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Stochastic differential equations on noncompact manifolds: moment stability and its topological consequences

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Summary. In this paper we discuss the stability of stochastic differential equations and the interplay between the moment stability of a SDE and the topology of the underlying manifold. Sufficient and necessary conditions are given for the moment stability of a SDE in terms of the coefficients. Finally we prove a vanishing result for the fundamental group of a complete Riemannian manifold in terms of purely geometrical quantities.

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1 Introduction

The aim of this paper is to give conditions under which a stochastic dynamical system is moment stable and to relate this to the topological properties of the underlying space, following an approach of Elworthy [5]. To be more precise we need the following set up.

A. Let M be a smooth manifold. Consider on M the stochastic differential equation (SDE) on M:

$$dx_t = X(x_t) \circ dB_t + A(x_t)\,dt. \tag{1}$$

Here B_t is a m-dimensional Brownian motion on a filtered probability space $\{\Omega, \mathscr{F}, \mathscr{F}_t, P\}$, X is C^3 from $\mathbb{R}^m \times M$ to the tangent bundle TM with $X(x)$: $\mathbb{R}^m \to T_xM$ a linear map for each x in M, and A is a C^2 vector field on M.

For each x in M there is a solution $\{F_t(x)\}$ to (1) starting from x (maybe with explosion time $\xi(x)$). If $\xi(x) = \infty$ a.s. for each x, we say the SDE is complete or has no explosion. There are also the standard transition semigroup P_t and its infinitesimal generator $\mathscr{A}$.

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Formally the derivative $v_t = TF_t(v_0)$ of F_t at x_0 in the direction v_0 satisfies the derivative stochastic differential equation on TM:

$$dv_t = \delta X(v_t) \circ dB_t + \delta A(v_t)\,dt, \tag{2}$$

where δX and δA are obtained by differentiate X and A respectively (with a twist). See [3]. Furthermore v_t has the same explosion time as x_t.

If M is given a Riemannian structure with Levi–Civita connection, then (2) is equivalent to (1) together with the covariant equation along the paths of $\{x_t \equiv F_t(x_0) \colon t \geqq 0\}$

$$Dv_t = \nabla X(v_t) \circ dB_t + \nabla A(v_t)\,\mathrm{d}t. \tag{3}$$

If the SDE is strongly complete, i.e. if there is a version of $\{F_t(x)\}$ which is jointly continuous in t and x for almost all $\omega \in \Omega$, then TF_t is the derivative of $F_t(x)$ in the classical sense [3]. But this is not needed here. What we really need is the concept of strong 1-completeness:

A SDE is *strongly 1-complete* if for each compact smooth curve in M, $\{F_t(x)\}$ has a version which is jointly continuous in t and x when restricted to the curve [11]. In this case $T_xF_t(v)$ is the derivative along a curve tangent to v_0 at x_0.

Let f be a bounded function with bounded continuous first derivative, we can differentiate P_tf to get

$$d(P_tf)(v) = \mathbb{E}\,df(TF_t(v)), \quad v \in T_xM \tag{4}$$

given strong 1-completeness and assuming $\sup_{x\in K}\mathbb{E}|T_xF_t|^{1+\delta}$ finite for all compact sets K and some $\delta > 0$. Note that given nonexplosion $\sup_{x\in K}\mathbb{E}|T_xF_t|^{1+\delta}$ is finite if $H_{1+\delta}$, as defined in Sect. 2, is bounded above. See [13].

Alternatively define a linear map δP_t on differential 1-forms by

$$\delta P_t(\phi)(x)(v) = \mathbb{E}\,\phi(TF_t(v))\chi_{t<\xi(x)}, \quad v \in T_xM, \tag{5}$$

which is formally a semigroup due to the Markov property of the solution flow. Furthermore if (4) holds, then

$$(\delta P_t)(df)(v) = \mathbb{E}\,df(TF_t(v)) = d(P_tf)(v).$$

This equality of δP_t and dP_t turns out to be a very interesting property of the SDE and will be extensively used in Sects. 3 and 4.

B. For p an integer and K a subset of the manifold, define the pth moment exponents as follows:

$$\mu_K(p) = \overline{\lim_{t\to\infty}}\,\sup_{x\in K}\frac{1}{t}\log\mathbb{E}|T_xF_t|^p. \tag{6}$$

The SDE is said to be *pth-moment stable* if $\mu_x(p) < 0$ for all x in M, *strongly pth-moment stable if* $\mu_K(p) < 0$ for all compact sets K. It is pth moment unstable if $\mu_x(p) > 0$ for all x.

