Effective Diffusions on the Orthonormal Frame Bundles

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Abstract

In this paper we study a model on the orthonormal frame bundle of a complete connected Riemannian manifold and investigate the effect of random perturbations. Although effective diffusions have been studied since the 60's, little is know or asked about the effective behaviour of the derivative flow which relates to the stability of the averaging procedure with respect to the initial data and numerical computations. We obtain an effective dynamic of both the original equation and its linearization.

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1 Introduction

If a family of conserved quantities $F_1, F_2, \dots, F_k$ for a Hamiltonian system is regular and defines an invariant manifold consider a perturbation transversal to the invariant manifold, of size ϵ. As ϵ tends to zero, the solution to the perturbed system converges to that of the original system. If seen on a large time scale, say on $[0, \frac{1}{\epsilon}]$, the ϵ-size perturbation are visible and the dynamics converges to an effective dynamics. In more complex system we would rely on the techniques from the theory of homogeneization. There is a vast literature on this and related topics and we refer to the following books and references therein: Arnold [1], Bensoussan-Lions-Papanicolau [2], Freidlin-Wentzell [13], and Stuart-Paviliotis [21]. The principal idea is to deduce long term trends of a complex system from that of a relatively simple one for which some observables are known.

Let $\mathcal{L}_0$ and $\mathcal{L}_1$ be second order differential operators with non-negative definite symbols and vanishing zero order terms. Consider the operator $\mathcal{L}^\epsilon = \frac{1}{\epsilon}\mathcal{L}_0 + \mathcal{L}_1$ as perturbation of $\mathcal{L}_0$. First we represent $\mathcal{L}_0$ in Hörmander form: $\mathcal{L}_0 = \frac{1}{2}\sum_{i=1}^m L_{X_i}L_{X_i} + L_{X_0}$, where X_i are vector fields and $\mathcal{L}_{X_i}$ denotes Lie differentiation in the direction of X_i. Together with a Hörmander form representation of $\mathcal{L}_1 = \frac{1}{2}\sum_{i=1}^m L_{Y_i}L_{Y_i} + L_{Y_0}$ we have a stochastic flow $\phi_t^\epsilon(x)$, solution to the stochastic differential equations (SDE) driven by the vector fields $X_i, X_0, \sqrt{\epsilon}Y_i, \epsilon Y_0$, $i = 1, \dots, m$, with initial point x and generator $\mathcal{L}^\epsilon$. Suppose that there is an effective dynamic for each starting point x. Denote by $\mu^\epsilon_{\frac{t}{\epsilon}}$ the probability distribution of $\phi^\epsilon_{\frac{t}{\epsilon}}$ which converges as $\epsilon \to 0$ to a limit law $\bar{\mu}^x$. Let us represent $\phi^\epsilon_{\frac{t}{\epsilon}}(x)$ and a random variable $\bar{\phi}_t(x)$ with probability distribution $\bar{\mu}^x_t$ on the same probability space with the property $\phi^\epsilon_{\frac{t}{\epsilon}}(x) \to \bar{\phi}_t(x)$ almost surely and we could compute the rate of convergence that depends on x. To consider uniform convergence in x we return to the original probability space and let $T_x\phi^\epsilon_t(v)$ be the solution to the corresponding linearised SDE with initial value v, otherwise known as the derivative flow. Then

$$\sup_{x,y\in K_1} \|\phi^\epsilon_{\frac{t}{\epsilon}}(x) - \phi^\epsilon_{\frac{t}{\epsilon}}(y)| \le \sup_{z\in K_2} \mathbb{E} \sup_{s\le t} \|T_z\phi^\epsilon_{\frac{t}{\epsilon}}\|.$$

where K_1 is a compact set containing the set K_1. The dependence of the convergence on x is seen to be controlled by $\sup_\epsilon \mathbb{E}\sup_{s\le t}\|T_z\phi^\epsilon_{\frac{t}{\epsilon}}\|$. We could not expect uniform control of the moments of the derivative flow over all ϵ. However it is reasonable to expect such control on the slow part of the motion. In this case we may estudy the convergence of the derivative flow and obtain effective derivative flow.

The question we study is related to the uniform convergence problem for solutions of SDEs, given by the Hörmander form representation of $\mathcal{L}^\epsilon$, with respect to the initial point as $\epsilon \to 0$. In terms of numerical computations, we ask the sensitivity of the limiting procedure with respect to the perturbation of the initial position. See R. Sowers [22] for the study of such a stability problem associated to an SDE on the cylinder, with fast drift. See also Sowers [23] and Kifer [15].

The state space of the proposed SDEs is the orthonormal frame bundle (OM) of a complete Riemannian manifold. The SDE we consider has both horizontal and vertical components. Horizontal SDEs on the orthonormal frame bundles haven been studied extensively and are originated from the study of the intrinsic geometry of elliptic stochastic differential equations and Malliavin Calculus. Since an elliptic SDEs on a manifold M relates naturally to the orthonormal frame bundle of M whose fibres at x are the space of isomorphisms from $\mathbf{R}^n$ with the Euclidean metric to T_xM with the Riemannian metric induced from the elliptic SDE, canonical geometric Brownian motions or other elliptic diffusions on a Riemannian manifold can be constructed as projection of the horizontal diffusion on the Orthonormal frame bundle. This also leads to construction of the development map, a fundamental tool in Malliavin calculus on path space. On the other hand the solution flow of a vertical diffusion can be considered as a random evolution of a linear frame. Orthonormal frames are used to define positive isotropic curvature in Brendle-Schoen [7] where they studied whether this condition is preserved by R. Hamilton's ODE.

The structure of the papers is as following. In section 2 we present the models, examples and the main theorem for model (2.1). In section 3.2-3.3 we present the background and geometric results needed for the study of the derivative processes. In section 3.4 we study model (3.1) and its special case (3.2), which concerns perturbation of vertical SDEs by horizontal vector fields. For the latter we investigate the conserved quantities of the slow variables, separating the slow and the fast variables and show that the family of slow variables, rescaled in time, converges to the stochastic Jacobi equation.

2 The Basic Perturbed System

Consider the space of isometries from $\mathbf{R}^n$ to T_xM, otherwise known as orthonormal frames. The orthonormal frame bundle $\pi : OM \to M$ is a principal bundle with group action $O(n)$ and the fibre at $u : \mathbf{R}^n \to T_{\pi(u)}M$ consisting of isometric linear maps from $\mathbf{R}^n$ to $T_{\pi(u)}M$. The right action is denoted by R_a. We are mainly interested in the component $SO(n)$ of $O(n)$ that contains the identity matrix. Some of the often used orthonormal frame bundles are those for $\mathbf{R}^n$, S^n and the hyperbolic space H^n which are respectively $\mathbf{R}^n \times SO(n)$, $SO(n+1)$ and the Lorentz group $SO(1,n)$.

Letting TOM be the tangent space of OM, denote by VT_uOM the naturally defined vertical tangent spaces which are the kernels of the linear map $T_u\pi$, the differential of π. We study the model $\mathcal{L}_0 + \epsilon\mathcal{L}_1$ where $\mathcal{L}_0$ is a vertical operator, elliptic on each fibre. To define a vertical operator intrinsically we use symbol $\sigma^{\mathcal{L}_0} : T^*OM \to TOM$ and the operator $\delta^{\mathcal{L}_0}$ associated to $\mathcal{L}_0$. The symbol of an operator is standard and the operator $\delta^{\mathcal{L}_0}$ is defined to be the unique operator from 1-forms to functions such that $\delta^{\mathcal{L}_0}(df) = \mathcal{L}_0 f$ and $\delta^{\mathcal{L}_0}(f\phi) = df\sigma^{\mathcal{L}_0}(\phi) + f\delta^{\mathcal{L}_0}(\phi)$ for all smooth functions f and differential form ϕ on OM, see page 4 of [11]. We say that $\mathcal{L}_0$ is vertical if $\delta^{\mathcal{L}_0}\phi = 0$ for all C^1 differential 1-form ϕ such that ϕ vanishes on the vertical bundle. By Proposition 1.4.7 in [11] $\mathcal{L}_0$ is vertical if $\mathcal{L}_0(f \circ \pi) = 0$ for all real valued C^2 function on M and it is vertical if and only if there are vertical vector fields A_i such that $\mathcal{L}_0 = \frac{1}{2}\sum_i L_{A_i}L_{A_i} + L_{A_0}$.

Model 1. Let $A_j, j = 0, 1, \dots m$, be a family of vector fields on OM with values in the vertical bundle $VTOM$ and $w_t = (w_t^1, \dots, w_t^m)$ an $\mathbf{R}^m$-valued Brownian motion. Let $\mathbb{X}_l, l = 0, 1, 2, \dots, p$, be a family of vector fields on OM and $B_t = (B_t^1, \dots, B_t^p)$ an $\mathbf{R}^p$-valued Brownian motion. Here $p, m \in \mathbb{N}$. Let $\tilde{L}^\epsilon$ be the infinitesimal generator of the perturbed SDE

$$\left\{ \begin{array}{ll} du_t^\epsilon & = \sqrt{\epsilon}\sum_{l=1}^p \mathbb{X}_l(u_t^\epsilon) \circ dB_t^l + \epsilon\mathbb{X}_0(u_t^\epsilon)dt + \sum_{j=1}^m A_j(u_t^\epsilon) \circ dW_t^j + A_0(u_t^\epsilon)dt, \\ u_0^\epsilon & = u_0. \end{array} \right. \tag{2.1}$$

Then $\tilde{L}^\epsilon = \mathcal{L}_0 + \epsilon\mathcal{L}_1$ for $\mathcal{L}_0 = \frac{1}{2}\sum L_{A_j}L_{A_j} + \mathcal{L}_{A_0}$ and $\mathcal{L}_1 = \frac{1}{2}\sum L_{\mathbb{X}_l}L_{\mathbb{X}_l} + L_{\mathbb{X}_0}$. Consider the diffusion $u^\epsilon_{\frac{t}{\epsilon}}$ whose generator is $\mathcal{L}^\epsilon = \frac{1}{\epsilon}\mathcal{L}_0 + \mathcal{L}_1$. Let $\{A_{i,j}\}$ be an orthonormal basis of $\mathfrak{so}(n)$, whose elements are skew symmetric matrices whose entries equal zero except at the (i,j)-th and (j,i)-th position and those non-zero entries, at (i,j) and (j,i) positions, are respectively $\frac{1}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}}$. We may however relabel the basis as $\{A_k\}$. Then $A_j = \sum_k \sigma_{k,j}A_k^*$ for some function $\sigma_{k,j}$ on OM, where A_k^* is the vertical fundamental vector field associated to A_k,

$$A_k^*(u) = \frac{d}{dt}|_{t=0} u \exp(tA_k).$$

At this point we would like to point out that although the Brownian motions (B_t) and (W_t) are independent there is nontrivial interaction between the perturbation and the unperturbed vector fields due to the fact that if $H(e)$ is a fundamental horizontal vector field then $[H(e), A^*]$ is a horizontal vector field and the vertical part of $[H_i, H_j]$ is given by the curvature $-2\Omega(H_i, H_j)$. These interactions become more apparent when the derivative processes are considered.

Let us first clarify the notion of vector fields or more intrinsically operators that are transversal to the vertical distribution $VTOM$. A linear connection ∇ on M corresponds to a right invariant smooth distribution on OM, denoted by $\{HTOM\}$ and the tangent bundle TOM can be written as a direct sum $T_uOM = HT_uOM \oplus VT_uOM$. The latter is also called an Ehresmann connection on OM viewed as a fibre bundle. By a transversal operator we mean one along the horizontal distribution of a connection.

There are two notable classes of transversal operators, in the first we assume that $\mathbb{X}_l$ are right invariant and in the second $\mathbb{X}_l$ are canonical horizontal vector fields.

