

# Intertwining and the Markov uniqueness problem on path spaces

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## Abstract

Techniques of intertwining by Itô maps are applied to uniqueness questions for the Gross-Sobolev derivatives that arise in Malliavin calculus on path spaces. In particular claims in our article [Elworthy-Li3] are corrected and put in the context of the Markov uniqueness problem and weak differentiability. Full proofs in greater generality will appear in [Elworthy-Li2].

## 1 Malliavin calculus on $\mathcal{C}_0\mathbb{R}^m$ and $\mathcal{C}_{x_0}M$ .

### 1.1 Notation

Let  $M$  be a compact Riemannian manifold of dimension  $n$ . Fix  $T > 0$  and  $x_0$  in  $M$ . Let  $\mathcal{C}_{x_0}M$  denote the smooth Banach manifold of continuous paths

$$\sigma : [0, T] \rightarrow M \text{ such that } \sigma_0 = x_0$$

furnished with its Brownian motion measure  $\mu_{x_0}$ . However most of what follows works for a class of more general, possibly degenerate, diffusion measures.

Let  $\mathcal{C}_0\mathbb{R}^m$  be the corresponding space of continuous  $\mathbb{R}^m$ -valued paths starting at the origin, with Wiener measure  $\mathbb{P}$ , and let  $H$  denote its Cameron-Martin space:  $H = L_0^{2,1}\mathbb{R}^m$  with inner product  $\langle \alpha, \beta \rangle_H = \int_0^T \langle \dot{\alpha}(s), \dot{\beta}(s) \rangle_{\mathbb{R}^m} ds$ .

As a Banach manifold  $\mathcal{C}_{x_0}M$  has tangent spaces  $T_\sigma M$  at each point  $\sigma$ , given by

$$T_\sigma M = \{v : [0, T] \rightarrow TM \mid v(0) = 0, v \text{ is continuous}, v(s) \in T_{\sigma(s)}M, s \in [0, T]\}.$$

Each tangent space has the uniform norm induced on it by the Riemannian metric of  $M$ . As an analogue of  $H$  there are the ‘Bismut tangent spaces’  $\mathcal{H}_\sigma$  defined by

$$\mathcal{H}_\sigma = \{v \in T_\sigma \mathcal{C}_{x_0}M \mid \parallel_s^{-1}v(s) \in L_0^{2,1}T_{x_0}M, 0 \leq s \leq T\}$$

where  $\parallel_s$  denotes parallel translation of  $T_{x_0}M$  to  $T_{\sigma(s)}M$  using the Levi-Civita connection.

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### 1.2 Malliavin Calculus on $\mathcal{C}_0\mathbb{R}^m$ .

To have a calculus on  $\mathcal{C}_0\mathbb{R}^m$  the standard method is to choose a dense subspace,  $\text{Dom}(d^H)$ , of Fréchet differentiable functions (or elements of the first chaos) in  $L^2(\mathcal{C}_0\mathbb{R}^m; \mathbb{R})$ . By differentiating in the H-directions we obtain the H-derivative operator  $d^H : \text{Dom}(d^H) \rightarrow L^2(\mathcal{C}_0\mathbb{R}^m; H^*)$ . By the Cameron -Martin integration by parts formula this operator is closable. Let  $d : \text{Dom}(d) \rightarrow L^2(\mathcal{C}_0\mathbb{R}^m; H^*)$  be its closure and write  $\mathbb{D}^{2,1}$  for its domain with its graph norm and inner product.

From work of Shigekawa and Sugita, [Sugita],  $\mathbb{D}^{2,1}$  does not depend on the (sensible) choice of initial domain  $\text{Dom}(d^H)$  and moreover if a function is weakly differentiable with weak derivative in  $L^2$ , in a sense described below, then it is in  $\mathbb{D}^{2,1}$ . In particular if  $\text{Dom}(d^H)$  consists of the polynomial cylindrical functions then  $\mathbb{D}^{2,1}$  contains the space  $\text{BC}^1$  of bounded functions with bounded continuous Fréchet derivatives.