C. Main results. We are mainly interested in non-degenerate stochastic differential equations. Recall that (1) is said to be a *Brownian system* (*with drift Z*) if it has generator $\frac{1}{2}\Delta(+Z)$ for Δ the Laplacian. It is called a *gradient Brownian system (with drift Z)* if X is given by an isometric immersion $j: M \to \mathbb{R}^m$, i.e. for each $e \in \mathbb{R}^m$ and $x \in M$, $X(x)(e)$ is given by $\nabla\langle j(x), e\rangle$. The solution flow to the SDE is then a Brownian flow or gradient Brownian flow respectively. If $Z = \nabla h$ for some function h on the manifold, then we have *h-Brownian systems.* Let Ric_x denote the Ricci curvature at x, and

$$h_p(x) = -p \inf_{|v| \leq 1} \left\{ \text{Ric}_x(v, v) - 2\langle \nabla Z(x)(v), v\rangle - \sum_1^m |\nabla X^i(v)|^2 - (p-2)\sum_1^m \frac{\langle \nabla X^i(v), v\rangle^2}{|v|^2} \right\}.$$

We say [7] that a function f is *strongly stochastically positive* if for all compact subsets K,

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \sup_{x\in K} \log \mathbb{E}\left(\exp\left(-1/2 \int_0^t f(F_s(x))\, ds\right)\right) < 0.$$

It is *stochastically positive* if the above holds for all $K = \{x\}$ for all x in M.

Theorem 2.2 *A complete gradient h-Brownian system is pth-moment stable if* $-h_p$ *is stochastically positive; strongly pth-moment stable if* $-h_p$ *is strongly stochastically positive.*

Theorem 4.1 *The fundamental group* $\pi_1(M)$ *of a complete Riemannian manifold vanishes if on it there is a strongly 1-complete strongly moment stable h-Brownian system with the property* $d(P_t f) = \delta P_t(df)$ *for all* f *in* C_K^∞*, the space of smooth functions with compact support.*

In particular,

Corollary 4.2 *The first fundamental group of a complete Riemannian manifold vanishes if there is a h-Brownian motion on it such that* $\sup_{x\in M} h_1(x)$ *is negative.*

In terms of the geometrical quantities of the manifold, we have: let r be the distance function between x and a fixed point of the manifold.

Corollary 4.3 *Let* M *be a closed submanifold of* $\mathbb{R}^m$ *with its second fundamental form* $|\alpha_x|$ *bounded by* $c[1 + \ln(1 + r(x))]^{1/2}$*. Let* h *be a smooth function with* $|\partial h/\partial r| \leq c[1 + r(x)]$ *and* $\text{Hess}(h) \leq c[1 + \ln(1 + r(x))]$*. Then* $\pi_1(M) = \{0\}$ *if* $-h_1$ *is strongly stochastically positive.*

For analogous results when M is compact, including the vanishing of higher homotopy groups see [1].

We also discuss the existence of moment stable Brownian systems on $\mathbb{R}^n$.

Corollary 3.2. *There is no moment stable Brownian system on* $\mathbb{R}^1$.

2 Conditions for strong moment stabilities

A. Let $\{e_1, e_2, \ldots, e_m\}$ be an orthonormal basis for $\mathbb{R}^m$. Write $X^i(x) = X(x)(e_i)$ and $B_t = (B_t^1, \ldots, B_t^m)$ giving m independent 1-dimensional Brownian motions. Equation (1) can be written as

$$dx_t = \sum_1^m X^i(x_t) \circ dB_t^i + A(x_t)\,dt$$

and (2) as

$$dv_t = \sum_1^m \nabla X^i(v_t) \circ dB_t^i + \nabla A(v_t)\,dt.$$

For $v \in T_xM$, define

$$H_p(x)(v, v) = 2\langle \nabla A(x)(v), v\rangle + \sum_1^m \langle \nabla^2 X^i(X^i, v), v\rangle + \sum_1^m \langle \nabla X^i(\nabla X^i(v)), v\rangle$$
$$+ \sum_1^m |\nabla X^i(v)|^2 + (p-2)\sum_1^m \frac{1}{|v|^2}\langle \nabla X^i(v), v\rangle^2. \tag{7}$$