Example 2.1 (The Right Invariant Case) Let $\mathcal{A}$ be any diffusion operator with the Hörmander form $\mathcal{A} = \frac{1}{2}\sum_i L_{X_i}L_{X_i} + L_{X_0}$ where X_l are vector fields on M. Let $\mathfrak{h}_u : T_{\pi(u)}M \to T_uOM$ be the horizontal lifting map determined by the connection ∇ on TM which is an isomorphism onto its image and is right invariant, $(R_a)_* h_u(v) = h_{ua}(v)$. Define $\mathbb{X}_l(u) = \mathfrak{h}_u(X_l(\pi(u)))$ and $\mathcal{L}_1 = \frac{1}{2}L_{\mathbb{X}_l}L_{\mathbb{X}_l} + L_{\mathbb{X}_0}$. Consider

$$du_t^\epsilon = \sqrt{\epsilon}\sum_{l=1}^p \mathbb{X}_l(u_t^\epsilon)\circ dB_t^l + \epsilon\mathbb{X}_0(u_t^\epsilon)dt + \sum_{j=1}^m A_j(u_t^\epsilon)\circ dW_t^j + A_0(u_t^\epsilon)dt.$$

Let u_t^ϵ be an $\mathcal{L}_0 + \epsilon\mathcal{L}_1$ diffusion and let $x_t^\epsilon = \pi(u_t^\epsilon)$. Then the law of x_t^ϵ is given by the SDE $dx_t^\epsilon = \sqrt{\epsilon}\sum_{l=1}^p X_l(x_t^\epsilon)\circ dB_t^l + \epsilon X_0(x_t^\epsilon)dt$. For all ϵ, $x^\epsilon_{\frac{t}{\epsilon}}$ is an $\mathcal{A}$-diffusion. The horizontal lift $\tilde{x}_{\frac{t}{\epsilon}}$ of $x_{\frac{t}{\epsilon}}$ is an $\mathcal{L}_1$-diffusion.

A diffusion operator $\mathcal{L}_1$ on OM is equi-variant if $(\mathcal{L}_1 f)(R_a u) = \mathcal{L}_1(f\circ R_a)(u)$ for any smooth function f on OM. In this case it defines an operator $\mathcal{A}$ on M such that $\mathcal{L}_1(f\circ\pi) = (\mathcal{A}f)\circ\pi$ for all real valued smooth functions f on M. If $\mathcal{A}$ is cohesive, i.e. $\sigma^{\mathcal{A}}$ has constant rank and $\mathcal{A}$ is along the image of $\sigma^{\mathcal{A}}$, then $\mathcal{L}_1 = \mathcal{L}_1^v + \mathcal{A}^H$ where $\mathcal{L}_1^v$ is vertical and $\mathcal{A}^H$ is the horizontal lift of $\mathcal{A}$ using the connection generated by $\mathcal{L}_1$ [11]. In this case

$$\frac{1}{\epsilon}\mathcal{L}_0 + \mathcal{L}_1 = \frac{1}{\epsilon}\mathcal{L}_0 + \mathcal{L}_1^v + \mathcal{A}^H$$

where $\mathcal{A}^H$ would have a Hörmander decomposition by right invariant horizontal vector fields. For simplicity we assume that $\mathcal{L}_0^v = 0$, *i.e.* $\mathcal{L}_1$ has no vertical part, so that we recover the model. To summarise if $\mathcal{L}_1$ is equi-variant we may seek a family of supplementary vector fields $\mathbb{X}_{p+1},\dots,\mathbb{X}_m$ so that $\{T\pi(\mathbb{X}_1),\dots,T\pi(\mathbb{X}_m)\}$ are elliptic in M and we may choose the connection to be that generated by $T\pi(\mathbb{X}_l)$, by the above discussion we are back to Example 2.1.

The second example is in terms of the fundamental horizontal vector fields. For each u, there is an isomorphism $H_u : \mathbf{R}^n \to HTOM$ such that $T\pi(H_u(e)) = u(e)$. The vector field $\{H(e)\}$ are the fundamental horizontal vector fields:

$$H_u(e) = \mathfrak{h}_u(u(e)).$$

Example 2.2 (The Rotational Invariant Case) Let $\{e_l\}$ be an o.n.b. of $\mathbf{R}^n$ and consider $\mathbb{X}(u) : \mathbb{L}(\mathbf{R}^n; T_uOM)$ which is smooth in u and satisfies that $\mathbb{X}(ug)(e) = TR_g\mathbb{X}_u(ge)$ for any $g\in O(n)$ and $T\pi\mathbb{X}(u)(e) = u(e)$. Define

$$H_uTOM = \text{span}\{\mathbb{X}(u)(e_1),\dots,\mathbb{X}(u)(e_n)\}.$$

Then $HTOM = \cup_u H_u TOM$ is right invariant and has dimension n. It defines a connection ∇ on M and a lifting map $\mathfrak{h}_u$. The vector fields $\mathbb{X}(u)(e) = \mathfrak{h}_u[u(e)] \equiv H_u(e)$ are the fundamental horizontal vector fields associated to ∇. Denote $H_l(u) := H(u)(e_l)$. Let $H_0(u) = H(u)(e_0)$ for some $e_0 \in \mathbf{R}^n$. Consider

$$du_t^\epsilon = \sqrt{\epsilon}\sum_{l=1}^n H_l(u_t^\epsilon)\circ db_t^l + \epsilon H_0(u_t^\epsilon)dt + \sum_{j=1}^m A_j(u_t^\epsilon)\circ dW_t^j + A_0(u_t^\epsilon)dt,$$

where $(b_t) = (b_t^1, \dots, b_t^n)$ is an $\mathbf{R}^n$-valued Brownian motion. Then the horizontal lift $\tilde{x}_t^\epsilon$ of $x_t^\epsilon = \pi(u_t^\epsilon)$ satisfies

$$d\tilde{x}_t^\epsilon = \sqrt{\epsilon}H(\tilde{x}_t^\epsilon)(g_t^\epsilon \circ db_t) + \epsilon H(\tilde{x}_t^\epsilon)(g_t^\epsilon e_0)dt, \tag{2.2}$$

where g_t^ϵ is defined by $u_t^\epsilon = \tilde{x}_t^\epsilon g_t^\epsilon$ and recall that $H(u)(e_l) = H_l(u)$. The SDE follows from computing $\circ d\tilde{x}_t^\epsilon = \circ d(u_t^\epsilon(g_t^\epsilon)^{-1})$ and from the observation that $\circ d\tilde{x}_t^\epsilon$ equals the horizontal part of the right hand side, so $d\tilde{x}_t^\epsilon = \sqrt{\epsilon}TR_{(g_t^\epsilon)^{-1}}H_l(u_t^\epsilon)\circ db_t^l + \epsilon TR_{(g_t^\epsilon)^{-1}}H_0(u_t^\epsilon)dt$.

If furthermore $A_j(u_t) = A_j^*(u_t)$, some $A_j \in \mathfrak{so}(n)$, we have

$$\begin{aligned} du_t^\epsilon &= \sqrt{\epsilon}\textstyle\sum_{l=1}^n H_l(u_t^\epsilon)\circ db_t^l + \epsilon H_0(u_t^\epsilon)dt + \sum_{j=1}^m A_j^*(u_t^\epsilon)\circ dW_t^j + A_0^*(u_t^\epsilon)dt \\ dg_t^\epsilon &= \textstyle\sum_j TR_{g_t^\epsilon}A_j \circ dW_t^j + TR_{g_t^\epsilon}A_0 dt, \end{aligned}$$

and g_t^ϵ is independent of ϵ. The equation for g_t follows from that the vertical part of $\circ d(\tilde{x}_t^\epsilon g_t^\epsilon)$ is $(g_t^{-1}\circ dg_t)^*(\tilde{x}_t)$. Apply the connection 1-form to the latter, taking into account of the adjoint invariance of ϖ, and compare the resulting quantity with the vertical part of the SDE for u_t^ϵ.

In this case $\tilde{x}_t^\epsilon$ is a Markov process on OM, observing that g_t^ϵ is independent of $\{b_t\}$. In the case of $e_0 = 0$ the law of $\tilde{x}_{\frac{t}{\epsilon}}^\epsilon$, and hence that of $x_{\frac{t}{\epsilon}}^\epsilon$, is independent of ϵ. Furthermore $\tilde{x}_t^\epsilon$ would be a Horizontal Brownian motion with projection x_t a Markov process and a Brownian motion on M. This is the Eells-Elworthy construction of Brownian motions [9]. The invariance is no longer true for $e_0 \neq 0$.

More generally if $\{\Phi_t(u)\}$ is a family of stochastic process on OM with the property that $\Phi_t(ug) \overset{law}{=} \Phi_t(u)\psi_t(g)$ for some $\psi_t(g) \in SO(n)$ and $\sigma\{\pi(\Phi_r(u))|r \le s\} = \sigma\{\Phi_r(u) : r \le s\}$, then $\pi(\Phi_t(u))$ is a Markov process. Denote by $Q_t(u_0, du)$ the law of $\Phi_t(u_0)$ and let $f : M \to \mathbf{R}$ be a Borel measurable function, $x_t = \pi(\Phi_t(u_0))$,

$$\mathbb{E}\{f(x_t)|\sigma\{x_r, r \le s\}\} = \int (f\circ\pi)(u)Q_{t-s}(\tilde{x}_s, du).$$

By the rotational invariance, $\int (f\circ\pi)(u)Q_{t-s}(\tilde{x}_s, du) = \int (f\circ\pi)(u\psi_s(g))Q_{t-s}(\tilde{x}_s g, du) = \int (f\circ\pi)(u)Q_{t-s}(\tilde{x}_s g, du)$, and so depends only on $\pi(\tilde{x}_s)$. In case of $e_0 = 0$, the flow of (2.2) satisfies the rotational invariance condition and the horizontal lift of x_t is a function of the path $(x_r, r \le t)$.

Example 2.3 Let $\alpha : M \times \mathbf{R}^n \to \mathbf{R}^n$ be a smooth map so that $\alpha(x) \in \mathbb{L}(\mathbf{R}^n; \mathbf{R}^n)$. Let$\{e_i\}_{i=1}^n$ be an o.n.b. of $\mathbf{R}^n$, $e_0 \in \mathbf{R}^n$, and $\mathbb{X}_i(u) = \mathfrak{h}_u[\alpha(\pi(u))e_i]$. Consider

$$du_t^\epsilon = \sqrt{\epsilon}\sum_{l=1}^n \mathfrak{h}_u[\alpha(\pi(u))e_l]\circ dB_t^l + \epsilon\mathfrak{h}_u[\alpha(\pi(u))e_0](u_t^\epsilon)dt + \sum_{j=1}^m A_j(u_t^\epsilon)\circ dW_t^j + A_0(u_t^\epsilon)dt.$$

The projection $x_t^\epsilon = \pi(u_t^\epsilon)$ satisfies:

$$dx_t^\epsilon = \sqrt{\epsilon}\sum_{l=1}^n u_t^\epsilon \alpha(x_t^\epsilon)(e_l) \circ dB_t^l + \epsilon u_t^\epsilon \alpha(x_t^\epsilon)(e_0)dt = \sqrt{\epsilon} u_t^\epsilon \alpha(x_t^\epsilon) \circ dB_t + \epsilon u_t^\epsilon \alpha(x_t^\epsilon)(e_0)dt.$$

Let $\tilde{x}_t^\epsilon$ be the horizontal lifting map of x_t^ϵ and g_t^ϵ be an element of $SO(n)$ determined by $u_t^\epsilon = x_t^\epsilon g_t^\epsilon$. Then $d\tilde{x}_t^\epsilon = \sqrt{\epsilon} H(\tilde{x}_t^\epsilon) g_t^\epsilon \alpha(x_t^\epsilon) \circ dB_t + \epsilon H(\tilde{x}_t^\epsilon) g_t^\epsilon \alpha(x_t^\epsilon)(e_0)dt$. When $\alpha(x)$ is not trivial letting $f : M \to \mathbf{R}$ be a smooth function, then the bounded variation term for $f(x_t)$ will involve $\sum_i \nabla df(u_t^\epsilon \alpha(x_t^\epsilon)e_i, u_t^\epsilon \alpha(x_t^\epsilon)e_i)$ which is no longer a trace. It will also involve the derivative of $\mathbb{X}_l$ and a computation for the laws are no longer possible. We will have to consider the system as the perturbation of the vertical SDE about which we know a lot more.

