamma(\mathcal{H}^{1*}_\cdot) \to L^2\left(C_0\left(\mathbb{R}^m\right); H^*\right),$$

which is the co-joint of $\overline{T\mathcal{I}(-)}$*. It is injective with closed range.*

The proof of the continuity of $\overline{T\mathcal{I}(-)}$ is given in the next section (§3D). Its surjectivity follows from the previous proposition. That its co-joint agrees with $\mathcal{I}^*$ on Ω^1 comes from the last few lines of the proof of that proposition. From this we have the existence of the claimed extension of $\mathcal{I}^*$ and its continuity, injectivity and the fact that it has closed range.

E. Let $\bar{d} : Dom(\bar{d}) \subset L^2(C_0(\mathbb{R}^m);\mathbb{R}) \to L^2(C_0(\mathbb{R}^m);H^*)$ be the usual closure of the H-derivative, as in Malliavin calculus. Let $\bar{d}$: $Dom(\bar{d}) \subset L^2(C_0(M);\mathbb{R}) \to L^2\Gamma\mathcal{H}^{1^*}$ be the closure of differentiation in $\mathcal{H}^1_\cdot$ directions defined on smooth cylindrical functions (to make a concrete choice). The existence of this closure is assured and well known by Driver's integration by parts formula. We note the following consequence of Theorem 2.2, although it is not needed in the following sections. In it we also use $\mathcal{I}$ to pull back functions on $C_{x_0}M$ by $\mathcal{I}^*(f)(\omega) = f(\mathcal{I}^*(\omega)) = f(x_\cdot(\omega))$.

Corollary 2.3. *With $\mathcal{I}^*$ defined on H-forms by Theorem 2.2 the compositions $\mathcal{I}^*\bar{d}$ and $\bar{d}\mathcal{I}^*$ are closed, densely defined operators on their domains in $L^2(C_{x_0}M;\mathbb{R})$ into $L^2(C_0(\mathbb{R}^m);H^*)$ and*

$$\mathcal{I}^*\bar{d} \subset \bar{d}\mathcal{I}^*. \tag{19}$$

Proof. Let Cyl denote the space of smooth cylindrical functions on $C_{x_0}M$. If $f \in Cyl$ it is standard that $\mathcal{I}^*(f) \in Dom(\bar{d})$ so $Cyl \subset Dom(\mathcal{I}^*\bar{d}) \cap Dom(\bar{d}\mathcal{I}^*)$ since by Theorem 2.2 $Dom(\mathcal{I}^*\bar{d}) = Dom(\bar{d})$. Moreover

$$\mathcal{I}^*df = \bar{d}\mathcal{I}^*f$$

by the chain rule. Since $\mathcal{I}^*$ is continuous on functions $\bar{d}I^*$ is closed and we have

$$\overline{\mathcal{I}^*d|_{Cyl}}^c = \overline{\bar{d}\mathcal{I}^*|_{Cyl}}^c \subset \bar{d}\mathcal{I}^*$$

where ^{c} denotes closure. Indeed $\mathcal{I}^*\bar{d}$ is closed since $\mathcal{I}^*$ is continuous with closed range on H-forms by the theorem so the closure $\overline{\mathcal{I}^*d|_{Cyl}}^c$ exists and is a restriction of $\mathcal{I}^*\bar{d}$. The result follows by showing this restriction is in fact equality: For this suppose $f \in Dom(\mathcal{I}^*\bar{d})$. Then $f \in Dom(\bar{d})$ so there exists $f_n \in Cyl$ with $f_n \to f$ in L^2 and $df_n \to \bar{d}f$. By continuity of $\mathcal{I}^*$ we have $\mathcal{I}^*(df_n) \to \mathcal{I}^*(\bar{d}f)$ so that $f \in Dom(\overline{\mathcal{I}^*d|_{Cyl}}^c)$. Q.E.D.

Taking co-joints, and defining $-div$ to be the co-joint of $\bar{d}$ on $C_{x_0}M$ and $C_0(\mathbb{R}^m)$, we have the following corollary. It formalises some of the arguments in [**EL96**] [**ELL99**].