If the SDE is a Brownian system with drift Z, then the first 3 terms of H_p becomes $-\operatorname{Ric}_x(v, v) + 2\langle \nabla Z(x)(v), v\rangle$.
By an Itô's formula from [5],

$$|v_t|^p = |v_0|^p + p\int_0^t |v_s|^{p-2} \sum_t^m \langle \nabla X^i(v_s), v_s\rangle\,dB_s^i$$
$$+ \frac{p}{2}\int_0^t |v_s|^{p-2} H_p(x_s)(v_s, v_s)\,ds.$$

Letting

$$M_t^p = \sum_1^m \int_0^t p\,\frac{\langle \nabla X^i(v_s), v_s\rangle_{x_s}}{|v_s|^2}\,dB_s^i,$$

$$a_t^p = \frac{p}{2}\int_0^t \frac{H_p(x_s)(v_s, v_s)}{|v_s|^2}\,ds,$$

and $\mathscr{E}(M_t^p) = \exp\left(M_t^p - \frac{\langle M^p, M^p\rangle_t}{2}\right)$, as in [13] we solve the equation for $|v_t|^p$ to get

$$|v_t|^p = |v_0|^p \mathscr{E}(M_t^p)\exp(a_t^p) \tag{8}$$

Cf. [14].
Clearly if H_p is negative, i.e. $H_p(x)(v, v) < -c^2|v|^2$ for some $c \neq 0$, then

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log \sup_{x\in M} \mathbb{E}|T_xF_t|^p < 0$$

as known [5]. But for gradient Brownian systems, we can do better.
B. Let $j: M \to \mathbb{R}^m$ be the isometric immersion giving rise to the gradient Brownian system and let ν_x be the space of normal vectors at x. There is the second fundamental form: $\alpha_x: T_xM \times T_xM \to \nu_x$ and the shape operator:

$$A_x: T_xM \times \nu_x \to T_xM$$

related by $\langle \alpha_x(v_1, v_2), w\rangle = \langle A_x(v_1, w), v_z\rangle$. If $Y(x): R^m \to v_x$ is the orthogonal projection, then

$$\nabla X^i(v) = A_x(v, Y(x)e_i)$$

as shown in [3, 4]. Denote by $|\alpha_x(v, \cdot)|_{\mathrm{H,S}}$ the corresponding Hilbert–Schmidt norm, and $|\cdot|_{v_x}$ the norm in v_x. Then

$$\sum_1^m |\nabla X^i(v)|^2 = |\alpha_x(v, -)|^2_{\mathrm{H,S}},$$

and

$$\sum_1^m \langle \nabla X^i(v), v\rangle^2 = |\alpha_x(v, v)|^2_{v^x},$$

for $v \in T_xM$. This gives [5]

$$\begin{aligned} H_p(x)(v, v) = {} & -\mathrm{Ric}_x(v, v) + 2\langle \nabla A(x)(v), v\rangle + |\alpha_x(v, \cdot)|^2_{\mathrm{H,S}} \\ & + \frac{(p-2)}{|v|^2} |\alpha_x(v, v)|^2_{v_x}. \end{aligned} \tag{9}$$

Let $h_p(x) = \sup_{|v| \leq 1} pH_p(x)(v, v)$, and $\underline{h}_p(x) = \inf_{|v| \leq 1} pH_p(x)(v, v)$.

Lemma 2.1 *Consider a gradient h-Brownian system. Assume nonexplosion. Then*

$$\mathbb{E}\left(\exp\left(\frac{1}{2}\int_0^t \underline{h}_p(F_s(x))\,dx\right)\right) \leqq \mathbb{E}|T_xF_t|^p \leqq n\mathbb{E}\left(\exp\left(\frac{1}{2}\int_0^t h_p(F_s(x))\,ds\right)\right).$$

Proof. Let $Y(x)^*: v_x \to \mathbb{R}^m$ be the adjoint of the orthogonal projection $Y(x)$, then $X(x)(Y(x)^*(\alpha_x(v, v)) \equiv 0$ and

$$\begin{aligned} \langle Y^*(\alpha_x(v, v)), dB_t\rangle &= \langle \alpha_x(v, v), Y(dB_t)\rangle \\ = \sum_1^m \langle A_x(v, Y(e_i)), v\rangle dB_t^i &= \sum_1^m \langle \nabla X^i(v), v\rangle dB_t^i \end{aligned}$$

from the definitions. Let

$$\tilde{B}_t = B_t - \int_0^t pY^*\left(\alpha_x\left(\frac{v_s}{|v_s|}, \frac{v_s}{|v_s|}\right)\right) ds.$$

Consider SDE (2) on TM with $\{B_t\}$ replaced by $\{\tilde{B}_t\}$:

$$d\tilde{v}_t = \delta X(\tilde{v}_t) \circ d\tilde{B}_t + \delta A(\tilde{v}_t)\,dt.$$

Then $\tilde{x}_t$ is the projection of $\tilde{v}_t$ to M and solves the stochastic differential equation

$$d\tilde{x}_t = X(\tilde{x}_t) \circ d\tilde{B}_t + A(\tilde{x}_t)\,dt, \tag{10}$$

which is just SDE (1) from $X(x)(Y(x)^*(\alpha_x(v, v))) \equiv 0$. So $\tilde{x}_t$ has the same distribution as x_t.