Example 2.4 (The Zero Curvature Case: Perturbation to the Horizontal Flow) Consider

$$du_t^\epsilon = H(u_t^\epsilon) \circ db_t + Z_0(u_t^\epsilon)dt + \sqrt{\epsilon} A_k^*(u_t^\epsilon) \circ dW_t^k + \epsilon Z_1(u_t^\epsilon)dt \tag{2.3}$$

where Z_0 is a horizontal vector field and Z_1 a vertical vector field. The Lie bracket of two horizontal vector fields is given by the curvature operator of the induced connection on M. The horizontal tangent bundle is Frobenius integrable if and only if the curvature vanishes. This is the case if M is a Lie group equipped with the Left or right invariant connection. How does the motion behave under a transversal force?

Let $\varpi : T_uOM \to \mathfrak{so}(n)$ be the connection 1-form, corresponding to the given Riemannian connenction ∇, which is determined by adjoint invariance and its values on fundamental vertical vector fields: $(R_g)^*\varpi = \mathrm{ad}(g^{-1})\varpi$ and $\varpi(A^*) \equiv A$. Let $\breve{\nabla}$ be the direct sum connection, $\breve{\nabla}_v U = \varpi^{-1} d(\varpi(U))(v) + \theta^{-1} d(\theta(U))(v)$, for any vector $v \in T_uOM$ and any vector field U on OM. Here θ is the connection 1-form. The connection $\breve{\nabla}$ is the metric connection associated to the SDEs below [10] :

$$du_t = \sqrt{\epsilon}\sum_{l=1}^n H_l(u_t) \circ db_t^l + \sum_{k=1}^{n(n-1)/2} A_k^*(u_t) \circ dW_t^k.$$

In the theorem below the assumption on the injectivity radius can be removed in the case of the projection a Brownian motion with bounded drift.

Theorem 2.5 *Let M be a complete Riemannian manifold with positive injectivity radius endowed with a connection ∇. Consider model (2.1). Let $\breve{\nabla}$ be a Riemannian connection on OM. Let u_t^ϵ be a solution to (2.1), $x_t^\epsilon = \pi(u_t^\epsilon)$ and $\tilde{x}_t^\epsilon$ the horizontal lift of x_t^ϵ. Assume that $\{\varpi_u[A_j(u)]\}_{j=1}^m$ spans $\mathfrak{so}(n)$. Assume that $\mathbb{X}_l$ and $|\breve{\nabla}_{\mathbb{X}_l}\mathbb{X}_l|$ have linear growth and that $\pi(u_t^\epsilon)$ does not explode. Define*

$$\begin{aligned} b_k &= \tfrac{1}{2}\textstyle\sum_{l=1}^p \int_{SO(n)} \langle \breve{\nabla}_{\mathbb{X}_l}\mathbb{X}_l(ug), H_k(ug)\rangle \, d\mu_u(g) + \int_{SO(n)} \mathbb{X}_0(ug)d\mu_u(g) \\ a_{i,j}(u) &= \tfrac{1}{2}\textstyle\int_{SO(n)} \sum_{l=1}^p \langle TR_g^{-1}\mathbb{X}_l(ug), H_i(u)\rangle \langle TR_g^{-1}\mathbb{X}_l(ug), H_j(u)\rangle \, d\mu_u(g). \end{aligned}$$

where μ_u is the invariant measure of the following SDE on $SO(n)$:

$$dg_t = \sum_{j=1}^{m} TR_{g_t}\varpi[A_j(ug_t)] \circ dW_t^j + TR_{g_t}\varpi[A_0(ug_t)]dt.$$

Then $\tilde{x}^\epsilon_{\frac{t}{\epsilon}}$ converges in law to that associated to the generator $\bar{\mathcal{L}}$,

$$\bar{\mathcal{L}}F(u) = \sum_{k=1}^{p} b_k(u)dF\left(H_k(u)\right) + \sum_{i,j=1}^{p} a_{i,j}(u)\breve{\nabla}dF(H_i(u), H_j(u)).$$

Proof Since $\tilde{x}^\epsilon_t$ and u^ϵ_t belong to the same fibre we may define $g^\epsilon_t \in SO(n)$ by $u^\epsilon_t = \tilde{x}^\epsilon_t g^\epsilon_t$. Use the relations $(R_g)_*A^* = [\mathrm{ad}(g^{-1})A]^*$ and

$$\frac{d}{ds}|_{s=t} u\exp(sA) = \frac{d}{dr}|_{r=0} u\exp(rA)\exp(tA) = TR_{\exp(tA)}A^*(u)$$

to obtain that

$$du^\epsilon_t = TR_{g^\epsilon_t}d\tilde{x}^\epsilon_t + (TL_{(g^\epsilon_t)^{-1}}dg^\epsilon_t)^*(\tilde{x}^\epsilon_t).$$

Comparing this with the original SDE for du^ϵ_t we obtain

$$dg^\epsilon_t = \sum_{j=1}^{m} TL_{g^\epsilon_t}\varpi[A_j(u^\epsilon_t)] \circ dw^j_t + TL_{g^\epsilon_t}\varpi[A_0(u^\epsilon_t)]dt. \tag{2.4}$$

On the other hand $dx^\epsilon_t = \sqrt{\epsilon}\sum_{l=1}^p T\pi(\mathbb{X}_l(u^\epsilon_t)) \circ dB^l_t + \epsilon T\pi(\mathbb{X}_0(u^\epsilon_t))dt$ so

$$d\tilde{x}^\epsilon_t = \mathfrak{h}_{\tilde{x}_t}(\circ dx^\epsilon_t) = \sqrt{\epsilon}\sum_{l=1}^{p} \mathfrak{h}_{\tilde{x}^\epsilon_t}[T\pi(\mathbb{X}_l(u^\epsilon_t))] \circ dB^l_t + \epsilon\mathfrak{h}_{\tilde{x}^\epsilon_t}[T\pi(\mathbb{X}_0(u^\epsilon_t))]dt.$$

In terms of the group action,

$$d\tilde{x}^\epsilon_t \quad = \quad \sqrt{\epsilon}\sum_{l=1}^{p} \mathfrak{h}_{\tilde{x}^\epsilon_t}[T_{\tilde{x}^\epsilon_t g^\epsilon_t}\pi(\mathbb{X}_l(\tilde{x}^\epsilon_t g^\epsilon_t)] \circ dB^l_t + \epsilon\mathfrak{h}_{\tilde{x}^\epsilon_t}[T_{\tilde{x}^\epsilon_t g^\epsilon_t}\pi(\mathbb{X}_0(\tilde{x}^\epsilon_t g^\epsilon_t)]dt.$$

Let μ^ϵ be the laws of the $\tilde{x}^\epsilon_t$. We first show that $\{\mu^\epsilon\}$ is tight. By Prohorov's theorem a family of probability measures is tight if it is relatively compact. Since $\tilde{x}^\epsilon_0 = u_0$ it suffices to estimate the modulus of continuity and show that for all positive numbers a, η, there exists $\delta > 0$ such that for all ϵ reasonably small, see Billingsley [6] Ethier-Kurtz[12],

$$P(\omega : \sup_{|s-t|<\delta} d(\tilde{x}^\epsilon_t, \tilde{x}^\epsilon_s) > a) < \eta.$$

Here d denotes a distance function on OM. The Riemannian distance function is not smooth on the cut locus. The cut locus of OM is in general not predictable by that of M. To avoid any assumption on the cut locus of OM we construct a new distance function that preserves the topology of OM.

Let $x \in M$ and $2a$ the minimum of 1 and the injectivity radius of M. Let $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a smooth concave function such that $\phi(r) = r$ when $r < a$ and $\phi(r) = 1$

when $r \geq 2a$, e.g. ϕ is the convolution of $\min(1, r)$ with a standard mollifier supported in the set $\{r : |r - \frac{3a}{2}| < a/2\}$. Let ρ and $\tilde{\rho}$ be respectively the Riemannian distance on M and OM. Then $\phi \circ \rho$ and $d := \phi \circ \tilde{\rho}$ are distance functions.

For $u \in \pi^{-1}(x)$,

$$\begin{aligned}
\phi \circ \tilde{\rho}(u, \tilde{x}_t^\epsilon) &= (\phi \circ \tilde{\rho})(u, \tilde{x}_0^\epsilon) + \int_0^t d(\phi \circ \tilde{\rho}) \left(\sqrt{\epsilon} \sum_{l=1}^p \mathfrak{h}_{\tilde{x}_s^\epsilon} [T\pi(\mathbb{X}_l(u_s^\epsilon))] \circ dB_s^l \right) \\
&\quad + \int_0^t \epsilon \, d(\phi \circ \tilde{\rho}) \, \mathfrak{h}_{\tilde{x}_s^\epsilon} [T\pi(\mathbb{X}_0(u_s^\epsilon))] \, ds \\
&= (\phi \circ \tilde{\rho})(u, \tilde{x}_0^\epsilon) + \int_0^t d(\phi \circ \rho) \left(\sqrt{\epsilon} \sum_{l=1}^p [T\pi(\mathbb{X}_l(u_s^\epsilon))] dB_s^l \right) \\
&\quad + \epsilon \sum_{l=1}^p \int_0^t \nabla d(\phi \circ \rho) \left(T\pi(\mathbb{X}_l(u_s^\epsilon)), T\pi(\mathbb{X}_l(u_s^\epsilon)) \right) \; ds \\
&\quad + \epsilon \int_0^t d(\phi \circ \rho) \left(\frac{1}{2} \sum_{l=1}^p \nabla_{T\pi(\mathbb{X}_l)} (T\pi \circ \mathbb{X}_l)(u_s^\epsilon) + T\pi(\mathbb{X}_0(u_s^\epsilon)) \right) ds.
\end{aligned}$$

Since $\phi \circ \rho$ has compact support and the vector fields concerned have linear growth, $|T\pi(\mathbb{X}_l(u_s^\epsilon))| \leq C(1 + \rho(u_s^\epsilon, u)) \leq [C + C\tilde{\rho}(\tilde{x}_s^\epsilon, \tilde{u}_s^\epsilon)] + C\tilde{\rho}(u, \tilde{x}_s^\epsilon)$ some $u \in OM$. The quantity $C + C\tilde{\rho}(\tilde{x}_s^\epsilon, \tilde{u}_s^\epsilon)$ is bounded from the compactness of $SO(n)$ and it follows that $\mathbb{E}[\phi \circ \tilde{\rho}(u, \tilde{x}_t^\epsilon)]^2) \leq C(\phi \circ \tilde{\rho})^2(u, \tilde{x}_0^\epsilon)\epsilon t$ for some constant C. By the Markov property and the estimates below the required tightness follows,

$$\mathbb{E}[\phi \circ \tilde{\rho}(\tilde{x}_{\frac{s}{\epsilon}}^\epsilon, \tilde{x}_{\frac{t}{\epsilon}}^\epsilon)]^2) \leq C_1 \mathbb{E}(\phi \circ \tilde{\rho})^2(\tilde{x}_{\frac{s}{\epsilon}}^\epsilon, \tilde{x}_0^\epsilon)(t - s) \leq C_1 |t - s|.$$

In the case of x_t^ϵ being a Brownian motion with a bounded drift we do not need to assume hypothesis on the injectivity radius for the tightness. See e.g. the estimates in [17]. Note that by the right invariance of the horizontal lift,

$$\mathfrak{h}_{\tilde{x}_s^\epsilon} [T_{\tilde{x}_s^\epsilon g_s^\epsilon} \pi(\mathbb{X}_l(\tilde{x}_s^\epsilon g_s^\epsilon)] = TR_{(g_s^\epsilon)^{-1}} \mathbb{X}_l(u_s^\epsilon).$$