Corollary 2.4. *The composition $div\,\overline{T\mathcal{I}(-)}$ and $\mathbb{E}\{div\,|\mathcal{F}^{x_0}\}$ are closed densely defined operators on $L^2(C_0(\mathbb{R}^m);H)$ into $L^2(C_{x_0}M;\mathbb{R})$. Moreover*

$$\mathbb{E}\{\,div\ |\,\mathcal{F}^{x_0}\} \subset div\,\overline{T\mathcal{I}(-)}. \tag{20}$$

Also by a comparison result of Hörmander, see [**Yos80**],Thm2, §6 of Chapter II, p79, (19) implies there exists a constant C such that

$$\int_{C_0(\mathbb{R}^m)} |\bar{d}\mathcal{I}^*(f)|^2_{H^*} d\mathbb{P} \leq C \left(\int_{C_0(\mathbb{R}^m)} |\mathcal{I}^*(\bar{d}f)|^2_{H^*} d\mathbb{P} + \|f\|^2_{L^2} \right) \tag{21}$$

for all $f \in Dom(\mathcal{I}^*\bar{d}) = Dom(\bar{d})$ in $L^2(C_{x_0}M;\mathbb{R})$.

Let $L^{2,1}$ denote the domain of the relevant $\bar{d}$ with its graph norm

$$\|f\|_{2,1} = \sqrt{\|\bar{d}f\|^2_{L^2} + \|f\|^2_{L^2}},$$

i.e. the usual Dirichlet space. The boundedness of $\mathcal{I}^*$ in the next corollary was proved directly for cylindrical functions (and hence for all $f \in L^{2,1}$) by Aida and Elworthy as the main step in their proof of the logarithmic Sobolev inequality on path spaces.

Corollary 2.5. *c.f.* [**AE95**] *The pull back determines a continuous linear map $\mathcal{I}^* : L^{2,1}(C_{x_0}M) \to L^{2,1}(C_0(\mathbb{R}^m))$. It is injective with closed range.*

Proof. Continuity and existence is immediate from (21) and the continuity of $\mathcal{I}^*$ in Theorem 2.2. Injectivity is clear. To show the range is closed suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence in $L^{2,1}(C_{x_0}M)$ with $\mathcal{I}^*(f_n) \to g$, in $L^{2,1}(C_0(\mathbb{R}^m))$ some g.

Then g is $\mathcal{F}^{x_0}$ measurable so $g = \mathcal{I}^*(\bar{g})$ some $\bar{g} \in L^2(C_{x_0}M)$. Moreover $f_n \to \bar{g}$ in $L^2(C_{x_0}M)$. We have $\bar{d}\mathcal{I}^*(f_n) \to \bar{d}g$. By (19) $\bar{d}\mathcal{I}^*(f_n) = \mathcal{I}^*(\bar{d}f_n)$ since $f_n \in Dom(\bar{d})$. From this we see $\bar{d}f_n$ converges in $L^2\Gamma\mathcal{H}$. because $\mathcal{I}^*$ has closed range in $L^2(C_0(\mathbb{R}^m);H)$ by Theorem 2.2. This shows $\bar{g} \in Dom(\bar{d})$ as required.

Remark: We chose the basic domain of d on functions of $C_{x_0}M$ to be smooth cylindrical functions but the results above would hold equally well with other choices e.g. the space BC^1 of functions $F : C_{x_0}M \to \mathbb{R}$ which are C^1 and have dF bounded on $C_{x_0}M$ using the Finsler structure of $C_{x_0}M$. The crucial conditions needed for $Dom(d)$ are that it is dense in $L^2(C_{x_0}M)$ and that $F \in Dom(d)$ implies $\mathcal{I}^*(F) \in Dom(\bar{d})$ on Wiener space. At present it seems unknown as to whether different choices give the same closure $\bar{d}$. This is essentially equivalent to knowing that the closed subspace $\mathcal{I}^*(L^{2,1}(C_{x_0}M))$ of $L^{2,1}(C_0(\mathbb{R}^m))$ is independent of the choice of $Dom(d)$. The obvious guess would be that it is and consists of the $\mathcal{F}^{x_0}$-measurable elements of $L^{2,1}(C_0(\mathbb{R}^m))$, but we do not pursue that here.

3. Decomposition of noise; proof of Theorem

A. Let $\mathbb{R}^m$ denote the trivial bundle $M \times \mathbb{R}^m \to M$. It has the sub-bundle $kerX$ and its orthogonal complement $KerX^\perp$. The projection onto these bundles induce connections on them, [**ELL96**] and these combine to give parallel translations

$$\tilde{//}_t(\sigma) : \mathbb{R}^m \to \mathbb{R}^m, \qquad 0 \leq t \leq T$$

along almost all $\sigma \in C_{x_0}M$, which map $KerX(x_0)$ to $KerX(\sigma(t))$ and preserve the inner product of $\mathbb{R}^m$. From [**EY93**] (extended to more general stochastic differential equations in [**ELL99**]), there is a Brownian motion $\{\tilde{B}_t : 0 \leq t \leq T\}$ on $KerX(x_0)^\perp$ and one, $\{\beta_t : 0 \leq t \leq T\}$ on $KerX(x_0)$ with the property that

1. $\tilde{B}.$ and $\beta.$ are independent;
2. $\sigma(\tilde{B}_s : 0 \leq s \leq t) = \sigma(x_s : 0 \leq s \leq t)$, $0 \leq t \leq T$ and in particular $\sigma(\tilde{B}_s : 0 \leq s \leq T) = \mathcal{F}^{x_0}$;
3. $dB_t = \tilde{//}_t d\tilde{B}_t + \tilde{//}_t d\beta_t$.