Then by (8) and the Cameron–Martin–Girsanov formula,

$$\begin{aligned}\mathbb{E}|v_t|^p &= |v_0|^p \mathbb{E}\exp\left(M_t^p - \frac{1}{2}\langle M^p, M^p\rangle_t\right)\exp\left(\int_0^t \frac{pH_p(x_s)(v_s, v_s)}{2|v_s|^2}\,ds\right)\\ &\leqq |v_0|^p \mathbb{E}\exp\left(M_t^p - \frac{1}{2}\langle M^p, M^p\rangle_t\right)\exp\left(\int_0^t \frac{h_p(x_s)}{2}\,ds\right)\\ &= |v_0|^p \mathbb{E}\exp\left(\int_0^t \frac{h_p(\tilde{x}_s)}{2}\,ds\right) = |v_0|^p \mathbb{E}\exp\left(\int_0^t \frac{h_p(x_s)}{2}\,ds\right).\end{aligned}$$

Thus

$$\mathbb{E}|T_x F_t|^p \leqq n\mathbb{E}\exp\left(\frac{1}{2}\int_0^t h_p(F_s(x))\,ds\right). \tag{11}$$

The other half of the required inequality follows in the same way. □

Theorem 2.2 *A gradient SDE is pth-moment stable if*

$$\overline{\lim_{t\to\infty}}\,\frac{1}{t}\log \mathbb{E}\exp\left(\frac{1}{2}\int_0^t h_p(F_s(x))\,ds\right) < 0 \tag{12}$$

for each x in M, and it is pth-moment unstable if

$$\overline{\lim_{t\to\infty}}\,\frac{1}{t}\log \mathbb{E}\exp\left(\frac{1}{2}\int_0^t \underline{h}_p(F_s(x))\,ds\right) > 0.$$

Similarly it is strongly pth-moment stable if

$$\overline{\lim_{t\to\infty}}\,\frac{1}{t}\sup_{x\in K}\log \mathbb{E}\exp\left(\frac{1}{2}\int_0^t h_p(F_s(x))\,ds\right) < 0$$

for each compact subset K.

3 The Recurrency of h-Brownian motions

A. Let $h: M \to \mathbb{R}$ be a smooth function on M and $\Delta^h = \Delta + 2L_{\nabla h}$ be the Bismut–Witten Laplacian for $L_{\nabla h}$ the Lie derivative in the direction of ∇h. Then it is known that Δ^h is an essentially self-adjoint operator. Its closure shall also be denoted by Δ^h. Moreover if $e^{(1/2)t\Delta^h}$ is the semigroup defined by the spectral theorem with $e^{(1/2)t\Delta^{h,1}}$ its restriction on differential 1-forms, then $d(e^{(1/2)t\Delta^h}f) = e^{(1/2)t\Delta^{h,1}}(df)$ for $f \in C_K^\infty$, the space of smooth functions with compact support. A SDE with generator $\frac{1}{2}\Delta^h$ is called an h-Brownian system, the solution is an h-Brownian motion.

As usual P_t is the transition semigroup associated with our SDE, which equals the heat semigroup $e^{(1/2)t\Delta^h}$ on bounded L^2 functions [8]. We shall show that a complete strongly moment stable h-Brownian motion is recurrent if $d(P_t f) = (\delta P_t)(df)$ holds on C_K^∞.