Let $F : OM \to \mathbf{R}$ be a smooth function. For $\breve{\nabla}$, the canonical direct sum connection on OM associated to ∇,

$$\begin{aligned}
F(\tilde{x}_t^\epsilon) &= F(u_0) + \sqrt{\epsilon} \sum_{l=1}^p \int_0^t dF \left(TR_{(g_s^\epsilon)^{-1}} \mathbb{X}_l(u_s^\epsilon) \right) dB_s^l \\
&\quad + \frac{1}{2} \epsilon \sum_{l=1}^p \int_0^t \breve{\nabla} dF \left(TR_{(g_s^\epsilon)^{-1}} \mathbb{X}_l(u_s^\epsilon), TR_{(g_s^\epsilon)^{-1}} \mathbb{X}_l(u_s^\epsilon) \right) ds \\
&\quad + \frac{1}{2} \epsilon \sum_{l=1}^p \int_0^t dF \left(\breve{\nabla}_{\mathbb{X}_l} \mathbb{X}_l(u_s^\epsilon) + \mathbb{X}_0(u_s^\epsilon) \right) ds.
\end{aligned}$$

By tightness $\tilde{x}_{\frac{t}{\epsilon}}^{\epsilon_n}$ converges in law to a probability measure μ for a sequence ϵ_n. Let $X.$ be the coordinate process on the path space and $\mathcal{G}_t = \sigma\{X_s : 0 \leq s \leq t\}$. Observe that

$$\int_{SO(n)} \langle TR_{g^{-1}} \mathbb{X}_l(ug), H_i(u)\rangle \langle TR_{g^{-1}} \mathbb{X}_l(ug), H_j(u)\rangle \mu_u(dg)$$

has at most quadratic growth since $SO(n)$ is compact and by the same argument $\int_{SO(n)} \sum_{l=1}^p \breve{\nabla} \mathbb{X}_l(\mathbb{X}_l)(ug)\mu_u(dg)$ has linear growth. To identify the limiting process it suffices to show that for all real valued smooth function F on OM with compact support, $\int \left(F(X_t) - F(X_s) - \int_s^t \bar{\mathcal{L}}F(X_r)dr \right) g \, d\mu^\epsilon$ converges to zero where g is any real-valued bounded $\mathcal{G}_s$-measurable function on the Wiener space.

Let z_t^n be a sequence of random variables whose law agrees with that of $\tilde{x}^{\epsilon_n}_{\frac{t}{\epsilon_n}}$ for some sequence ϵ_n and z_t^n converges almost surely. Let g be a $\{z_s^n, s \le t\}$-adapted bounded function. For $t \ge s$,

$$\mathbb{E}g \left(F(z_t^n) - F(z_s^n) - \int_s^t \bar{\mathcal{L}}F(z_r^n)dr \right) = \mathbb{E}\left[g \int_s^t (\mathcal{A}^{\epsilon_n} F - \bar{\mathcal{L}}F)(z_r^n)dr \right] \to 0,$$

where $\mathcal{A}^{\epsilon_n}F$ is given by the bounded variation part in the formula for $F(\tilde{x}_t^\epsilon)$. The convergence holds since $SO(n)$ is compact and also the invariant measure $\mu_{SO(n)}$ for the elliptic left invariant SDE is ergodic. The proof is standard and follows from the Lemma below. See e.g. Hasminskii [14], Papanicolaou-Stroock-Varadhan [23]. □

Lemma 2.6 *Let f be a bounded function with bounded derivative then*

$$\int_s^t \mathcal{A}^\epsilon f(\tilde{x}^\epsilon_{\frac{r}{\epsilon}})dr = \int_s^t \bar{\mathcal{L}} f(\tilde{x}^\epsilon_{\frac{r}{\epsilon}})dr + R(f, \epsilon, s, t)$$

where $\left(\mathbb{E} \sup_{s \le t} |R(f, \epsilon, s, t)|^\beta \right)^{\frac{1}{\beta}} \le C(t)\epsilon^{\frac{1}{3}}$ *for any* $\beta > 1$.

The proof is completely analogous to that of Lemma 3.2 in [18]. In sub-intervals whose length is very small compared to $1/\epsilon$ we consider $\tilde{x}_s^\epsilon$ as constants, and apply the ergodic theorem on each interval. With the size of the sub-intervals chosen correctly, the sum over all sub-intervals of the limits forms a Riemann sum. The proof follows from the Cocycle property of the flows, estimates for the rate of convergence in the ergodic theorem and the regularity of the function $\mathcal{A}^\epsilon f$.

3 Effective Diffusion for the Linearised SDE

We first explain our problem and terminology in terms of an SDE on $\mathbf{R}^d$ with smooth coefficients: $dx_t = \sum_j \sigma_j(x_t) \circ dB_t^j + \sigma_0(x_t)dt$. Suppose that there is a global smooth solution to the SDE. For almost surely all ω, for each initial value $x \in \mathbf{R}^d$ the solution $\phi_t(x, \omega)$ exists for all time and the derivative $(D\phi_t)_x(v)$ of the function $x \mapsto \phi_t(x, \omega)$ exists almost surely for all $v \in \mathbf{R}^d$. It is a solution to the linearised SDE: $dv_t = \sum_j (D\sigma_j)_{\phi_t(x)}(v_t) \circ dB_t^j + (D\sigma_0)_{\phi_t(x)}(v_t)dt$. More generally independent of the existence of a global smooth solution we call the solution to the linearised SDE *the derivative flow* of the original SDE. On a manifold M we must choose a connection to differentiate the vector fields σ_j. The solution with starting point $v \in T_xM$ will be denoted by $T\phi_t(v)$.

We wish to answer the following question: Given a perturbed SDE, what information could we deduce from solution of the un-perturbed linearised SDE? In this section we study the model presented in Example 2.2. We first explain the significance of the derivative flow for ODE's on OM and introduce the notation.

3.1 The Model

For computational simplification we assume the following model. Let $\{e_i\}$ be an orthonormal basis of $\mathbf{R}^n$. Consider the fundamental horizontal vector fields $H_i = H(e_i)$. For $e_0 \in \mathbf{R}^n$ let $H_0 = H(e_0)$ and $\mathcal{L}_1 = \frac{1}{2}\sum_{i=1}^n \mathcal{L}_{H_i}\mathcal{L}_{H_i} + L_{H_0}$. The operator $\mathcal{L}_0 + \epsilon\mathcal{L}_1$ corresponds to the following stochastic differential equation,

$$du_t^\epsilon = \sqrt{\epsilon} H(u_t^\epsilon) \circ db_t + \epsilon H(u_t^\epsilon)(e_0)dt + \sum A_k(u_t^\epsilon) \circ dW_t^k + A_0(u_t^\epsilon)dt \tag{3.1}$$

where $b_t = (b_t^1, \dots, b_t^n)$ is an $\mathbf{R}^n$-valued Brownian motion and $\{W_t^k\}$ are independent 1-dimensional Brownian motions, independent of b_t. Denote by $\phi_t^\epsilon(u)$ the solution with initial point u and let $u_t^\epsilon = \phi_t(u_0)$. We will see later that we should take A_k to be of the form $\sum_j \sigma_{k,j} A_j^*$ where $\sigma_{k,j}$ are constants in which case $A_k(u) = (\sum \sigma_{k,k} A_j)^*$ is a fundamental vertical vector field and we are led to consider

$$\left\{ \begin{array}{ll} du_t^\epsilon & = \sqrt{\epsilon} H(u_t^\epsilon) \circ db_t + \epsilon H(u_t^\epsilon)(e_0)dt + \sum_k A_k^*(u_t^\epsilon) \circ dW_t^k + A_0^*(u_t^\epsilon)dt \\ u_0^\epsilon & = u_0. \end{array} \right. \tag{3.2}$$

for A_k elements of $\mathfrak{so}(n)$.

3.2 Some Notation

Let ∇ be a metric connection with torsion $\mathcal{T}$ and parallel transport denoted by $/\!\!/_.$. Denote by $\frac{DV}{dt}$ the covariant derivative of a vector field V along a path σ, $\frac{DV}{dt} = /\!\!/_t(\sigma)\frac{d}{dt}/\!\!/_t^{-1}(\sigma)$. Let DV_t be the stochastic covariant differential: $DV_t = /\!\!/_t(\sigma)d/\!\!/_t^{-1}(\sigma)$.

To linearise the SDE on the orthonormal frame bundle we must choose a linear connection on OM. As it turns out there is a good choice of such connection which is the trivial connection $\breve{\nabla}$ induced by the global parallelism induced by the canonical 1-form θ and the connection 1-form ϖ, viewing OM as a Riemannian manifold. Denote by $\breve{/\!\!/}_t$, $\breve{D}$ and $\breve{\mathcal{T}}$ the corresponding parallel transport, the stochastic covariant differential and the torsion tensor. The connection $\breve{\nabla}$ has, in general, a non-vanishing torsion $\breve{\mathcal{T}}$.

Let us first consider a vertical fundamental vector field A^* where $A \in \mathfrak{so}(n)$. The solution to $\dot{u}_t = A^*(u_t)$ is $u_t = u_0 \exp(tA)$ which stays in the same fibre as u_0. The derivative flow of $\pi(u_t)$ is constant in t. The horizontal flow is more interesting. For $e \in \mathbf{R}^n$ consider the horizontal vector field $V(u) = H(u)(e)$. Let $\phi_t^e(u)$ be its integral curve with initial value u. Then $\gamma_t^{e,u} = \pi(\phi_t^e(u))$ is a ∇-geodesic, starting from $\pi(u)$ and with initial speed $u^{-1}(e)$. Let $V \in T_uOM$ and define

$$T\phi_t^e(V) = \frac{d}{dr}\Big|_{r=0} \phi_t^e(u_r),$$

where $r \in I \mapsto u_r \in OM$ is a smooth curve with $u_0 = u$ and $\frac{d}{dr}|_{r=0} u_r = V$. Here I is a valid interval of $\mathbf{R}$ containing 0. Then $T\phi_t^e(V)$ is the derivative of ϕ_t^e in the direction of V. Its projection is the Jacobi field $J_t^e(V)$, for the variation of geodesics γ^{e,u_r}, with $J_0^e(V)=V$ and $\frac{D}{dt}|_{t=0}J_t^e(V)=u(\varpi(V)e)+\mathcal{T}(u(e),T\pi(V))$.

3.3 Derivation of the Equations

For the derivation it is simpler to consider the stochastic differential equation

$$du_t=\sum\sigma_l(u_t)\circ dB_t^l+\sigma_0(u_t)dt$$

on OM where σ_l are arbitrary smooth vector fields on OM and B_t^l independent Brownian motions. The SDE has pathwise uniqueness and let $\Phi_t(u)$ be the solution with initial value u and write $u_r=\Phi_r(u_0)$. Let $T\Phi_t(V)$ be its derivative flow with initial value V and write $V_t=T\Phi_t(V_0)$ for $V_0\in T_{u_0}OM$. Since the canonical forms θ and ϖ give an absolute parallelism, it is desirable to obtain equations for $\eta_t:=\theta(V_t)$ and $S_t:=\varpi(V_t)$.