Let $L^2(\mathcal{F}^{x_0}; \mathbb{R})$, $L^2(\beta; \mathbb{R})$ etc. denote the Hilbert subspaces of $L^2(C_0(\mathbb{R}^m); \mathbb{R})$ etc. consisting of elements measurable with respect to $\mathcal{F}^{x_0}$ or $\sigma\{\beta_s : 0 \leq s \leq T\}$. By (2) above $L^2(\mathcal{F}^{x_0}; \mathbb{R}) = L^2\left(\tilde{B}; \mathbb{R}\right)$. As before we can identify $L^2(\mathcal{F}^{x_0}; \mathbb{R})$ with $L^2(C_{x_0}M; \mathbb{R})$.

LEMMA 3.1. *The map $f \otimes g \mapsto f(\cdot)g(\cdot)$ determines an isometric isomorphism*

$$L^2(\mathcal{F}^{x_0}; \mathbb{R}) \hat{\otimes} L^2(\beta; \mathbb{R}) \to L^2(C_0(\mathbb{R}^m); \mathbb{R})$$

where $\hat{\otimes}$ denotes the usual Hilbert space completion.

Proof. This is immediate from the independence of $\tilde{B}.$ and β, the fact that $\tilde{B}. \times \beta : C_0(\mathbb{R}^m) \to C_0(KerX(x_0)^\perp) \times C_0(KerX(x_0))$ generates $\mathcal{F}$, and the well known tensor product decomposition of L^2 of a product space. Q.E.D.

B. Let us recall the representation theorem for Hilbert space valued Wiener functionals:

LEMMA 3.2 (L^2 representation theorem). *Let $\{\beta_t, 0 \leq t \leq T\}$ be an $m-n$ dimensional Brownian motion and H a separable Hilbert space. Let K be the Hilbert space of β-predictable $\mathbb{L}(\mathbb{R}^{m-n}; H)$ valued process with*

$$\|\alpha\|_K = \sqrt{\int_0^T \mathbb{E}\, \|\alpha_r(\cdot)\|^2_{\mathbb{L}(\mathbb{R}^{m-n}, H)}\, dr} < \infty$$

(using the Hilbert-Schmidt norm on $\mathbb{L}(\mathbb{R}^{m-n}, H)$). Then the map

$$\begin{aligned} H \times K &\to L^2(\beta; H) \\ (h, \alpha) &\mapsto \int_0^T \alpha_r(\cdot)(d\beta_r) + h \end{aligned}$$

is an isometric isomorphism.

Proof. That it preserves the norm is a basic property of the Hilbert space valued Itô integral. To see that it is surjective just observe that the image of the set of $\alpha \in K$ of the form

$$\alpha_s(\omega) = g(\omega) h_s$$

for $g \in L^2(\beta; \mathbb{R})$, $h \in H$ is total in $L^2(\beta; H)$ by the usual representation theorem for real valued functionals and the isometry of $L^2(\beta; \mathbb{R}) \hat{\otimes} H$ with $L^2(\beta; H)$. Q.E.D.

LEMMA 3.3. *The map*

$$\begin{aligned} L^2(\beta; \mathbb{R}) \hat{\otimes} H &\longrightarrow L^2\Gamma\left(\mathcal{H}^1_{\cdot}\right) \\ g \otimes h &\mapsto \overline{T\mathcal{I}(gh)} \end{aligned}$$

is continuous linear.