B. But the assumption on the completeness of the h-Brownian motion is not really an extra condition since $d(P_t f) = (\delta P_t)(df)$ for f in C_K^∞ implies nonexplosion if $\mathbb{E}|T_x F_t|\chi_{t<\xi(x)} < \infty$ on an open set U and for $t < t_0$ for some constant t_0. In fact [11] it is known that an h-Brownian motion on a complete Riemannian manifold does not explode if there is an open set U and a number $t_0 > 0$ such that

$$|e^{(1/2)t\Delta^{h,1}} df|_x \leqq c_t(x)|df|_\infty, x \in U, \quad t \leqq t_0.$$

Here $c_t(x)$ is a constant depending possibly on t and x but not on the function f in C_K^∞. But if $d(P_t f) = \delta P_t(df)$ we can obtain an upper bound for the L^∞ norm of the heat semigroup on differential 1-forms in terms of $|TF_t|$:

$$|e^{(1/2)t\Delta^{h,1}}(df)(v)| = |d(P_t f)(v)| = |\mathbb{E} df(TF_t(v))\chi_{t<\xi}|$$

$$\leqq |df|_\infty |v| \mathbb{E}|T_x F_t|\chi_{t<\xi}$$

and so take $c_t(x) = \mathbb{E}|T_x F_t|\chi_{t<\xi}$.

We should note here a more standard non-explosion criteria which can be deduced by the same type of argument. Using another expression of the heat semigroup on 1-forms: $e^{(1/2)t\Delta^{h,1}}(df) = \mathbb{E}\, df(W_t^h)\chi_{t<\xi}$ for W_t^h the Hessian flow (see [4], 360–362), we conclude: an h-Brownian motion does not explode if

$$\mathbb{E} \sup_{t \leqq T} \exp\left(\frac{1}{2}\int_0^t \rho^h(x_s)\,ds\right)\chi_{t<\xi(x_0)} < \infty, \quad x \in U,$$

for some open set U, constant $T > 0$. Here

$$\rho^h(x) = -\inf_{|v|=1, v \in T_x M} \{\mathrm{Ric}_x(v,v) - 2\mathrm{Hess}(h)(v,v)\}. \tag{13}$$

This extends Bakry's result [1]: an h-Brownian motion does not explode if ρ^h is bounded above.

C. On the other hand given nonexplosion, there is an easy test for $d(P_t f) = \delta P_t(df)$ for Brownian systems with drift Z [11]. We only need to check $\mathbb{E}\sup_{s \leqq t}|T_x F_s| < \infty$. This very condition on $|T_x F_s|$ also gives strong 1-completeness and can be realized by assuming $H_1(x)$ given by (7) is bounded above. Note that H_p is nondecreasing as p increases, and for $p \geqq 1$

$$H_p(x)(v,v) \geqq -\mathrm{Ric}_x(v,v) + 2\langle \nabla Z(x)v, v\rangle.$$

So for h-Brownian motions the condition that H_p is bounded above for some $p \geqq 1$ gives nonexplosion, and therefore strong 1-completeness and $d(P_t f) = \delta P_t(df)$ for all functions $f \in C_K^\infty$.

D. Let M be a complete Riemannian manifold and dx its Riemannian volume element. Then it is known that $e^{2h}dx$ is the invariant measure for the h-Brownian motion. The manifold is said to have finite h-volume if $\int_M e^{2h(x)}dx < \infty$. Arguing by contradiction we see that the h-Brownian motion on M is recurrent if M has finite h-volume.

Proposition 3.1 *Suppose there is a h-Brownian system such that* $d(P_t f) = \delta P_t(df)$ *on* C_K^∞ *and*

$$\int_0^\infty \sup_{x \in K} \mathbb{E}|T_x F_t| \chi_{t<\xi(x)} \, dt < \infty .$$

Then M has finite h-volume.

For the proof, we mimic Bakry [1]. Let $\{h_n\}$ be a sequence of increasing functions bounded between 0 and 1 with limit 1 and $|\nabla h_n| \leq 1/n$. For $f \in C_K^\infty$, let $P_\infty f$ be the limit of $e^{(1/2)t\Delta^h} f$ as t goes to infinity. Then $P_\infty f$ is harmonic and thus a constant. Suppose the h-volume is not finite, then $P_\infty f$ has to be zero. We shall show that this is impossible.
Take $g \in C_K^\infty$, then

$$\int_M (P_\infty f - f) g e^{2h} dx = \lim_{t \to \infty} \int_M (P_t f - f) g e^{2h} dx.$$