Lemma 3.1 *Let $\eta_t=\theta(T\Phi_t(V_0))$ and $S_t:=\varpi(T\Phi_t(V_0))$ be the horizontal and vertical components of the derivative process. They satisfy the following equations with initial value $\eta_0=\theta(V_0)$, $S_0=\varpi(V_0)$,*

$$\begin{aligned}d\eta_t &= \theta\breve{\nabla}_{\theta^{-1}\eta_t+\varpi^{-1}S_t}\sigma_l\circ dB_t^l+\theta\breve{\nabla}_{\theta^{-1}\eta_t+\varpi^{-1}S_t}\sigma_0 dt\\ &\quad+\theta\breve{\mathcal{T}}\left(\sigma_l(u_t)\circ dB_t^l+\sigma_0(u_t)dt,\theta^{-1}\eta_t+\varpi^{-1}S_t\right)\\ dS_t &= \varpi\breve{\nabla}_{\theta^{-1}\eta_t+\varpi^{-1}S_t}\sigma_l\circ dB_t^l+\varpi\breve{\nabla}_{\theta^{-1}\eta_t+\varpi^{-1}S_t}\sigma_0 dt\\ &\quad+\varpi\breve{\mathcal{T}}\left(\sigma_l(u_t)\circ dB_t^l+\sigma_0(u_r)dt,\theta^{-1}\eta_t+\varpi^{-1}S_t\right)\end{aligned}$$

Proof Let K_s^l be the integral curve of the vector field σ_l and TK_s^l its derivative so

$$\frac{d}{ds}K_s^l=\sigma_l(K_s^l),\qquad \frac{\breve{D}}{ds}TK_s^l=\breve{\nabla}\sigma_l(K_s^l)+\breve{\mathcal{T}}(\sigma_l(K_s^l),TK_s^l).$$

Recall that $\eta_t=\theta(T\Phi_t(V_0))$ and we write $V_t=T\Phi_t(V_0)$. It follows that

$$\theta(V_t)=\theta(V_0)+\int_0^t\frac{d}{ds}\Big|_{s=0}\theta\left(TK_s^l(V_r)\right)\circ dB_r^l+\int_0^t\frac{d}{ds}\Big|_{s=0}\theta\left(TK_s^0(V_r)\right)dr.$$

Since

$$d(\theta V)(\cdot)=\theta\left(\theta^{-1}d(\theta V)(\cdot)+\varpi^{-1}d(\varpi V)(\cdot)-\varpi^{-1}d(\varpi V)(\cdot)\right)=\theta\breve{\nabla}.V,$$

we have

$$\theta(TK_s^l(V_r))=\theta\left(\frac{\breve{D}}{ds}TK_s^l(V_r)\right)=\theta\left(\breve{\nabla}\sigma_l(K_s^l(V_r))+\breve{\mathcal{T}}(\frac{d}{ds}K_s^l(u_r),TK_s^l(V_r))\right).$$

Consequently $\frac{d}{ds}\Big|_{s=0}\theta\left(TK_s^l(V_r)\right)=\theta\left(\breve{\nabla}\sigma_l(V_r)+\breve{\mathcal{T}}(\sigma_l(u_r),V_r)\right)$. It follows that

$$d\eta_t \;=\; \theta\breve{\nabla}\sigma_l(T\Phi_t(V))\circ dB_t^l+\theta\breve{\nabla}\sigma_0(T\Phi_t(V))\circ dt+\theta\breve{T}(\sigma_l(u_r)\circ dB_t^l+\sigma_0(u_r)dt,T\Phi_t(V))$$

Observe that $T\Phi_t(V) = \theta^{-1}\eta_t + \varpi^{-1}S_t$ to obtain the required formula for η_t and the required formula for S_t can be deduced in a similar way. $\square$

Denote by $\Theta : \wedge^2 TOM \to \mathbf{R}^n$ and $\Omega : \wedge^2 TOM \to \mathfrak{so}(n)$ the torsion form and the curvature form.

Proposition 3.2 *The torsion of the canonical direct sum connection $\breve{\mathcal{T}}$ is given by, for $v_1, v_2 \in T_uOM$, hv_1, hv_2 their horizontal components,*

$$\begin{aligned}\breve{\mathcal{T}}(v_1,v_2) = \quad & 2\Omega(hv_1,hv_2)^* + 2H(u)\Theta(hv_1,hv_2) - [\varpi(v_1),\varpi(v_2)]^* \\ & -H(u)\varpi(v_1)\theta(v_2) + H(u)\varpi(v_2)\theta(v_1).\end{aligned}$$

Proof Denote by d exterior differentiation and we use the following convention: $d\psi(v_1,v_2) = L_{v_1}(\psi(V_2)) - L_{v_2}(\psi(V_1)) - \psi([V_1,V_2])(u)$, where ψ is a differential one form, and V_1, V_2 are vector fields with values v_1, v_2 at u. In Kobayashi-Nomizu [16], $d\psi(v_1,v_2) = \frac{1}{2}\left(L_{v_1}(\psi(V_2)) - L_{v_2}(\psi(V_1)) - \psi([V_1,V_2])\right)(u)$. Define the bundle map $X : P \times \mathbf{R}^n \times \mathfrak{so}(n) \to TOM$ by $X_u(e,A) = \theta_u^{-1}(e) + \varpi_u^{-1}(A)$ and $Y_u(v) = (\theta_u(v), \varpi_u(v))$. We consider the canonical direct sum connection $\breve{\nabla}$ on OM as that induced by X in the sense of [10], $\breve{\nabla}_w U = X(u_0)D(Y_\cdot(w)U(\cdot))(v)$ for $w \in T_{u_0}OM$. By formula (2.2.3) there, for $v_1, v_2 \in T_{u_0}OM$,

$$\begin{aligned}\breve{\mathcal{T}}(v_1,v_2) &= X(u_0)dY(v_1,v_2) \\ &= X(u_0)\left(d\varpi(v_1,v_2) + d\theta(v_1,v_2)\right) \\ &= \left(d\varpi(v_1,v_2)\right)^* + H(u_0)\left(d\theta(v_1,v_2)\right).\end{aligned}$$

The required identity follows from the structure equations: $d\varpi = -2\varpi \wedge \varpi + 2\Omega$, $d\theta = -2\varpi \wedge \theta + 2\Theta$. $\square$

Let $\phi_t^\epsilon(\cdot,\omega)$ be the flow of (3.1) and u_t^ϵ the solution with initial value u_0. Let $\breve{D}V_t^\epsilon$ denote $\breve{/\!/}_t(u_\cdot^\epsilon)d\breve{/\!/}_t^{-1}(u_\cdot^\epsilon)V_t^\epsilon$, the stochastic covariant derivative of a stochastic process V_t^ϵ along the path $u_\cdot^\epsilon$. Then

$$\begin{aligned}\breve{D}V_t^\epsilon &= \sqrt{\epsilon}\sum_{i=1}^n \breve{\nabla}H_i(V_t^\epsilon)\circ db_t^i + \sum_k \breve{\nabla}A_k(V_t^\epsilon)\circ dW_t^k + \breve{\nabla}A_0(V_t^\epsilon)dt + \breve{\mathcal{T}}\left(\circ du_t^\epsilon, V_t^\epsilon\right) \\ &= \sum_k \breve{\nabla}A_k(V_t^\epsilon)\circ dW_t^k + \breve{\nabla}A_0(V_t^\epsilon)dt + \breve{\mathcal{T}}\left(\circ du_t^\epsilon, V_t^\epsilon\right)\end{aligned}$$

is the linearised SDE to (3.1). Let $T\phi_t^\epsilon(V)$ denote the solution with initial speed $V \in VT_{u_0}OM$. Let $\eta_t^\epsilon = \theta(T\phi_t^\epsilon(V))$ and $S_t^\epsilon = \varpi(T\phi_t^\epsilon(V))$. Then $(u_t^\epsilon)\eta_t^\epsilon = T\pi(T\phi_t^\epsilon(V))$ is the derivative flow of $\pi(u_t^\epsilon)$. Write $A_k(u) = \sum \sigma_{j,k}(u)A_j^*(u)$ where $A_j \in \mathfrak{so}(n)$.

Lemma 3.3 *Consider the model (3.1). The triple $(u_t^\epsilon, \eta_t^\epsilon, S_t^\epsilon)$ is a Markov process on $OM \times \mathbf{R}^n \times \mathfrak{so}(n)$ satisfying:*

$$\begin{aligned}du_t^\epsilon &= \sqrt{\epsilon}H(u_t^\epsilon)\circ db_t + \epsilon H(u_t^\epsilon)(e_0)dt + \sigma_{j,k}(u_t^\epsilon)A_j^*(u_t^\epsilon)\circ dW_t^k + \sigma_{j,0}(u_t^\epsilon)A_j^*(u_t^\epsilon)dt \\ d\eta_t^\epsilon &= \sqrt{\epsilon}2\Theta(H(u_t^\epsilon)\circ db_t, \theta^{-1}\eta_t^\epsilon) + \sqrt{\epsilon}S_t^\epsilon\circ db_t - A_k\eta_t^\epsilon\circ dW_t^k \\ &\quad +\epsilon 2\Theta(H(u_t^\epsilon)(e_0), \theta^{-1}\eta_t^\epsilon)dt - A_0\eta_t^\epsilon dt + \epsilon S_t^\epsilon e_0 dt \\ dS_t^\epsilon &= \sqrt{\epsilon}2\Omega(H(u_t^\epsilon)\circ db_t, \theta^{-1}\eta_t^\epsilon) - \sigma_{j,k}(u_t^\epsilon)[A_j, S_t^\epsilon]\circ dW_t^k + \epsilon 2\Omega(H_0(u_t^\epsilon)(e_0), \theta^{-1}\eta_t^\epsilon)dt \\ &\quad -\sigma_{j,0}(u_t^\epsilon)[A_j, S_t^\epsilon]dt + D\sigma_{j,k}(H(u_t^\epsilon)\eta_t^\epsilon + (S_t^\epsilon)^*)A_j\circ dW_t^k + D\sigma_{j,0}(H(u_t^\epsilon)\eta_t^\epsilon + (S_t^\epsilon)^*)A_j dt.\end{aligned}$$

Furthermore assume that $\sigma, |D\sigma_{j,k}|$ *are bounded and the manifold is stochastically complete, there is a global solution of the system of SDEs. If furthermore the torsion* $\mathcal{T}$ *is bounded and skew symmetric, and* $|\Omega|$ *and* $|\sum \nabla_{H_i}\Omega(H_i,\cdot)|$ *are bounded, then* $\mathbb{E}\sup_{r\le t}(|\eta_r^\epsilon|^2+|S_r^\epsilon|^2)$ *is finite for all* $t>0$ *and the map from* $u\mapsto \phi_t^\epsilon(u)$ *is smooth.*

Proof The equations follow from Lemma 3.1 and Proposition 3.2, taking into account that $\breve\nabla H_i=0$ and $\breve\nabla A_k^*=0$ and that

$$\begin{aligned} d\eta_t^\epsilon &= \theta\breve{\mathcal{T}}(\circ du_t^\epsilon, \theta^{-1}\eta_t^\epsilon+\varpi^{-1}S_t^\epsilon),\\ dS_t^\epsilon &= \varpi\breve{\mathcal{T}}(\circ du_t^\epsilon, \theta^{-1}\eta_t^\epsilon+\varpi^{-1}S_t^\epsilon)+D\sigma_{j,k}(T\phi_t^\epsilon(V))A_j\circ dW_t^k+D\sigma_{j,0}(T\phi_t^\epsilon(V))A_j dt. \end{aligned}$$

Let $u_t^\epsilon=\tilde x_t^\epsilon g_t^\epsilon$ where $\tilde x_t^\epsilon$ is the horizontal lift of x_t with initial value u_0 and $g_t^\epsilon\in SO(n)$. Then

$$dg_t^\epsilon=\sigma_{j,k}(u_t^\epsilon)A_j g_t^\epsilon\circ dW_t^k+\sigma_{j,0}(u_t^\epsilon)A_j g_t^\epsilon dt.$$