### 1.3 Malliavin Calculus on $\mathcal{C}_{x_0}M$ .

If  $f : \mathcal{C}_{x_0}M \rightarrow \mathbb{R}$  is Fréchet differentiable with differential  $(df)_\sigma : T_\sigma\mathcal{C}_{x_0}M \rightarrow \mathbb{R}$  at the point  $\sigma$ , define  $(d^H f)_\sigma : \mathcal{H}_\sigma \rightarrow \mathbb{R}$  by restriction. Choosing a suitable domain  $\text{Dom}(d^H)$  in  $L^2$  the integration by parts results of [Driver] imply closability and we obtain a closed operator  $d : \text{Dom}(d) \subset L^2(\mathcal{C}_{x_0}M; \mathbb{R}) \rightarrow L^2\mathcal{H}^*$ , for  $L^2\mathcal{H}^*$  the space of  $L^2$ -sections of the dual 'bundle'  $\mathcal{H}^*$  of  $\mathcal{H}$ . Let  $\mathbb{D}^{2,1}$  or  $\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M; \mathbb{R})$  denote the domain of this  $d$  furnished with its graph norm and inner product. Possible choices for the initial domain  $\text{Dom}(d^H)$  include the following:

- (i)  $C^\infty$  Cyl, the space of  $C^\infty$  cylindrical functions;
- (ii)  $\text{BC}^1$ , the space of  $\text{BC}^1$  bounded functions with first Fréchet derivatives bounded;
- (iii)  $\text{BC}^\infty$ , the space of infinitely Fréchet differentiable functions all of whose derivatives are bounded .

One fundamental question is whether such different choices of the initial domain lead to the same space  $\mathbb{D}^{2,1}$ . At the time of writing this question appears to still be open. There is a gap in the proof suggested in [Elworthy-Li3] as will be described in §2.3 below. However the techniques given there do show that choices (i) and (iii) above lead to the same  $\mathbb{D}^{2,1}$ .

From now on we shall assume that choice (i) has been taken. We use  $\nabla : \text{Dom}(d) \rightarrow L^2\mathcal{H}$  defined from  $d$  using the canonical isometry of  $\mathcal{H}_\sigma$  with its dual space  $\mathcal{H}_\sigma^*$ . This requires the choice of a Riemannian structure on  $\mathcal{H}$ ; for this see below. Let  $\text{div} : \text{Dom}(\text{div}) \subset L^2\mathcal{H} \rightarrow L^2(\mathcal{C}_{x_0}M; \mathbb{R})$  denote the adjoint of  $-\nabla$ . Then if  $f \in \text{Dom}(d)$  and  $v \in \text{Dom}(\text{div})$  we have

$$\int df(v)d\mu_{x_0} = - \int f \text{div}(v)d\mu_{x_0} = \int \langle \nabla f, v \rangle .d\mu_{x_0}.$$

Using these we get the self-adjoint operator  $\Delta$  defined to be  $\text{div} \nabla$ . Another basic open question is whether this is essentially self-adjoint on  $C^\infty$  Cyl. From the point of view of stochastic analysis it would be almost as good for it to have Markov Uniqueness. Essentially this means that there is a unique diffusion process on  $\mathcal{C}_{x_0}M$  whose generator

$\mathcal{A}$  agrees with  $\Delta$  on  $C^\infty$  cylindrical functions, see [Eberle]. Another characterisation of this is given below.

Finally there is the question of the existence of ‘local charts’ for  $\mathcal{C}_{x_0}M$  which preserve, at least locally, this sort of differentiability. The stochastic development maps  $\mathfrak{D} : \mathcal{C}_0\mathbb{R}^m \rightarrow \mathcal{C}_{x_0}M$  appear not to have this property, [XD-Li]. The Itô maps we use seem to be the best substitute for such charts.