But

$$\begin{aligned}
\int_M e^{(1/2)t\Delta^h} f - f) g e^{2h} dx &= \frac{1}{2} \int_0^t \int_M \Delta^h (e^{(1/2)s\Delta^h} f) g e^{2h} dx\, ds \\
&= \frac{1}{2} \int_0^t \int_M \langle d(e^{(1/2)s\Delta^h} f), dg \rangle e^{2h} dx\, ds \\
&= \frac{1}{2} \int_0^t \int_M \langle e^{(1/2)s\Delta^{h,1}} (df), dg \rangle e^{2h} dx\, ds \\
&= \frac{1}{2} \int_0^t \int_M \langle df, e^{(1/2)s\Delta^{h,1}} (dg) \rangle e^{2h} dx\, ds \\
&= \frac{1}{2} \int_0^t \int_{\mathrm{Supp}(f)} \langle df, e^{(1/2)s\Delta^{h,1}} (dg) \rangle e^{2h} dx\, ds,
\end{aligned}$$

since $e^{(1/2)t\Delta^{h,1}}$ is self-adjoint. Here $\mathrm{Supp}(f)$ denotes the support of f. Note that there is no explosion by the previous argument. Replacing $e^{(1/2)t\Delta^{h,1}}(dg)$ by $\mathbb{E} dg(TF_t)$ in the above calculation, we get

$$\left| \int_M (P_t f - f) g e^{2h} dx \right| \leq \frac{1}{2} |\nabla g|_\infty \int_0^t \sup_{x \in \mathrm{Supp}(f)} \mathbb{E}(|T_x F_s|) ds \int_M |\nabla f| e^{2h} dx.$$

Letting $g = h_n$ and taking t to infinity,

$$\left| \int_M - f e^{2h} dx \right| \leq \lim_{n \to \infty} \frac{1}{2n} |\nabla f|_{L^1} \int_0^\infty \sup_{x \in \mathrm{Supp}(f)} \mathbb{E}(|T_x F_s|) ds = 0.$$

Since we can choose a function $f \in C_K^\infty$ with $\int_M f e^{2h} dx \neq 0$, this gives a contradiction. □

E. Note that in [10] it was showed that for M compact moment stability is impossible for Brownian systems if $\mathrm{Ric}_x \leq 0$ at all points. In fact, by (8), there is no moment stable SDE on a manifold if $H_p(x)$ defined by (7) is non-negative and if $\mathbb{E}\exp(\frac{1}{2}\int_0^t |\nabla X(x_s)|^2\, ds) < \infty$ for all t. However

$$H_p(x)(v, v) \geqq 2\langle \nabla A(x)(v), v\rangle$$
$$+\sum_1^m \langle \nabla^2 X^i(X^i, v), v\rangle + \sum_1^m \langle \nabla X^i(\nabla X^i(v), v\rangle,$$

by (7) and the right hand side equals $2\langle \nabla Z(x)(v), v\rangle - \mathrm{Ric}_x(v, v)$ for Brownian systems with drift Z. For R^n the condition $\mathbb{E}\exp(\frac{1}{2}\int_0^t |\nabla X(x_s)|^2\, ds) < \infty$ for all t is satisfied if X and A have linear growth and $|DX|$ and $|DA|$ have sub-logarithmic growth. See [13].

Furthermore a complete gradient Brownian system (with drift) is not moment stable if $H_p \geqq 0$, by Lemma 2.1. In particular, c.f. [5]:

Corollary 3.2 *There is no moment stable Brownian system on* $\mathbb{R}^1$.

When the Ricci curvature is not non-positive, we have an analogous result here: let $r(x) = d(x, x_0)$ the distance of x from a fixed point x_0 of M,

Corollary 3.3 *Suppose M is complete and non-compact with*

$$\mathrm{Ric}_x > -\frac{n}{n-1}\frac{1}{r(x)^2}, \quad r(x) > r_0, \quad x \in M$$

for a constant $r_0 > 0$. Then a Brownian system on M cannot be strongly moment stable and have $d(P_t f) = \delta P_t(df)$ for all $f \in C_K^\infty$.

This is an application of the following from [C2]: The volume of M is infinite for noncompact manifolds with the above condition on the Ricci curvature. However we do not have a good extension of the corollary to h-Brownian systems (unless in the trivial case when the function h is bounded).

As a consequence there is no strongly moment stable Brownian system on $\mathbb{R}^n$ with $\sum_1^m |DX^i(x)(v)|^2 - \sum_1^m \frac{1}{|v|^2}\langle DX^i(x)(v), v\rangle^2$ bounded, for this gives $d(P_t f) = \delta P_t(df)$ for $f \in C_K^\infty$ by part C of this section.