If $\sigma_{j,k}(u)$ are independent of u, g_t^ϵ is independent of ϵ. In all cases there is a global solution g_t^ϵ as $SO(n)$ is a compact manifold. If x_t^ϵ exists for all time then so are $u_t^\epsilon=\tilde x_t g_t^\epsilon$, η_t^ϵ and S_t^ϵ as the equations for the latter two are linear and the coefficients are smooth in u_t^ϵ. Since

$$\begin{aligned} dx_t^\epsilon &= \sqrt{\epsilon}u_t^\epsilon\circ db_t+\epsilon u_t^\epsilon e_0\,dt\\ &= \sqrt{\epsilon}\tilde x_t^\epsilon g_t^\epsilon\circ db_t+\epsilon\tilde x_t^\epsilon g_t^\epsilon e_0\,dt, \end{aligned}$$

the geometric Brownian motion on M does not explode, and the drift term in the SDE for x_t^ϵ is bounded, x_t^ϵ does not explode by Girsanov transform, taking into account that $e^{\int_0^t\langle e_0,db_s\rangle-\frac{1}{2}|e_0|^2t}$ is a martingale. Since $\mathcal{T}$ is skew symmetric, we have the following equations:

$$\begin{aligned} d|\eta_t^\epsilon|^2 &= 2\sqrt{\epsilon}\langle S_t^\epsilon e_l,\eta_t^\epsilon\rangle db_t^l+2\epsilon\langle S_t^\epsilon e_0,\eta_t^\epsilon\rangle\,dt+\epsilon\langle 2\Omega(H(u_t^\epsilon)e_l,\theta^{-1}\eta_t^\epsilon)e_l,\eta_t^\epsilon\rangle\,dt\\ &\quad +\epsilon\langle S_t^\epsilon e_l, 2\Theta(H(u_t^\epsilon)e_l,\theta^{-1}\eta_t^\epsilon)\rangle\,dt+\epsilon\,|S_t^\epsilon|^2\,dt\\ &= 2\sqrt{\epsilon}\langle S_t^\epsilon e_l,\eta_t^\epsilon\rangle db_t^l+2\epsilon\langle S_t^\epsilon e_0,\eta_t^\epsilon\rangle dt-\epsilon\,\mathrm{Ric}(u_t^\epsilon\eta_t^\epsilon,u_t^\epsilon\eta_t^\epsilon)dt\\ &\quad +\epsilon\langle u_t^\epsilon\eta_t^\epsilon,\mathcal{T}(u_t^\epsilon e_l,u_t^\epsilon S_t^\epsilon e_l)\rangle dt+\epsilon\,|S_t^\epsilon|^2\,dt\\ d|S_t^\epsilon|^2 &= 2\epsilon\langle S_t^\epsilon, 2\Omega(H_0(u_t^\epsilon)(e_0),\theta^{-1}\eta_t^\epsilon)\rangle dt+2D\sigma_{j,0}(T\phi_t^\epsilon(V))\,\langle S_t^\epsilon,A_j\rangle dt\\ &\quad +2\sqrt{\epsilon}\langle S_t^\epsilon,2\Omega(H(u_t^\epsilon)\,db_t,\theta^{-1}\eta_t^\epsilon)\rangle+\epsilon|R(u_t^\epsilon e_l,u_t^\epsilon\eta_t^\epsilon)|^2dt\\ &\quad +\epsilon\langle S_t^\epsilon, 2\sum_{l=1}^n\nabla_{H_l(u_t^\epsilon)}\Omega(H_l(u_t^\epsilon),\theta^{-1}\eta_t^\epsilon)\rangle\,dt+\epsilon\langle S_t^\epsilon,(u_t^\epsilon)^{-1}R\left(u_t^\epsilon e_l,\mathcal{T}(u_t^\epsilon e_l,u_t^\epsilon\eta_t^\epsilon)+u_t^\epsilon S_t^\epsilon e_l\right)\rangle\,dt\\ &\quad +2D\sigma_{j,k}(T\phi_t^\epsilon(V))\langle S_t^\epsilon,A_j\rangle dW_t^k+\sum_{k,j}|D\sigma_{j,k}(T\phi_t^\epsilon(V))|^2\,dt\\ &\quad -D\sigma_{j,k}\left(H(u_t^\epsilon)(A_k\eta_t^\epsilon)+\sigma_{i,k}[A_i,S_t^\epsilon]^*\right)\,\langle S_t^\epsilon,A_j\rangle dt\\ &\quad +D\sigma_{i,k}\left(H(u_t^\epsilon)\eta_t^\epsilon+(S_t^\epsilon)^*\right)D\sigma_{j,k}(A_i^*)\,\langle S_t^\epsilon,A_j\rangle dt. \end{aligned}$$

The required moment estimates easily follow. Similar estimates can now be made for the pth moment, $p>1$, and the required continuity of ϕ_t^ϵ in the initial data follows from Theorem 4.1 in [17]. $\square$

3.4 Effective Diffusions Associated to the Derivative Flow

Let us first analyse the unperturbed system which would be used as basis for analysis of the perturbed system. Denote $\langle A, B\rangle$ by the inner product of two matrices in $\mathfrak{so}(n)$, considered as a subspace of $\mathbf{R}^{n^2}$. So $\langle A, B\rangle = \mathrm{tr}\, B^\tau A$ where tr stands for the trace of a matrix.The inner product is adjoint invariant and hence induce a bi-invariant Riemannian metric on $SO(n)$ with the corresponding normalised volume measure which we denote by $\mu_{SO(n)}$. Denote by Conj the group homomorphism given by conjugation so $\mathrm{Conj}(O)M = OMO^{-1}$ and denote $\mathrm{ad}(g)(O) = [g, O]$ for $g \in \mathfrak{so}(n)$.

Lemma 3.4 *We consider the model (3.2) and recall that $A_k \in \mathfrak{so}(n)$. Let K_t be the solution of $dK_t = \sum_k K_t A_k dW_t^k + K_t A_0 dt$ with K_0 the unit matrix. The unperturbed system,*

$$\begin{array}{rcl} du_t & = & \sum_k A_k^*(u_t) \circ dW_t^k + A_0^*(u_t)dt \\ D\eta_t & = & -\sum A_k \eta_t \circ dW_t^k - A_0 \eta_t dt \\ DS_t & = & -\sum [A_k, S_t] \circ dW_t^k - [A_0, S_t]dt, \end{array}$$

does not explode. The OM-valued process u_t is elliptic on each fibre $\pi^{-1}(\pi(u_0))$ and $u_t = u_0 K_t$. In the case of $\{A_k\}$ forming an orthonormal basis of $\mathfrak{so}(n)$ and $A_0 = 0$, K_t is the left invariant Brownian motion on $SO(n)$. The norm of the derivative flow $|T\phi_t(V)|^2 = |S_t|^2 + |\eta_t|^2$ is a conserved quantity, and so are the eigenvalues of S_t. The eigenvectors v_t of S_t evolve according to the SDE on the complex space $\mathcal{C}^n$:

$$dv_t = -\sum_k A_k v_t \circ\ dW_t^k - A_0 v_t dt. \tag{3.3}$$

Furthermore $S_t = \mathrm{Conj}(K_t^{-1})(S_0)$ and $\eta_t = K_t^{-1}\eta_0$.

Proof Letting $u_t = u_0 g_t$ then $\frac{d}{dt} u_0 g_t = (g_t^{-1}\dot g_t)^*(u_0 g_t)$ and $\varpi_{u_t}(\frac{d}{dt} u_0 g_t) = g_t^{-1}\dot g_t$. From $\varpi(du_t) = \varpi(A_k^*(u_t)) \circ dW_t^k + \varpi(A_0^*(u_t))dt = A_k \circ dW_t^k + A_0(u_t)dt$, it follows that $dg_t = \sum g_t A_k \circ dW_t^k + g_t A_0 dt$. The process g_t is solution to an elliptic SDE, as action by left multiplication is free, proving the first assertion. Note that

$$\begin{array}{rcl} d\,\mathrm{trace}\, S_t^n & = & \mathrm{trace}\, dS_t^n = n\, \mathrm{trace}(S_t^{n-1} \circ dS_t) \\ & = & -n\, \mathrm{trace}(S_t^{n-1}[A_k, S_t]) \circ dW_t^k = 0, \end{array}$$

and S_t is isospectral. Let $v_0 \in \mathcal{C}^n$ be an eigenvector of S_0 and v_t a solution to (3.3). A vector v_t in $\mathbf{R}^n$ is an eigenvector of S_t corresponding to λ if and only if $S_0 v_0 = \lambda v_0$ and $\lambda dv_t = d(S_t v_t)$. On one hand we have:

$$\begin{array}{rcl} d(S_t v_t) & = & -[A_k, S_t]v_t \circ dW_t^k - [A_0, S_t]v_t dt - S_t A_k v_t \circ dW_t^k - S_t A_0 v_t dt \\ & = & -\sum_k A_k S_t v_t \circ dW_t^k - A_0 S_t v_t dt. \end{array}$$

On the other hand $d(\lambda v_t) = -\sum_k A_k \lambda v_t \circ dW_t^k - \lambda A_0 v_t dt$. Both $S_t v_t$ and λv_t satisfy the same equation with the same initial value and hence are the same. Furthermore

$$\begin{array}{rcl} d\,\mathrm{Conj}(K_t^{-1})S_0 & = & \mathrm{Conj}(K_t^{-1})\mathrm{ad}(K_t \circ dK_t^{-1})(S_0) \\ & = & -\mathrm{Conj}(K_t^{-1})\mathrm{ad}\left(K_t \sum_k A_k K_t^{-1} \circ dW_t^k + K_t A_0 K_t^{-1})dt\right)(S_0) \\ & = & -\sum_k [A_k, \mathrm{Conj}(K_t^{-1})(S_0)] \circ dW_t^k - [A_0, \mathrm{Conj}(K_t^{-1})(S_0)]dt. \end{array}$$

We used the elementary identity, $[\text{Conj}(O)(M), \text{Conj}(O)(N)] = \text{Conj}(O)[M, N]$ for $O, M, N \in SO(n)$. Hence $S_t = \text{Conj}(K_t^{-1})(S_0)$, similarly $\eta_t = K_t^{-1}\eta_0$. □

The conserved quantities we observed in the lemma, the eigenvalues of S_t, the norm of η_t and the projection of u_t, are all of the conserved quantities of the system. Note that if A_1 and A_2 have the same spectrum they are related by $A_1 = \text{Conj}(B)A_2$ for some $B \in O(n)$ and $SO(n)$ has a maximal torus whose elements are of the form $\mathfrak{so}(2)$ piles on the diagonal and every element of $SO(n)$ is conjugate to an element of the torus.

Consider $SO(n)$ as a Lie transformation group on the Lie group $SO(n)$ on the right. The action is conjugation

$$(g, O) \in SO(n) \times SO(n) \mapsto \text{Conj}_O(g^{-1}) \in SO(n).$$

Denote by X_A the left invariant vector field induced by $A \in \mathfrak{so}(n)$ and $X_A(O) = -[A, O]$ and the SDE for S_t can be considered as a left invariant SDE on $SO(n)$.

The vector fields $X_{A_{i,j}}$ correspond to an orthonormal basis of $\mathfrak{so}(n)$ may vanish at some point as the action is not free on $\mathfrak{so}(n)$ and at different points A the dimension of $\text{Span}\{X_{A_{i,j}}\}$ may vary. For any $g \in SO(n)$, the Lie-brackets of the vector fields generated by A and B from $\mathfrak{so}(n)$ are given by $[\text{ad}(A), \text{ad}(B)](g) = \text{ad}([A, B])(g) = [[A, B], g]$. Since $[A_{i,j}, A_{k,l}]$ is either 0 or of the form $A_{m,n}$ where m, n equal to one of the four numbers: i, j, k, l, the set of vector fields generated by $A_{i,j}$ is integrable and generate a foliation which is of *non-constant rank* in general.

Let $\phi_t^\epsilon(u)$ be the solution flow of (3.2). Recall that for $u_0 \in OM$ and $V_0 \in T_{u_0}M$ we let $S_t^\epsilon = \varpi(T\phi_t^\epsilon(V_0))$ and $\eta_t^\epsilon = \theta(T\phi_t^\epsilon(V_0))$, the vertical and the horizontal part of the derivative process $\mathcal{T}\phi_t^\epsilon(V_0)$.