## 2 The approach via Itô maps and main results.

### 2.1 Itô maps as a charts

As in [Aida-Elworthy] and [Elworthy-LeJan-Li] take an SDE on  $M$

$$dx_t = X(x_t) \circ dB_t, \quad 0 \leq t \leq T \quad (2.1)$$

with our given initial value  $x_0$ . Here  $(B_t, 0 \leq t \leq T)$  is the canonical Brownian motion on  $\mathbb{R}^m$  and  $X(x)$  is a linear map from  $\mathbb{R}^m$  to the tangent space  $T_x M$  for each  $x$  in  $M$ , smooth in  $x$ . Choose the SDE with the properties:

SDE1 The solutions to (1) are Brownian motions on  $M$ .

SDE2 For each  $e \in \mathbb{R}^m$  the vector field  $X(-)e$  has covariant derivative which vanishes at any point  $x$  where  $e$  is orthogonal to the kernel of  $X(x)$ .

This can be achieved, for example, by using Nash’s theorem to obtain an isometric immersion of  $M$  into some  $\mathbb{R}^m$  and taking  $X(x)$  to be the orthogonal projection onto the the tangent space; see [Elworthy-LeJan-Li].

Let  $\mathcal{I} : \mathcal{C}_0\mathbb{R}^m \rightarrow \mathcal{C}_{x_0}M$  denote the Itô map  $\omega \mapsto x.(\omega)$  with  $\mathcal{I}_t(\omega) = x_t(\omega)$ . Then  $\mathcal{I}$  maps  $\mathbb{P}$  to  $\mu_{x_0}$ . Set

$$\mathcal{F}^{x_0} = \sigma\{x_s : 0 \leq s \leq T\}$$

$$\mathbb{D}_{\mathcal{F}^{x_0}}^{2,1} = \{f : \mathcal{C}_0\mathbb{R}^m \rightarrow \mathbb{R} \text{ s.t. } f \in \mathbb{D}^{2,1} \text{ and } f \text{ is } \mathcal{F}^{x_0} \text{-measurable}\}.$$

Also consider the isometric injection  $\mathcal{I}^* : L^2(\mathcal{C}_{x_0}M; \mathbb{R}) \rightarrow L^2(\mathcal{C}_0\mathbb{R}^m; \mathbb{R})$  given by  $f \mapsto f \circ \mathcal{I}$ .

### 2.2 Basic results.

**Theorem 1** [Elworthy-Li1] *The map  $\mathcal{I}^*$  sends  $\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M; \mathbb{R})$  to  $\mathbb{D}_{\mathcal{F}^{x_0}}^{2,1}$  with closed range.*

**Theorem 2** *Markov uniqueness holds if and only if  $\mathcal{I}^*[\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M; \mathbb{R})] = \mathbb{D}_{\mathcal{F}^{x_0}}^{2,1}$ .*

**Theorem 3** *If  $f : \mathcal{C}_0\mathbb{R}^m \rightarrow \mathbb{R}$  is in  $\text{Dom}(\Delta)$  and  $\mathcal{F}^{x_0}$ -measurable then  $f$  belongs to  $\mathcal{I}^*[\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M; \mathbb{R})]$ .*

From Theorem 3 we see that  $\text{BC}^2 \subset \mathbb{D}^{2,1}$  on  $\mathcal{C}_{x_0}M$ . Theorem 2 is a consequence of Theorem 4 below.

**Problem 1** Is the set  $\{f : \mathcal{C}_0\mathbb{R}^m \rightarrow \mathbb{R} \text{ s.t. } f \text{ is in } \text{Dom}(\Delta) \text{ and } \mathcal{F}^{x_0}\text{-measurable}\}$  dense in  $\mathbb{D}_{\mathcal{F}^{x_0}}^{2,1}$ ?

Problem1 is open. An affirmative answer would imply Markov uniqueness by the theorems above.

### 2.3 A stronger possibility.

**Problem 2** If  $f \in \mathbb{D}^{2,1}$  does  $\mathbb{E}\{f|\mathcal{F}^{x_0}\} \in \mathbb{D}^{2,1}$  ?

Problem 2 is open: there is a gap in the ‘proof’ in [Elworthy-Li3]. It is true for  $f$  an exponential martingale or in a finite chaos space. An affirmative answer would imply an affirmative answer to Problem 1 and Markov uniqueness.