Remark: Note that

$$\overline{T\mathcal{I}(gh)}_\sigma = \mathbb{E}\left\{T\mathcal{I}(gh) \mid x_\cdot = \sigma\right\} = \mathbb{E}\left\{gT\mathcal{I}(h) \mid x_\cdot = \sigma\right\}.$$

Proof. By the representation theorem, a typical element u of $L^2(\beta; H)$ has the form

$$u_t = h_t + \int_0^T \alpha_r(t)\,(d\beta_r), \qquad 0 \le t \le T$$

for $h \in H = L_0^{2,1}(\mathbb{R}^m)$ and $\alpha \in K$; writing $\alpha_r(t)(e)$ for $\alpha_r(e)_t$, $e \in KerX(x_0)^\perp \approxeq \mathbb{R}^{m-n}$. Now by equation (4)

$$\begin{aligned} &\overline{T\mathcal{I}_t\left(\int_0^T \langle \alpha_r(\cdot), d\beta_r\rangle\right)} \\ = \; &\mathbb{E}\left\{\; \mathbb{E}\left\{T\mathcal{I}_t\left(\int_0^T \alpha_r(\cdot)(d\beta_r)\right) \mid \mathcal{F}^{x_0} \vee \mathcal{F}_t \mid \mathcal{F}^{x_0}\right\}\right\} \\ = \; &\mathbb{E}\left\{T\xi_t \int_0^t T\xi_s^{-1} X(x_s)\left(\int_0^t \dot{\alpha}_r(\cdot)(d\beta_r)\right)_s ds \mid \mathcal{F}^{x_0}\right\}, \end{aligned}$$

where $\dot{\alpha}_r(s)$ means the derivative with respect to s. Set

$$\begin{aligned} u_t &= T\xi_t \int_0^t (T\xi_s)^{-1} X(x_s) \left(\int_0^t \dot{\alpha}_r(\cdot)(d\beta_r) \right)_s ds \\ &= T\mathcal{I}_t \left(\int_0^t \alpha_r(\cdot) d\beta_r \right) \end{aligned}$$

Then, by equation (7), writing $\nabla_v X(e) = \nabla X(v)(e) = \nabla X^e(v)$ we see that u_t has covariant Itô differential given by

$$\begin{aligned} Du_t &= \nabla X(u_t) dB_t - \frac{1}{2} Ric^{\#}(u_t) dt + X(x_t) \left(\int_0^t \dot{\alpha}_r(\cdot)(d\beta_r) \right)_t dt \\ &\quad + T\mathcal{I}_t \left(\alpha_t(\cdot)(d\beta_t) \right) + \frac{1}{2} \sum_{i=1}^{m-n} \nabla X^{e^i} \left(T\mathcal{I}_t \left(\alpha_t(\cdot)(e^i) \right) \right) dt \end{aligned}$$

where $e^1, e^2, \dots, e^{m-n}$ is an orthonormal basis of $KerX(x_0)^\perp$.

By properties (1) and (2) of β and $\tilde{B}$ given in §3A above we can argue as in [**EY93**] [**ELL99**] to see that $\bar{u}_t := \mathbb{E}\{u_t \mid \mathcal{F}^{x_0}\}$ satisfies

$$D\bar{u}_t = -\frac{1}{2} Ric^{\#} \left(\bar{u}_t \right) dt + \frac{1}{2} \sum_{i=1}^{m-n} \nabla X^{e^i} \left(\overline{T\mathcal{I}_t \left(\alpha_t \left(\cdot \right) \left(e^i \right) \right)} \right) dt.$$

Thus

$$\begin{aligned} 4\, \mathbb{E} \left| \bar{u}_\cdot \right|^2_{\mathcal{H}^1} &= 4\, \mathbb{E} \int_0^T \left| \frac{\mathbb{D} \bar{u}_t}{\partial t} \right|^2 dt \\ &= \mathbb{E} \int_0^T \left| \sum_{i=1}^{m-n} \nabla X^{e^i} \left(\mathbb{E} \left\{ T\mathcal{I}_r \left(\alpha_r \left(\cdot \right) \left(e^i \right) \right) \mid \mathcal{F}^{x_0} \right\} \right) \right|^2 dr \\ &\le \int_0^T \mathbb{E} \left[\mathbb{E} \left\{ \sum_{i=1}^{m-n} \left| \nabla X^{e^i} \left(T\mathcal{I}_r \left(\alpha_r \left(\cdot \right) \left(e^i \right) \right) \right) \right| \mid \mathcal{F}^{x_0} \right\} \right]^2 dr \\ &\le const \int_0^T \mathbb{E} \left[\sum_{i=1}^{m-n} \mathbb{E} \left\{ \left\| \nabla X^{e^i} \left(T\mathcal{I}_r(-) \right) \right\|^2_{\mathbb{L}(H; T_{x_r} M)} \mid \mathcal{F}^{x_0} \right\} \cdot \right. \\ &\qquad \left. \mathbb{E} \left\{ \left| \alpha_r(\cdot)(e^i) \right|^2_H \mid \mathcal{F}^{x_0} \right\} \right] dr \\ &\le const \sup_{0 \le r \le T} \mathbb{E} \left(||T\mathcal{I}_r||^2_{\mathbb{L}(H; T_{x_r} M)} \right) \int_0^T \sum_1^{m-n} \mathbb{E} |\alpha_r(\cdot)(e^i)|^2_H dr \\ &= const \left\| \int_0^T \alpha_r \left(\cdot \right) d\beta_r \right\|^2_{L^2}, \end{aligned}$$

since $T\mathcal{I} : H \to TC_{x_0}M$ is well known to be in L^2, using the Finsler metric of $C_{x_0}M$, as can be seen from formula (4) and standard estimates for $|T\xi_t|$, $|T\xi_t|^{-1}$, e.g. [**Bis81b**], [**Kif88**], or use [**Elw82**]. Q.E.D.