Theorem 3.5 *Consider model (3.2). Assume the curvature R and the torsion $\mathcal{T}$ together with their first derivatives are bounded. Let K_t be the solution of $dK_t = \sum_k K_t A_k dW_t^k + K_t A_0 dt$ with K_0 the unit matrix. Assume that $x_t^\epsilon = \pi(u_t^\epsilon)$ does not explode. Let $\tilde{x}_t^\epsilon$ be the horizontal lift of x_t^ϵ through u_0.*

The stochastic process $(\tilde{x}_{\frac{t}{\epsilon}}, \text{Conj}(K_{\frac{t}{\epsilon}})S_{\frac{t}{\epsilon}}^\epsilon, (K\eta^\epsilon)_{\frac{t}{\epsilon}})$ converges weakly to the law of the solution of the SDEs:

$$\begin{aligned} du_t &= H(u_t) \circ d\tilde{b}_t, \\ dS_t &= (u_t)^{-1} R(u_t \circ d\tilde{b}_t, u_t\eta_t)u_t, \\ d\eta_t &= (u_t)^{-1}\mathcal{T}(u_t \circ d\tilde{b}_t, u_t\eta_t) + S_t \circ d\tilde{b}_t. \end{aligned}$$

In particular the horizontal part of u_t^ϵ converges in law to the Horizontal Brownian motion, and the modified derivative flow converges to the stochastic Jacobi equation.

Proof Define the $\mathfrak{so}(n)$-valued stochastic process: $\tilde{S}_t^\epsilon = \text{Conj}(K_t)S_t^\epsilon$, with $\tilde{S}_0^\epsilon = S_0$ and $\tilde{\eta}_t^\epsilon = K_t\eta_t^\epsilon$. By Lemma 3.3,

$$\begin{aligned} d\tilde{S}_t^\epsilon &= \text{Conj}(K_t)\left(\text{ad}(K_t^{-1} \circ dK_t)(S_t^\epsilon)\right) + \text{Conj}(K_t)\left(-[A_k, S_t^\epsilon] \circ dW_t^k - [A_0, S_t^\epsilon]dt\right) \\ &\quad + \text{Conj}(K_t)\left(\sqrt{\epsilon}2\Omega(H(u_t^\epsilon) \circ db_t + H(u_t^\epsilon)(e_0)dt, \theta^{-1}\eta_t^\epsilon)\right) \\ &= \sqrt{\epsilon}\,\text{Conj}(K_t)2\Omega(H(u_t^\epsilon) \circ db_t, \theta^{-1}\eta_t^\epsilon) + \epsilon\,\text{Conj}(K_t)2\Omega(H(u_t^\epsilon)e_0, \theta^{-1}\eta_t^\epsilon) \\ &= \sqrt{\epsilon}(\tilde{x}_t^\epsilon)^{-1} R_{x_t^\epsilon}(\tilde{x}_t^\epsilon K_t \circ db_t, \tilde{x}_t^\epsilon\tilde{\eta}_t^\epsilon)\tilde{x}_t^\epsilon + \epsilon(\tilde{x}_t^\epsilon)^{-1} R_{x_t^\epsilon}(\tilde{x}_t^\epsilon K_t e_0, \tilde{x}_t^\epsilon\tilde{\eta}_t^\epsilon)\tilde{x}_t^\epsilon dt \end{aligned}$$

Since $R(v_1, v_2)w = 2u[\Omega(v_1^*, v_2^*)(u^{-1}w)]$ and $\mathcal{T}(v_1, v_2) = u(2\Theta(v_1^*, v_2^*))$ where v_i^* are any vectors whose projections to TM are $v_i \in T_{\pi(u)}M$, so

$$\begin{array}{rcl} d\tilde{\eta}_t^\epsilon &=& \sqrt{\epsilon}K_t 2\Theta(H(u_t^\epsilon)\circ db_t, \theta^{-1}\eta_t^\epsilon) + \sqrt{\epsilon}K_t S_t^\epsilon \circ db_t \\ && +\epsilon K_t 2\Theta(H(u_t^\epsilon)(e_0), \theta^{-1}\eta_t^\epsilon)dt + \epsilon K_t S_t^\epsilon e_0 dt \\ &=& \sqrt{\epsilon}(\tilde{x}_t^\epsilon)^{-1}\mathcal{T}_{x_t^\epsilon}(\tilde{x}_t^\epsilon K_t \circ db_t, \tilde{x}_t^\epsilon\tilde{\eta}_t^\epsilon) + \sqrt{\epsilon}\tilde{S}_t^\epsilon K_t \circ db_t \\ && +\epsilon(\tilde{x}_t^\epsilon)^{-1}\mathcal{T}_{x_t^\epsilon}(\tilde{x}_t^\epsilon K_t e_0, \tilde{x}_t^\epsilon\tilde{\eta}_t^\epsilon)dt + \epsilon\tilde{S}_t^\epsilon K_t e_0 dt. \end{array}$$

Since the SDE for K_t^{-1} is driven by W_t^k, in the above equations $K_t \circ db_t = \circ(K_t db_t)$. Note that $K_t db_t \overset{law}{=} db_t$. Let $\tilde{b}_t = \int_0^{\frac{t}{\epsilon}} K_s db_s$, $\bar{S}_t^\epsilon = \tilde{S}_{\frac{t}{\epsilon}}^\epsilon$, $y_t^\epsilon = x_{\frac{t}{\epsilon}}^\epsilon$, $\tilde{y}_t^\epsilon = \tilde{x}_{\frac{t}{\epsilon}}^\epsilon$, $\bar{\eta}_t^\epsilon = \tilde{\eta}_{\frac{t}{\epsilon}}^\epsilon$. We see that

$$\begin{array}{rcl} d\tilde{y}_t^\epsilon &=& H(y_t^\epsilon)\circ d\tilde{b}_t + H(y_t^\epsilon)(K_{\frac{t}{\epsilon}}e_0)dt \\ d\bar{S}_t^\epsilon &=& (\tilde{y}_t^\epsilon)^{-1}R_{y_t^\epsilon}(\tilde{y}_t^\epsilon \circ d\tilde{b}_t, \tilde{y}_t^\epsilon\bar{\eta}_t^\epsilon)\tilde{y}_t^\epsilon + (\tilde{y}_t^\epsilon)^{-1}R_{y_t^\epsilon}(\tilde{y}_t^\epsilon K_{\frac{t}{\epsilon}}e_0, \tilde{y}_t^\epsilon\bar{\eta}_t^\epsilon)\tilde{y}_t^\epsilon dt \\ d\bar{\eta}_t^\epsilon &=& (\tilde{y}_t^\epsilon)^{-1}\mathcal{T}_{y_t^\epsilon}(\tilde{y}_t^\epsilon \circ d\tilde{b}_t, \tilde{y}_t^\epsilon\bar{\eta}_t^\epsilon) + \bar{S}_t^\epsilon \circ d\tilde{b}_t \\ && +(\tilde{y}_t^\epsilon)^{-1}\mathcal{T}_{y_t^\epsilon}(\tilde{y}_t^\epsilon K_{\frac{t}{\epsilon}}e_0, \tilde{y}_t^\epsilon\bar{\eta}_t^\epsilon)dt + \bar{S}_t^\epsilon K_{\frac{t}{\epsilon}}e_0 dt. \end{array}$$

The stochastic integral terms are invariant and we do not have to transform them into Itô form. The conditions on the curvature and the torsion implies that the law of $(\tilde{y}_t, \bar{S}_t^\epsilon, \bar{\eta}_t^\epsilon)$ is tight. Estimates on the modulus of continuity of $\tilde{y}_t^\epsilon$ is the same as that of y_t^ϵ which is a Brownian motion with a bounded drift for which there are nice estimates, see e.g. [17]. To see that the limiting law is as stated in the theorem, we note that the convergence of $\tilde{y}_t^\epsilon$ is shown in Proposition 2.5 and the remaining two terms are flat space valued and can be deduced from Lemma 2.6. The limiting equations are:

$$\begin{array}{rcl} du_t &=& H(u_t)\circ d\tilde{b}_t + Z(u_t)dt, \\ dS_t &=& (u_t)^{-1}R(u_t \circ d\tilde{b}_t, u_t\eta_t)u_t + M(u_t, \eta_t)dt, \\ d\eta_t &=& (u_t)^{-1}\mathcal{T}(u_t \circ d\tilde{b}_t, u_t\eta_t) + S_t \circ d\tilde{b}_t + F(u_t, \eta_t, S_t)dt. \end{array}$$

where

$$\begin{array}{rcl} Z(u) &=& \int_{SO(n)} H(u)(Ke_0)\, d\mu(K), \\ M(u,\eta) &=& \int u^{-1}R_{\pi(u)}(uKe_0, u\eta)u\, d\mu(K), \\ F(u,\eta,S) &=& \int_{SO(n)}[u^{-1}\mathcal{T}(uKe_0, u\eta) + SKe_0]\, d\mu(K). \end{array}$$

Finally observe that the average of the vector Ke_0 with respect to $\mu(K)$ vanishes and by linearity $Z \equiv 0$, $M \equiv 0$ and $F \equiv 0$. □

The stability of the limiting system is indeed well understood. In this case

$$\begin{array}{rcl} d|\eta_t|^2 &=& 2\langle S_t e_l, \eta_t\rangle db_t^l - \mathrm{Ric}(u_t\eta_t, u_t\eta_t)dt + \langle u_t\eta_t, \mathcal{T}(u_t e_l, u_t S_t e_l)\rangle dt + |S_t|^2\, dt \\ d|S_t|^2 &=& 2\langle S_t, 2\Omega(H(u_t)\, db_t, \theta^{-1}\eta_t)\rangle + |R(u_t e_l, u_t\eta_t)|^2\, dt \\ && +\langle S_t, 2\nabla_{H_l(u_t)}\Omega(H_l(u_t), \theta^{-1}\eta_t)\rangle\, dt + \langle u_t S_t, R(u_t e_l, \mathcal{T}(u_t e_l, u_t\eta_t))\rangle\, dt \\ && +\langle u_t S_t, R(u_t e_l, u_t S_t e_l)\rangle\, dt. \end{array}$$

In the case of ∇ being the Levi-Civita connection of constant sectional curvature K, $\langle R(X,Y)Z, W\rangle = K\langle Z, Y\rangle\langle X, W\rangle - K\langle Z, X\rangle\langle Z, W\rangle$ and $\mathrm{Ric}(X,Y) =$

$(n-1)K\langle U, V\rangle$ so $|R(u_te_l, u_t\eta_t)|^2 = 2K\,\mathrm{Ric}(u_t\eta_t, u_t\eta_t)$, $\langle u_tS_t, R(u_te_l, u_tS_te_l)\rangle = -2K|S_t|^2$ and

$$\begin{aligned} d[|S_t|^2+|\eta_t|^2] &= 2\langle S_te_l, \eta_t\rangle db_t^l + 2\langle S_t, 2\Omega(H(u_t)\,db_t, \theta^{-1}\eta_t)\rangle \\ &\quad +(2K-1)\,\mathrm{Ric}(u_t\eta_t, u_t\eta_t)dt + (1-2K)|S_t|^2dt \\ &= \text{martingale} + (2K-1)(n-1)K|\eta_t|^2\,dt + (1-2K)|S_t|^2dt. \end{aligned}$$

We see that $\frac{1}{t}\log\mathbb{E}|T\Phi_t(V)|^2 > 0$, in the case of $K < 0$. For more exploration of moment stability see [19].

Letting $c_t = |S_t|^2 + |\eta_t|^2$, the Lyapunov exponent is defined by $\lim_{t\to\infty}\frac{1}{t}\log c_t$ if it exists. To estimate the Lyapunov exponent one may first obtain a formula for $\log c_t^2$ and then use the martingale convergence theorem. For this we observe that $2\langle S_t, 2\Omega(H(u_t)\,db_t, \theta^{-1}\eta_t)\rangle = 4K\langle S_t\eta_t, db_t\rangle$ and $d\langle c^2, c^2\rangle_t = -\frac{2(1-2K)|S_t\eta_t|^2}{c_t^2}dt$. It follows that $\frac{d}{dt}(|S_t|^2 + 2K|\eta_t|^2) = 0$. In the case of K is strictly positive the manifold is compact, it was shown in Carverhill-Elworthy [8] that the Lyapunov spectrum is well defined and the sum of the Lyapunov exponents is zero . The case of positive but not strictly positive sectional curvature is treated in Liao [20]. See Baxendale [5] for a case study of an SDE on the orthonormal frame bundle of negative curvature and Carverhill-Elworthy [8] for a detailed analysis of surfaces of constant curvature. The general case of non-vanishing torsion is yet to be treated.