4 Vanishing of $\pi_1(M)$

As pointed out by Elworthy [6], the stability of a stochastic flow is directly related to the topological properties of the underlying manifold. In particular there is the following theorem: A Brownian motion on a compact Riemannian manifold cannot be moment stable if the first fundamental group $\pi_1(M)$ of M is not trivial. It is proved with the help of the following: On a compact manifold with non-vanishing $\pi_1(M)$ there is a loop of strictly positive minimal length in its homotopy class. This is not true in general for noncompact manifold. However for M not compact but with positive injective radius, a similar argument shows [11] that $\pi_1(M) = \{0\}$ if there is a strongly 1-complete h-Brownian system such that for all compact subsets K, $\lim_{t\to\infty} \sup_{x\in K} \mathbb{E}|T_x F_t| = 0$. The key for a proof for general non-compact manifolds is the recurrency of h-Brownian motions. Recall that an h-Brownian motion has a natural invariant measure $e^{2h}dx$. Furthermore if the invariant measure is finite and μ is the normalized invariant measure, we have the recurrency:

$$\lim_{t\to\infty} P_t(\chi_K)(x) = \mu(K).$$

Theorem 4.1 *Let M be a complete Riemannian manifold. Suppose there is a strongly* 1*-complete h-Brownian system such that*

$$\int_0^\infty \sup_{x\in K} \mathbb{E}|T_xF_t|\,dt < \infty.$$

Then the first homotopy group $\pi_1(M)$ vanishes if $dP_tf = \delta P_t(df)$ for $f \in C_K^\infty$ (or more generally if M has finite h-volume).

Proof. Take σ to be a C^1 loop parametrized by arc length. Then $F_t\circ\sigma$ is a C^1 loop homotopic to σ by the strong 1-completeness. Let $\ell(\sigma_t)$ denote the length of $F_t(\sigma)$ with $\ell_0 = \ell(\sigma_0)$.

If we can show $F_t\circ\sigma$ is contractible to a point in M with probability bigger than zero for some $t > 0$, then the theorem is proved from the definition: $\pi_1(M) = 0$ if every continuous loop is contractible to one point.
First we claim there is a sequence of numbers $\{t_j\}$ converging to infinity such that

$$\mathbb{E}\ell(\sigma_{t_j}) \to 0. \tag{14}$$

Since

$$\int_0^\infty \mathbb{E}\ell(\sigma_t)\,dt \leqq \int_0^\infty \mathbb{E}\left(\int_0^{\ell_0} |T_{\sigma(s)}F_t|\,ds\right)dt$$

$$\leqq \ell_0 \int_0^\infty \sup_s \mathbb{E}|T_{\sigma(s)}F_t|\,dt < \infty.$$

So $\underline{\lim}_{t\to\infty} \mathbb{E}\ell(\sigma_t) = 0$, giving (14). Therefore $\ell(\sigma_{t_j}) \to 0$ in probability.

Note that by Proposition 3.1 we have finite h-volume if $dP_tf = (\delta P_t)(df)$ for $f \in C_K^\infty$. Let μ be the normalized invariant measure on M for the process. Let K be a compact set in M containing the image set of the loop σ and which has measure $\mu(K) > 0$. Let $a > 0$ be the infimum over $x \in K$ of the injectivity radius at x.
Then first by (14), there is a number N such that for $j > N$,

$$P\{\ell(\sigma_{t_j}) \geqq \tfrac{1}{2}a\} < \tfrac{1}{4}\mu(K).$$

And then by ergodicity,

$$\lim_{t\to\infty} P\{F_t(x) \in K\} = \mu(K)$$

for $x \in M$. Take a point $\tilde{x}$ in the image of the loop σ. There exists a number N_1 such that if $j > N_1$, then

$$P\{F_{t_j}(\tilde{x}) \in K\} > \tfrac{1}{2}\mu(K).$$

Thus

$$P\{\ell(\sigma_{t_j}) < \tfrac{1}{2}a, F_{t_j}(\tilde{x}) \in K\} = P\{F_{t_j}(\tilde{x}) \in K\} - P\{F_{t_j}(\tilde{x}) \in K, \ell(\sigma_{t_j}) \geqq \tfrac{1}{2}a\}$$
$$\geqq P\{F_{t_j}(\tilde{x}) \in K\} - P\{\ell(\sigma_{t_j}) \geqq \tfrac{1}{2}a\}$$
$$> \frac{\mu(K)}{4}.$$

But by the definition of injectivity radius, there is a coordinate chart containing a geodesic ball of radius a around $F_{t_j}(\tilde{x})$ if $F_{t_j}(\tilde{x})$ belongs to K. So the whole loop $F_{t_j} \circ \sigma$ is contained in the same chart and thus contractible to one point with probability $> \mu(k)/4$. □

It is clear from the proof that we do not need to assume either the SDS is a h-Brownian system or the equivalence of dP_t and δP_t provided there is the recurrency.