D. Completion of Proof of Theorem 2.2

$$\overline{T\mathcal{I}()} : L^2\left(C_0\left(\mathbb{R}^m\right); H\right) \longrightarrow L^2\Gamma\left(\mathcal{H}^1\right)$$

is continuous.

Proof. Consider the sequence of maps

$$\begin{aligned} L^2(C_0(\mathbb{R}^m); H) \;\approxeq\; & L^2(C_0(\mathbb{R}^m); \mathbb{R}) \hat{\otimes} H \\ & \approxeq L^2(\mathcal{F}^{x_0}; \mathbb{R}) \hat{\otimes} L^2(\beta; \mathbb{R}) \hat{\otimes} H \\ & \xrightarrow{1 \otimes \overline{TI(-)}} \quad L^2(\mathcal{F}^{x_0}; \mathbb{R}) \hat{\otimes} L^2\Gamma(\mathcal{H}^1_{\cdot}) \\ & \xrightarrow{f \otimes v \mapsto fv} \quad L^2\Gamma(\mathcal{H}^1_{\cdot}) \end{aligned}$$

where the first is the natural isomorphism and the second the isomorphism given by Lemma 3.1. The maps factorise our map $\overline{T\mathcal{I}(-)}$ and are all continuous, using Lemma 3.3. Q.E.D.

4. Exterior differentiation and the space $\mathcal{H}^2_\sigma$

A. From $T_\omega\mathcal{I} : H \to T_{x.(\omega)}C_{x_0}M$ we can form the linear map of 2-vectors

$$\wedge^2(T_\omega\mathcal{I}) : \wedge^2 H \to \wedge^2 T_{x.(\omega)}C_{x_0}M$$

determined by

$$h^1 \wedge h^2 \mapsto T_\omega\mathcal{I}(h^1) \wedge T_\omega\mathcal{I}(h^1).$$

Here as always, we use the usual Hilbert space cross norm on $H \otimes H$ and $\wedge^2 H$ is the corresponding Hilbert space completion, while for $T_\sigma C_{x_0}M \otimes T_\sigma C_{x_0}M$ we use the greatest cross norm (in order to fit in with the usual definition of differential forms as alternating continuous multilinear maps). To see that this map exists and is continuous using these completions we can use [**CC79**], or directly use the characterisation of $\wedge^2 H$ as a subspace of the functions $\underline{h} : [0,T] \times [0,T] \to \mathbb{R}^m \otimes \mathbb{R}^m$ such that

$$\underline{h}(s,t) = \int_0^s \int_0^t k(s_1, t_1) ds_1 dt_1$$

for some $k \in L^2\left([0,T] \times [0,T]; \mathbb{R}^m \otimes \mathbb{R}^m\right)$, and the characterisation of $\wedge^2 T_\sigma C_{x_0}M$ as a space of continuous functions V into $TM \times TM$ with $(s,t) \mapsto V_{(s,t)} \in T_{\sigma(s)}M \otimes T_{\sigma(t)}M$.

For $\underline{h} \in \wedge^2 H$ we can form

$$\overline{\wedge^2(T\mathcal{I})}(\underline{h})_\sigma \;:=\; \mathbb{E}\,\{\wedge^2(T\mathcal{I})(\underline{h})\;\mid x.(\omega)=\sigma\} \in \wedge^2(T_\sigma C_{x_0}M)$$

almost all $\sigma \in C_{x_0}M$. This gives a continuous linear map

$$\overline{\wedge^2(T\mathcal{I})}_\sigma :=\; \overline{\wedge^2(T\mathcal{I})}()_\sigma :\; \wedge^2 H \;\to \wedge^2(T_\sigma C_{x_0}M).$$

Let $\mathcal{H}^2_\sigma$ be its image with induced Hilbert space structure (i.e. determined by its linear bijection with $\wedge^2 H/(Ker\overline{\wedge^2(T\mathcal{I})}_\sigma)$). We quote the following without giving its proof here. For the sequel the important point is that the spaces $\mathcal{H}^2_\sigma$ are determined only by the Riemannian structure of M and are independent of the embedding used to obtain $\mathcal{I}$. Note also that in general they will be distinct from $\wedge^2\mathcal{H}^1_\sigma$.