3.5 Two Related Models

Below we point out two further cases of interest. However due to the similarity in the methodology we omit the computational detail.

Case 1 (The righ invariance case of Example 2.1). Consider a family of right invariant vector fields $\{\mathbb{X}_l, l=0,1,\dots,m\}$ such that $\{\mathbb{X}_1,\dots,\mathbb{X}_m\}$ spans $HTOM$ and the SDE

$$du_t^\epsilon = \sqrt{\epsilon}\mathbb{X}_l(u_t^\epsilon)\circ dB_t^l + \epsilon\mathbb{X}_0(u_t^\epsilon)dt + A_k^*(u_t^\epsilon)\circ dW_t^k + A_0^*(u_t^\epsilon)dt$$

In this case define $\mathbb{X} : OM\times\mathbf{R}^m \to HTOM$ by $\mathbb{X}(u)(f_l) = \mathbb{X}_l(u)$ where $\{f_l\}$ is an o.n.b. of $\mathbf{R}^m$, hence $\mathbb{X}(u)(e) := \sum\mathbb{X}_l(u)\langle e, f_l\rangle$. Let $\mathbb{Y}_u(v) = (\mathbb{X}^*(u), 0)$ for $\mathbb{X}^*(u)$ the adjoint of $\mathbb{X}(u)$. Instead of the canonical direct sum connection on OM we employ the connection induced by $\mathbb{X}$ and ϖ, whose torsion is

$$\breve{\mathcal{T}}(V_1, V_2) = \mathbb{X}d\mathbb{Y}(hV_1, hV_2) + (d\varpi(V_1, V_2))^*$$

Define $X_l = \pi_*(\mathbb{X}_l)$. Since the projection of $\mathbb{X}(d\mathbb{Y})$ is the torsion $\mathcal{T}$ of ∇ and $\breve{\nabla}\mathbb{X}_l = 0, l = 1,\dots,m$, we see that

$$\begin{aligned} d\eta_t^\epsilon &= \theta\breve{\mathcal{T}}(\circ du_t^\epsilon, T\phi_t^\epsilon(V_0)) = \theta\,(\mathbb{X}d\mathbb{Y})(h\circ du_t^\epsilon, \theta^{-1}\eta_t^\epsilon) \\ &= \sqrt{\epsilon}(u_t^\epsilon)^{-1}\mathcal{T}(X_l(x_t^\epsilon)\circ dB_t^l, u_t^\epsilon\eta_t^\epsilon) + \epsilon(u_t^\epsilon)^{-1}\mathcal{T}(T\pi(\mathbb{X}_0)(u_t^\epsilon), u_t^\epsilon\eta_t^\epsilon)\,dt \\ dS_t &= \varpi\breve{\mathcal{T}}(\circ du_t^\epsilon, T\phi_t^\epsilon(V_0)) = d\varpi(\circ du_t^\epsilon, T\phi_t^\epsilon(V_0)) \\ &= -[A_k, S_t]\circ dW_t^k - [A_0, S_t]dt + 2\sqrt{\epsilon}\Omega(\mathbb{X}_l(u_t^\epsilon)\circ db_t^l + \epsilon\mathbb{X}_0(u_t^\epsilon)dt, \theta^{-1}\eta_t^\epsilon). \end{aligned}$$

Note that the equation for η_t^ϵ is autonomous given u_t^ϵ. Furthermore let $x_t^\epsilon = \pi(u_t^\epsilon)$ and $u_t^\epsilon = \tilde{x}_t^\epsilon g_t^\epsilon$ then

$$dg_t^\epsilon = \sum A_k g_t^\epsilon \circ dW_t^k + A_0 g_t^\epsilon dt$$

Let $\tilde{\eta}_t^\epsilon = (g_t^\epsilon)^{-1}\eta_t^\epsilon$ and $\tilde{S}_t^\epsilon = \text{Conj}(g_t^{-1})S_t$, then

$$\begin{aligned}
d\tilde{x}_t^\epsilon &= \sqrt{\epsilon}\sum \mathbb{X}_l(\tilde{x}_t^\epsilon) \circ dB_t^l + \epsilon T R_{g_t^\epsilon}\mathbb{X}_0(\tilde{x}_t^\epsilon)dt \\
d\tilde{\eta}_t^\epsilon &= \sqrt{\epsilon}(\tilde{x}_t^\epsilon)^{-1}\mathcal{T}(X_l(x_t^\epsilon) \circ dB_t^l, \tilde{x}_t^\epsilon\tilde{\eta}_t^\epsilon) + \epsilon(\tilde{x}_t^\epsilon)^{-1}\mathcal{T}(T\pi(\mathbb{X}_0)(\tilde{x}_t^\epsilon g_t^\epsilon), \tilde{x}_t^\epsilon\tilde{\eta}_t^\epsilon)dt \\
d\tilde{S}_t^\epsilon &= \sqrt{\epsilon}(\tilde{x}_t^\epsilon)^{-1}R(X_l(x_t^\epsilon) \circ dB_t^l, \tilde{x}_t^\epsilon\tilde{\eta}_t^\epsilon)\tilde{x}_t^\epsilon + \epsilon(\tilde{x}_t^\epsilon)^{-1}R(T\pi\mathbb{X}_0(\tilde{x}_t^\epsilon g_t^\epsilon), \tilde{x}_t^\epsilon\tilde{\eta}_t^\epsilon)\tilde{x}_t^\epsilon\, dt.
\end{aligned}$$

Then $(\tilde{x}_{\frac{t}{\epsilon}}^\epsilon, \tilde{\eta}_{\frac{t}{\epsilon}}^\epsilon, \tilde{S}_{\frac{t}{\epsilon}}^\epsilon)$ converges in law to that of the following SDE:

$$\begin{aligned}
du_t &= \sum \mathbb{X}_l(u_t) \circ d\tilde{B}_t^l + \int_{SO(N)} T R_g \mathbb{X}_0(u_t)d\mu(g)\, dt \\
d\eta_t &= u_t^{-1}\mathcal{T}(X_l(\pi(u_t)) \circ d\tilde{B}_t^l, u_t\eta_t) + \int_{SO(n)} u_t^{-1}\mathcal{T}(T\pi(\mathbb{X}_0)(u_t g)\mu(dg), u_t\eta_t)\, dt \\
dS_t &= u^{-1}R(X_l(\pi(u_t)) \circ d\tilde{B}_t^l, u_t\eta_t)u_t + \int_{SO(n)} u_t^{-1}R(T\pi\mathbb{X}_0(u_t g), u_t\eta_t)u_t d\mu(g)\, dt,
\end{aligned}$$

where $\tilde{B}_t^l$ is a family of independent Brownian motions. For computation of the Lyapunov exponent of the horizontal flow over Lie groups and symmetric spaces see [20].

Case 2 (Model in Example 2.4). When the curvature $R \equiv 0$ we may consider this model. The horizontal tangent bundle is integrable and there are sub-manifolds $\{N_u, u \in OM\}$ of OM whose tangent spaces are exactly $HTOM$ restricted to N_u. Let μ_u be the volume measure on N_u for the induced Riemannian metric. The movement governed by $du_t = H(u_t) \circ db_t + Z_0(u_t)dt$ has no vertical component and hence the vertical part S_t of its derivative process is a conserved quantity and its horizontal part η_t is a martingale $\eta_0 + S_0 b_t$, in the case of $\mathcal{T} = 0$ and $Z_0 = 0$. Consider

$$du_t^\epsilon = H(u_t^\epsilon) \circ db_t + Z_0(u_t^\epsilon)dt + \sqrt{\epsilon}A_k^*(u_t^\epsilon) \circ dW_t^k + \epsilon Z_1(u_t^\epsilon)dt. \tag{3.4}$$

Denote by ϕ_t^ϵ the solution flow and $u_t^\epsilon = \phi_t^\epsilon(u_0)$. For some $V_0 \in T_{u_0}OM$ let $\eta_t^\epsilon = \theta(T\phi_t^\epsilon(V_0))$ and $S_t^\epsilon = \varpi(T\phi_t^\epsilon(V_0))$. Then

$$\begin{aligned}
dS_t^\epsilon &= -\sqrt{\epsilon}[A_k, S_t^\epsilon] \circ dw_t^k - \epsilon[\varpi(Z_1)(u_t^\epsilon), S_t^\epsilon]dt \\
d\eta_t^\epsilon &= \Theta(H(u_t^\epsilon) \circ db_t + Z_0(u_t^\epsilon)dt, \theta^{-1}\eta_t^\epsilon) + S_t^\epsilon \circ db_t + S_t^\epsilon(\theta Z_1)(u_t^\epsilon)dt \\
&\quad -\sqrt{\epsilon}A_k\eta_t^\epsilon \circ dw_t^k - \epsilon(\varpi Z_1)(u_t^\epsilon)\eta_t^\epsilon dt.
\end{aligned}$$

A stochastic averaging can also obtained for S_t^ϵ. If $Z_0 = Z_1 = \mathcal{T} = 0$ then $dS_t^\epsilon = -\sqrt{\epsilon}[A_k, S_t^\epsilon]dW_t^k - \frac{1}{2}\epsilon[A_k^2, S_t^\epsilon]dt$, and $d\eta_t^\epsilon = S_t^\epsilon db_t - \sqrt{\epsilon}A_k\eta_t^\epsilon dw_t^k - \frac{1}{2}\epsilon\sum_k A_k^2\eta_t^\epsilon dt$.

Alternatively for $A_k = \sum \sigma_{k,j}(u)A_j^*(u)$, consider

$$du_t^\epsilon = \mathbb{X}_l(u_t^\epsilon) \circ dB_t^l + \mathbb{X}_0(u_t^\epsilon)dt + \sqrt{\epsilon}A_k(u_t^\epsilon) \circ dW_t^k + \epsilon A_0(u_t^\epsilon)dt.$$

Let $x_t^\epsilon = \pi(u_t^\epsilon)$ and $u_t^\epsilon = \tilde{x}_t^\epsilon g_t^\epsilon$ then $d\tilde{x}_t^\epsilon = \sum \mathbb{X}_l(\tilde{x}_t^\epsilon) \circ dB_t^l + TR_{g_t^\epsilon}\mathbb{X}_0(\tilde{x}_t^\epsilon)dt$ is elliptic on the foliated manifold and the rotation and the vertical derivative flow over $[0, \frac{t}{\epsilon}]$ can be described by the average of the relevant vector fields over the its invariant measure μ_g, assuming the necessary technical conditions. Note that

$$\begin{aligned}
d\eta_t^\epsilon &= \theta\breve{\mathcal{T}}(\circ du_t^\epsilon, T\phi_t^\epsilon(V_0)) = \theta\mathbb{X}d\mathbb{Y}(h \circ du_t^\epsilon, \theta^{-1}\eta_t^\epsilon)\\
&= (u_t^\epsilon)^{-1}\mathcal{T}(X_l(u_t^\epsilon) \circ dB_t^l + T\pi(\mathbb{X}_0)(u_t^\epsilon)dt, u_t^\epsilon\eta_t^\epsilon),\\
dS_t^\epsilon &= \varpi\breve{\mathcal{T}}(\circ du_t^\epsilon, T\phi_t^\epsilon(V_0)) = d\varpi(\circ du_t^\epsilon, T\phi_t^\epsilon(V_0))\\
&= -\sqrt{\epsilon}\sigma_{k,j}[A_j, S_t^\epsilon] \circ dW_t^k - \epsilon\sigma_{0,j}[A_j, S_t^\epsilon]dt + \mathbb{X}_0(u_t^\epsilon)dt, \theta^{-1}\eta_t^\epsilon).
\end{aligned}$$

Now η_t^ϵ is a fast variable in general, while S_t^ϵ is clearly a slow variable.

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