### 2.4 Markov uniqueness and weak differentiability

Let  $\mathbb{D}^{2,1}\mathcal{H}$  and  $\mathbb{D}^{2,1}\mathcal{H}^*$  be the spaces of  $\mathbb{D}^{2,1}$ -H-vector fields and H-1-forms on  $\mathcal{C}_{x_0}M$ , respectively, with their graph norms (see details below). Write:

$$\begin{aligned} \text{Cyl}^0\mathcal{H}^* &= \text{linear span } \{gdk|g, k : \mathcal{C}_{x_0}M \rightarrow \mathbb{R} \text{ are in } C^\infty \text{ Cyl}\} \\ W^{2,1} &= \text{Dom}(d^* | \mathbb{D}^{2,1}\mathcal{H}^*)^* \\ {}^0W^{2,1} &= \text{Dom}(d^* | \text{Cyl}^0\mathcal{H}^*)^*. \end{aligned}$$

Then  $\mathbb{D}^{2,1} \subseteq W^{2,1} \subseteq {}^0W^{2,1}$ . From [Eberle] we have:

$$\text{Markov uniqueness} \iff \mathbb{D}^{2,1} = {}^0W^{2,1} \quad (2.2)$$

We claim:

**Theorem 4** A.  $f \in W^{2,1}$  on  $\mathcal{C}_{x_0}M \iff \mathcal{I}^*(f) \in W^{2,1}$  on  $\mathcal{C}_0\mathbb{R}^m$ .

B.

$$W^{2,1} = {}^0W^{2,1}.$$

If  $f \in W^{2,1}$  it has a ‘‘weak derivative’’  $df \in L^2\Gamma\mathcal{H}$  defined by  $\int df(V)d\mu_{x_0} = -\int f \text{div } V d\mu_{x_0}$  for all  $V \in \mathbb{D}^{2,1}\mathcal{H}$ . See §3.4 below where the proof of Proposition 9 also demonstrates one of the implications of Theorem 4A.

An important step in the proof of part B is the analogue of a fundamental result of [Kree-Kree] for  $\mathcal{C}_0\mathbb{R}^m$ :

**Theorem 5** *The divergence operator on  $\mathcal{C}_{x_0}M$  restricts to give a continuous linear map  $\text{div} : \mathbb{D}^{2,1}\mathcal{H} \rightarrow L^2(\mathcal{C}_{x_0}M; \mathbb{R})$ .*

## 3 Some details and comments on the proofs.

We will sketch some parts of the proofs. The full details will appear, in greater generality, in [Elworthy-Li2].

### 3.1 To prove Theorem 3.

For  $f : \mathcal{C}_0\mathbb{R}^m \rightarrow \mathbb{R}$  in  $\mathbb{D}^{2,1}$  take its chaos expansion

$$f = \sum_{k=1}^{\infty} I^k(\alpha^k) = \sum_{k=1}^N I^k(\alpha^k) + R_{N+1} \quad (3.1)$$

say. This converges in  $\mathbb{D}^{2,1}$  as is well known, eg see [Nualart].

Set  $\mathbb{E}\{I^k(\alpha^k)|\mathcal{F}^{x_0}\} = J^k(\alpha^k)$ . Then

$$\mathbb{E}\{f|\mathcal{F}^{x_0}\} = \sum_{k=1}^{\infty} J^k(\alpha^k) \quad (3.2)$$

The right hand side converges in  $L^2$ . An equivalent problem to Problem 2 is:

**Problem 3** Does the right hand side of equation (4) always converge in  $\mathbb{D}^{2,1}$ ?

If  $f$  is  $\mathcal{F}^{x_0}$ -measurable and in the domain of  $\Delta$  it is not difficult to show that there is convergence in  $\mathbb{D}^{2,1}$ , using the Lemma below. Moreover  $\sum_{k=1}^N J^k(\alpha^k) \in \mathcal{I}^*[\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M; \mathbb{R})]$ . Therefore by Theorem 1 we see  $f \in \mathcal{I}^*[\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M; \mathbb{R})]$ . Again this uses the basic result (c.f. [Elworthy-Yor], [Aida-Elworthy], [Elworthy-LeJan-Li]).