THEOREM 4.1. [**EL**] *The space $\mathcal{H}^2_\sigma$ consists of elements of $\wedge^2 T_\sigma C_{x_0}M$ of the form $V + Q(V)$ where $V \in \wedge^2\mathcal{H}^1_\sigma$ and $Q : \wedge^2\mathcal{H}^1_\sigma \to \wedge^2_\sigma C_{x_0}M$ is the continuous linear map determined by*

$$Q(V)_{(s,t)} = (1\otimes W^s_t)\; W^{(2)}_s \int_0^s (W^{(2)}_r)^{-1}\;\mathcal{R}(V_{(r,r)})\;dr, \qquad 0\le s\le t\le T$$

where (i) $W^t_s = W_t\;W_s^{-1} : T_{\sigma(s)}M \to T_{\sigma(t)}M$ for W_t as in §1B,

(ii) $W^{(2)}_t : \wedge^2 T_{x_0}M \to \wedge^2 T_{\sigma(t)}M$ is the the damped translation of 2-vectors on M given by

$$\begin{aligned}\frac{D}{\partial t}W^{(2)}_t(u) &= -\frac{1}{2}\mathcal{R}^{(2)}_{\sigma(t)}\left(W^{(2)}_t(u)\right)\\ W^{(2)}_0(u) &= u, \qquad\qquad u\in\wedge^2 T_{x_0}M,\end{aligned}$$

for $\mathcal{R}^{(2)}_{\sigma(t)}$ the Weitzenböck curvature on 2-vectors e.g. see [**IW89**], [**Elw88**],[**EY93**], [**ELL99**], [**Ros97**] *(where it is called 'the curvature endomorphism') and*

(iii) $\mathcal{R} : \wedge^2 TM \to \wedge^2 TM$ denotes the curvature operator.

In this theorem we have used the identification of $\wedge^2 T_\sigma C_{x_0}M$ with elements $V_{(s,t)} \in T_{\sigma(s)}M\otimes T_{\sigma(t)}$, continuous in (s,t), and with the natural symmetry property. Thus $Q(V)$ is determined by the values $Q(V)_{(s,t)}$ for $s<t$.

B. By restriction, 2-forms on $C_{x_0}M$ i.e. sections of the dual bundle to $\wedge^2 TC_{x_0}M$ can be considered as sections of $(\mathcal{H}^2_\cdot)^*$. Let $Dom(d) \subset L^2\Gamma(\mathcal{H}^1_\cdot)^*$ be a dense linear subspace consisting of differential 1-forms ϕ on $C_{x_0}M$ restricted to the $\mathcal{H}^1_\sigma$ such that the exterior differential $d\phi$, defined by (1), restricts to give an L^2 section of $(\mathcal{H}^2_\cdot)^*$. For example Dom(d) could consist of C^∞ cylindrical forms on $C_{x_0}M$ or C^1 forms which are bounded together with $d\phi$, using the Finsler norms. This gives a densely defined operator

$$d \ : \ Dom(d) \subset L^2\Gamma(\mathcal{H}_\cdot^{1^*}) \to L^2\Gamma(\mathcal{H}_\cdot^{2^*}).$$

We will also assume that each $\phi \in Dom(d)$ satisfies $\mathcal{I}^*(\phi) \in Dom(\bar{d})$ and $\mathcal{I}^*(d\phi) = \bar{d}\mathcal{I}^*(\phi)$, where $\mathcal{I}^*(d\phi) := d\phi(\wedge^2(T\mathcal{I})(-))$ and $\bar{d}$ refers to the closure of exterior differentiation on Wiener space,

$$\bar{d} \ : \ Dom(\bar{d}) \subset L^2(C_0(\mathbb{R}^m); H^*) \to L^2(C_0(\mathbb{R}^m); \wedge^2 H^*)$$

as defined in [**Shi86**] or [**Mal97**]. This condition is easily seen to be satisfied by the two examples of $Dom(d)$ just mentioned, for example by approximation of $\mathcal{I}$.