The following is known for arbitrary systems on compact manifolds [5]:

Corollary 4.2 *Let M be a complete Riemannian manifold. Consider a h-Brownian system on M. Then $\pi_1(M) = \{0\}$, if $H_1(v, v) < -c^2|v|^2$ for some constant $c \neq 0$.*

Proof. By Sect. 3, the condition on H_1 gives us the required strong 1-completeness and $dP_t = \delta P_t$ on C_K^∞.

Corollary 4.3 *Let M be a closed submanifold of R^n with its second fundamental form α_x bounded by $c[1 + \ln(1 + r(x))]^{1/2}$. Let h be a smooth function on M with $|\partial h/\partial r|_x \leqq c|1 + r(x)|$ and $Hess(h)(x) \leqq c[1 + \ln(1 + r(x))]$. Then the first fundamental group vanishes if $-h_1$ is strongly stochastically positive.*

Proof. Under these conditions, the gradient h-Brownian system on M is strongly complete and $d(P_t f) = \delta P_t(df)$ for all bounded functions with bounded first derivative [13].

When M is compact, this gives $\pi_1(M) = \{0\}$ if $\Delta^h + h_1 < 0$. See [9]. Note in [12], it was shown that a compact manifold M has finite fundamental group under a weaker assumption: $\Delta^h + \rho^h < 0$.

Remark. A SDE is strongly p-complete if its solution $\{F_t(x)\}$ is jointly continuous in t and x a.s. when restricted to any smooth singular simplices. If on M there is a strongly p-complete SDE which is also strongly pth-moment stable, then all bounded closed p-forms are exact. Consequently the natural map from the pth real cohomology with compact support to the pth real cohomology is trivial for such manifolds [11].

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References

[1] Bakry, D.: Un critére de non-explosion pour certaines diffusions sur une varieté Riemannienne complète. C. R. Acad. Sc. Paris **303,** Série I(1), 23–26 (1986)

[2] Cheeger, M., Gromov, M., Taylor, M.: Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. J. Diff. Geom. **17,** 15–53 (1982)

[3] Elworthy, K.D.: Stochastic Differential Equations on Manifolds. (Lect. Notes Series 70) Cambridge: Cambridge University Press 1982

[4] Elworthy, K.D.: Geometric aspects of diffusions on manifolds. In: Hennequin, P.L. (ed.) Ecole d'Eté de Probabilités de Saint-Flour XV-XVII, 1985, 1987 (Lect. Notes Math., vol. 1362, p. 276–425) Berlin Heidelberg New York: Springer 1988

[5] Elworthy, K.D., Stochastic flows on Riemannian manifolds. In: M.A. Pinsky and V. Wihstutz (eds.) Diffusion processes and related problems in analysis, volume II. Birkhauser Progress in Probability, pp. 37–72. Boston: Birkhauser, 1992

[6] Elworthy, K.D.: Stochastic differential geometry. Bull. Sci. Math. **117,** 2^e série, 7–28 (1993)

[7] Elworthy, K.D., Li, X.-M, Rosenberg, S.: Curvature and topology: spectral positivity. In: Methods and Applications of Global Analysis (Voronezh Series on New Developments in Global Analysis series) Voronezh, Russia: Voronezh University Press, 1993

[8] Elworthy, K.D., Rosenberg, S.: Generalized Bochner theorems and the spectrum of complete manifolds. Acta Appl. Math. **12,** 1–33 (1988)

[9] Elworthy, K.D., Rosenberg, S.: Homotopy and homology vanishing theorems and the stability of stochastic flows. Warwick Preprints: 13/1994

[10] Elworthy, K.D., Yor, M.: Conditional expectations for derivatives of certain stochastic flows. In: J. Azéma, P.A. Meyer, and M. Yor (eds.) Sem. de Prob. XXVII (Lect. Notes Math., vol. 1557, pp. 159–172) Berlin Heidelberg New York: Springer, 1993

[11] Li, X.-M.: Stochastic Flows on Noncompact Manifolds. Ph.D. thesis, University of Warwick, 1992

[12] Li, X.-M.: On extension of Myers' theorem. Warwick Preprints: 18/1993 Bull. London Math. Soc. (to appear)

[13] Li, X.-M.: Strong p-completeness and the existence of smooth flows on noncompact manifolds. Warwick preprints: 28/1993. Probab. Theory Relat. Fields (to appear)

[14] Taniguchi, S.: Stochastic flows of diffeomorphisms on an open set of R^n. Stochastics Stochastics Rep. **28**, 301–315 (1989)