**Lemma 6** Let  $K^\perp(x) : \mathbb{R}^m \rightarrow \mathbb{R}^m$  denote the orthogonal projection onto the orthogonal complement of the kernel of  $X(x)$  for each  $x$  in  $M$ . Suppose  $(\alpha_s, 0 \leq s \leq T)$  is progressively measurable, locally square integrable and  $\mathbb{L}(\mathbb{R}^m; \mathbb{R}^p)$ -valued. Then

$$\mathbb{E} \left\{ \int_0^T \alpha_s(dB_s) \middle| \mathcal{F}^{x_0} \right\} = \int_0^T \mathbb{E} \{ \alpha_s | \mathcal{F}^{x_0} \} K^\perp(x_s) dB_s.$$

### 3.2 The Riemannian structure for $\mathcal{H}$ .

Let  $\text{Ric}^\sharp : TM \rightarrow TM$  correspond to the Ricci curvature tensor of  $M$ , and  $W_s : T_{x_0}M \rightarrow T_{x_s}M$  the damped, or ‘Dohrn-Guerra’, parallel translation, defined for  $v_0$  in  $T_{x_0}M$  by

$$\begin{aligned} \frac{\mathbb{D}W_s(v_0)}{ds} &= 0 \\ W_0(v_0) &= v_0. \end{aligned}$$

Here  $\frac{D}{ds} = \frac{D}{ds} + \frac{1}{2}\text{Ric}^\sharp$ . Define  $\langle v^1, v^2 \rangle_\sigma = \int_0^T \langle \frac{D}{ds} v^1, \frac{D}{ds} v^2 \rangle_{\sigma_s} ds$  and let  $\nabla$  denote the damped Markovian connection of [Cruzeiro-Fang]; see [Elworthy-Li2] for details.

For each  $0 \leq t \leq T$  the Itô map  $\mathcal{I}_t : H \rightarrow T_{x_t}M$  is infinitely differentiable in the sense of Malliavin Calculus, with derivative  $T_\omega \mathcal{I}_t : H \rightarrow T_{x_t(\omega)}M$  giving rise to a continuous linear map  $T_\omega \mathcal{I} : H \rightarrow T_{x_t(\omega)}M$  defined almost surely for  $\omega \in \mathcal{C}_0\mathbb{R}^m$ . For  $\sigma \in \mathcal{C}_{x_0}M$  define  $\overline{T\mathcal{I}}_\sigma : H \rightarrow \mathcal{H}_\sigma$  by

$$\overline{T\mathcal{I}}_\sigma(h)_s = \mathbb{E}\{T\mathcal{I}_s(h) | x_s = \sigma\}.$$

From [Elworthy-LeJan-Li] this does map into the Bismut tangent space and gives an orthogonal projection onto it. It is given by

$$\frac{\mathbb{D}}{ds} \overline{T\mathcal{I}}_\sigma(h)_s = X(\sigma(s))(\dot{h}_s)$$

and has right inverse  $Y_\sigma : \mathcal{H}_\sigma \rightarrow H$  given by

$$Y_\sigma(v)_t = \int_0^t Y_{\sigma(s)} \left( \frac{\mathbb{D}}{ds} v_s \right) ds,$$

for  $Y_x : T_xM \rightarrow \mathbb{R}^m$  the right inverse of  $X(x)$  defined by  $Y_x = X(x)^*$ .

It turns out, [Elworthy-Li2], that for suitable H-vector fields  $V$  on  $\mathcal{C}_{x_0}M$ , the covariant derivative is given by  $\nabla_u V = \overline{T\mathcal{I}}_\sigma(d(Y_\sigma(V(-)))_\sigma(u))$ , for  $u \in T_\sigma \mathcal{C}_{x_0}M$ , and we define  $V$  to be in  $\mathbb{D}^{2,1}\mathcal{H}$  iff  $\sigma \mapsto Y_\sigma(V(\sigma))$  is in  $\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M; H)$ .