Theorem 4.2. *The operator d from $Dom(d)$ in $L^2\Gamma(\mathcal{H}_\cdot^{1^*}) \to L^2\Gamma(\mathcal{H}_\cdot^{2^*})$ is closable with closure a densely defined operator $\bar{d}$,*

$$\bar{d} : Dom(\bar{d}) \subset L^2\Gamma(\mathcal{H}_\cdot^{1^*}) \to L^2\Gamma(\mathcal{H}_\cdot^{2^*})$$

Proof. Suppose $\{\phi_n\}_{n=1}^\infty$ is a sequence in Dom(d) converging in $L^2\Gamma(\mathcal{H}_\cdot^{1^*})$ to 0 with $d\phi_n \to \psi$ in $L^2\Gamma(\mathcal{H}_\cdot^{2^*})$ for some ψ. It suffices to show $\psi = 0$. As usual the proof uses integration by parts; the following method is derived from one used for scalars as in [**ABR89**], [**Ebe**]. Let $\lambda : C_{x_0}M \to \mathbb{R}$ be C^∞ and cylindrical and let $\underline{h} \in \wedge^2 H$. Then

$$\begin{aligned}
&\int_{C_{x_0}M} \lambda(\sigma)\psi\left(\overline{\wedge^2(T\mathcal{I})}(\underline{h})(\sigma)\right)\ d\mu_{x_0} \\
= \ & \lim_{n\to\infty} \int_{C_{x_0}M} \lambda(\sigma)d\phi_n\left(\overline{\wedge^2(T\mathcal{I})}(\underline{h})(\sigma)\right)\ d\mu_{x_0} \\
= \ & \lim_{n\to\infty} \int_{C_{x_0}M} d\phi_n\left(\lambda(\sigma)\overline{\wedge^2(T\mathcal{I})}(\underline{h})(\sigma)\right)\ d\mu_{x_0} \\
= \ & \lim_{n\to\infty} \int_{C_0(\mathbb{R}^m)} (d\phi_n)\left(\lambda(x.(\omega)) \wedge^2 (T_\omega\mathcal{I})(\underline{h})\right)\ d\,\mathbb{P}(\omega) \\
= \ & \lim_{n\to\infty} \int_{C_0(\mathbb{R}^m)} \mathcal{I}^*(d\phi_n)\left(\lambda(x.(\omega))\underline{h}\right)\ d\,\mathbb{P}(\omega) \\
= \ & \lim_{n\to\infty} \int_{C_0(\mathbb{R}^m)} \bar{d}(\mathcal{I}^*\phi_n)\left(\lambda(x.(\omega))\underline{h}\right)\ d\,\mathbb{P}(\omega) \\
= \ & \lim_{n\to\infty} \int_{C_0(\mathbb{R}^m)} (\mathcal{I}^*\phi_n)\left(div\lambda(x.(\omega))\underline{h}\right)\ d\,\mathbb{P}(\omega) \\
= \ & 0
\end{aligned}$$

by the continuity of $\mathcal{I}^*$ in Theorem 2.2. Here we used the property that $\mathcal{I}^*(d\phi_n) = \bar{d}(\mathcal{I}^*\phi_n)$ for $\phi_n \in Dom(d)$, and have let

$$-div : Dom(div) \subset L^2(C_0(\mathbb{R}^m); \wedge^2 H) \to L^2(C_0(\mathbb{R}^m); H)$$

denote the co-joint of $\bar{d}$. From [**Shi86**], $\mathcal{I}^*(\lambda)\underline{h} \in Dom(div)$, as can be seen explicitly in this simple situation.

Since the smooth cylindrical functions are dense in L^2 the above shows that $\psi(\overline{\wedge^2(T\mathcal{I})}(\underline{h})_\sigma) = 0$ for almost all σ. Since $\underline{h}$ was arbitrary, $\overline{\wedge^2(T\mathcal{I})}(-)_\sigma$ maps onto $\mathcal{H}^2_\sigma$ by definition, and H is separable, this implies $\psi = 0$ a.s. as required. Q.E.D.

Let $\bar{d} \equiv \bar{d}^1$ be the closure of d, using the previous theorem, and let $d^* = (\bar{d}^1)^*$ be its adjoint.

From now on we shall assume that $Dom(d)$ was chosen so that it contains smooth cylindrical one forms. Since the basic domain of d on functions was taken to be smooth cylindrical functions this implies that d maps $Dom(d)$ to $Dom(d)$. This property making the following crucial fact almost immediate:

Proposition 4.3. *If $f : C_{x_0}M \to \mathbb{R}$ is in $Dom(\bar{d})$ then $\bar{d}f \in Dom(\bar{d}^1)$ and $\bar{d}^1\bar{d}f = 0$.*

Proof. Let $\{f_n\}_{n=1}^\infty$ be a sequence in $D(d)$ converging in L^2 to f with $df_n \to \bar{d}f$ in $L^2\Gamma\mathcal{H}^1_\cdot$. Then $d(df_n) = 0$, since $d^2 = 0$, each n, so $\bar{d}(\bar{d}f) = 0$. Q.E.D.