### 3.3 Continuity of the divergence

There is also a continuous linear map  $\overline{T\mathcal{I}(-)} : L^2(\mathcal{C}_0\mathbb{R}^m; H) \rightarrow L^2\mathcal{H}$  defined by  $\overline{T\mathcal{I}(U)}(\sigma)_s = \mathbb{E}\{T_- \mathcal{I}_s(U(-)) | x_-(\cdot) = \sigma\}$ , [Elworthy-Li1]. Another fundamental and easily proved result is

**Proposition 7** *Suppose the  $H$ -vector field  $U$  on  $\mathcal{C}_0\mathbb{R}^m$  is in  $\text{Dom}(\text{div})$ . Then  $\overline{T\mathcal{I}(U)}$  is in  $\text{Dom}(\text{div})$  on  $\mathcal{C}_{x_0}M$  and*

$$\mathbb{E}\{\text{div } U | \mathcal{F}^{x_0}\} = (\text{div } \overline{T\mathcal{I}(U)}) \circ \mathcal{I} \quad (3.3)$$

Theorem 5 follows easily from Proposition 7 by observing that if  $V \in \mathbb{D}^{2,1}\mathcal{H}$  then, from Theorem 1,  $\mathcal{I}^*(\mathbf{Y}_-V(-)) \in \mathbb{D}^{2,1}$ . By [Kree-Kree] this implies that  $\mathcal{I}^*(\mathbf{Y}_-(V(-)))$  is in  $\text{Dom}(\text{div})$ . Since

$$\overline{T\mathcal{I}(\mathcal{I}^*(\mathbf{Y}_-(V(-))))} = V$$

Proposition 7 assures us that  $V \in \text{Dom}(\text{div})$ . Moreover

$$\text{div } V(x_0) = \mathbb{E}\{\text{div } \mathcal{I}^*(\mathbf{Y}_-(V(-))) | \mathcal{F}^{x_0}\}. \quad (3.4)$$

Theorem 4A can be deduced from Proposition 7 together with:

**Lemma 8** *The set of  $H$ -vector fields  $V$  on  $\mathcal{C}_0\mathbb{R}^m$  such that  $\overline{T\mathcal{I}(V)} \in \mathbb{D}^{2,1}\mathcal{H}$  is dense in  $\mathbb{D}^{2,1}$ .*

### 3.4 Intertwining and weak differentiability.

To see how weak differentiability relates to intertwining by our Itô maps we have:

**Proposition 9** *If  $f \in W^{2,1}$  it has weak derivative  $df$  given by*

$$(df)_\sigma = \mathbb{E}\{d(\mathcal{I}^*(f))_\omega | x_-(\omega) = \sigma\} \mathbf{Y}_\sigma \quad (3.5)$$

**Proof** Let  $V \in \mathbb{D}^{2,1}\mathcal{H}$ . Then for  $f \in W^{2,1}$ , by equation (3.4) and then by Theorem 4A,

$$\begin{aligned} \int_{\mathcal{C}_{x_0}M} f \text{div}(V) d\mu &= \int_{\mathcal{C}_0\mathbb{R}^m} \mathcal{I}^*(f) \text{div}(V) \circ \mathcal{I} d\mathbb{P} \\ &= \int_{\mathcal{C}_0\mathbb{R}^m} \mathcal{I}^*(f) \text{div } \mathcal{I}^*(\mathbf{Y}_-(V(-))) d\mathbb{P} \\ &= - \int_{\mathcal{C}_0\mathbb{R}^m} d(\mathcal{I}^*(f))_\omega (\mathbf{Y}_{x_-(\omega)}(V(x_-(\omega)))) d\mathbb{P}(\omega) \\ &= - \int_{\mathcal{C}_{x_0}M} \mathbb{E}\{d(\mathcal{I}^*(f))_\omega | x_-(\omega) = \sigma\} \mathbf{Y}_\sigma(V(\sigma)) d\mu_{x_0}(d\sigma) \end{aligned}$$

as required.  $\square$

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