The proposition enables us to define the first L^2 cohomology group $L^2H^1(C_{x_0}M)$ by

$$L^2H^1(C_{x_0}M) = \frac{Ker\ \bar{d}^1}{Image\ \bar{d}}.$$

Remark: On functions $\bar{d}$ has closed range by the existence of a spectral gap for d^*d on functions, proved by Fang[**Fan94**] and the functional analytical argument of H. Donnelly [**Don81**] Proposition 6.2.

5. A Laplacian on 1-forms and Hodge decomposition Theorem

A. For completeness we go through the formal argument which gives a self-adjoint 'Laplacian' on 1-forms and a Kodaira-Hodge decomposition. Define an operator $\tilde{d}$ on $L^2(C_{x_0}M;\mathbb{R}) \oplus L^2\Gamma(\mathcal{H}^{1^*}_\cdot) \oplus L^2\Gamma(\mathcal{H}^{2^*}_\cdot)$ to itself by

$$Dom(\tilde{d}) = Dom(\bar{d}) \oplus Dom(\bar{d}^1) \oplus L^2\Gamma(\mathcal{H}^{2^*}_\cdot)$$

and $\tilde{d}(f,\theta,\phi) = (0, df, \bar{d}\theta)$ for $(f,\theta,\phi) \in Dom(\tilde{d})$.

The operator $\tilde{d}+\tilde{d}^*$ has domain the intersection of the two domains, i.e.

$$Dom(\tilde{d}+\tilde{d}^*) = Dom(\bar{d}) \oplus (Dom(\bar{d}^1) \cap Dom(d^*)\,) \oplus Dom({\bar{d}^1}^*).$$

From Driver's integration by parts formula it is known that $Dom(d^*)$ contains all smooth cylindrical 1-forms, as does $Dom(\bar{d}^2)$ by assumption. Thus $\tilde{d}+\tilde{d}^*$ has dense domain. It is clearly symmetric. Furthermore, using Proposition 4.3, the consequent orthogonality of $Image(\bar{d})$ and $Image(d^1)^*$, and decomposition

$$L^2\Gamma(\mathcal{H}_\cdot^{1^*}) = (Ker\bar{d}^1 \cap Ker d^*) \oplus \overline{Image\ \bar{d}} \oplus \overline{Image\ d^{1*}},$$

we can see that it is self-adjoint. By Von-Neumann's theorem [**RW75**] Theorem X25, $(\tilde{d}+\tilde{d}^*)^2$ is also self-adjoint (and in particular has dense domain). From Proposition 4.3 we see that for (f,θ,ϕ) in its domain

$$(\tilde{d}+\tilde{d}^*)^2(f,\theta,\phi) = (\bar{d}^*df, ({\bar{d}^1}^*\bar{d}^1 + \bar{d}\bar{d}^*)\theta, \bar{d}^1{\bar{d}^1}^*\phi)$$

and

$$Dom(\tilde{d}+\tilde{d}^*)^2 = Dom(\bar{d}^*d) \oplus Dom({\bar{d}^1}^*\bar{d}^1 + \bar{d}\bar{d}^*) \oplus Dom(\bar{d}^1{\bar{d}^1}^*).$$

In particular if we set

$$\Delta^1 := {\bar{d}^1}^*\bar{d}^1 + \bar{d}\bar{d}^*$$

we obtain a nonnegative self-adjoint operator on $L\Gamma(\mathcal{H}_\cdot^{1^*})$. Since

$$Ker(\tilde{d}+\tilde{d}^*)^2 \;=\; Ker(\tilde{d}+\tilde{d}^*) \;=\; Ker\ \tilde{d} \cap Ker\ \tilde{d}^*$$

we see that $\phi \in L^2\Gamma(\mathcal{H}_\cdot^{1^*})$ is harmonic, i.e. $\phi \in Ker\Delta^1$, if and only if $\bar{d}^1\phi = 0$ and $\bar{d}^*\phi = 0$.

As remarked in §4 we know $\bar{d}$ has closed range. Thus:

THEOREM 5.1. *The space $L^2\Gamma(\mathcal{H}^{1^*})$ of $\mathcal{H}$ 1-forms has the decomposition*

$$L^2\Gamma(\mathcal{H}_\cdot^{1^*}) = Ker\Delta^1 \oplus Image\ \bar{d} \oplus \overline{Image\ d^{1*}}.$$

In particular every cohomology class in $L^2H^1(C_{x_0}M)$ has a unique representative in $Ker\Delta^1$.

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