Hodge de Rham decomposition for an L^2 space of differfential 2-forms on path spaces

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For a compact Riemannian manifold the space L^2A of L^2 differential forms decomposes into the direct sum of three spaces, the Hodge decomposition,

$$L^2A = Imd \oplus Imd^* \oplus H$$

where d stands for exterior differential, d^* its adjoint and H the space of L^2 harmonic differential forms. This decomposition identifies the space of harmonic forms with de Rham cohomology groups, which are in this case topological invariants. If M is a noncompact finite dimensional complete Riemannian manifold, the above decomposition holds with a modification. From it information on the geometry and sometimes the topology of the underlying spaces can be obtained. In this paper we shall be concerned with manifolds of continuous paths over complete Riemannian manifolds M, of which Wiener space is an example. Here we consider only L^2 theory. This paper is based on [1] and [2]. For works related to smooth cohomologies please see the series of papers by Léandre, e.g. [3], [4].

For $T > 0$, let $C_{x_0}M$ stand for the space of continuous paths $\sigma : [0, T] \to M$ starting from a fixed point x_0 in M. As functions interesting to us, such as stochastic integrals, are not smooth functions we are forced to consider H-derivatives when differentiations are concerned and thus the concept of linear forms on H-vectors, the so called H-forms, needs to be introduced. Analysis on Wiener space Ω in the sense of Malliavin Calculus relies on a Hilbert subspace, the Cameron Martin space $L_0^{2,1}$ of the Banach space Ω. The Hodge decomposition for Wiener space has been obtained by Shigekawa [5] for H-forms. Results are also obtained for Lie groups by Fang and Franchi [6]. Arai and Mitoma have generalized Shigekawa's work on Hodge decomposition theorem in [7] and [8], but still in the context of linear spaces.

For path spaces, the role of Cameron-Martin space is replaced by 'Bismut tangent spaces', for $\sigma \in C_{x_0}M$,

$$H^1_\sigma =: \{//^\sigma k \ \mid \ k \in L_0^{2,1}(T_{\sigma_0}M)\}.$$

Here $//^\sigma$ stands for parallel translations along σ using the Levi-Civita connection on M. The use of different connections induces different spaces H^1_σ. Write H^1 for the Hilbert bundle with fibres H^1_σ. One technical difference between the Wiener space and the more general path spaces is that H^1 is not in general an integrable bundle. In another word given two H^1 valued vector fields, their Lie bracket may not lie in

H^1, a fact which has been observed by many people. In [9] Driver has given explicit calculations of Lie brackets of H^1-valued vector fields.

Let ϕ be a smooth differential form on $C_{x_0}M$. Its exterior derivative is given by

$$d\phi(u \wedge v) = L_u(\phi(v)) - L_v(\phi(u)) - \phi([u,v]).$$

Here L stands for Lie differentiation and $[u,v]$ the Lie bracket of the vector fields u and v. Denote by ΓH^1 the space of sections of H^1. If ϕ were not a smooth form, e.g. $\phi = df$ for df the H-derivative of a function f then $d\phi$ is not defined by this formula when $[u,v]$ does not belong to ΓH^1.

In [1] we have introduced a series of Hilbert spaces $\mathcal{H}^q_\sigma$ of q-vectors. Let $X : M \times R^m \to TM$ be a surjective bundle map, inducing the Riemannian metric on M. For convenience we shall assume from now on M is compact and that the linear connection induced by X, as in [10], is the Levi-Civita connection on M. The example to have in mind is when X is given by an isometric immersion of M to R^m and $X(x)(e), e \in R^m$ is the orthogonal projection of e to T_xM. Consider the canonical probability space over R^m and denote by (B_t) the canonical Brownian motion. Let $(x_t(\omega)\colon\ 0 \le t \le T, \omega \in \Omega)$ be the solution to the Stratonovich stochastic differential equation

$$dx_t = X(x_t) \circ dB_t \tag{1}$$

on M starting from x_0. It gives rise to an Itô map:

$$\mathcal{I} : C_0\left(R^m\right) \to C_{x_0}M,$$

by $\mathcal{I}(\omega)_t = x_t(\omega)$ and a measure μ_{x_0} on the path space. The Itô map has H-derivatives which shall be denoted by $T\mathcal{I}(-)$. Set

$$\mathcal{H}^q_\sigma = \{E\{\wedge^q T\mathcal{I}(h) \mid x_\cdot = \sigma\} \qquad \mid\ h \in \wedge^q L^{2,1}_0(R^m)\} \tag{2}$$

Equip $\mathcal{H}^q_\sigma$ with the induced Hilbert space structure. In fact $\mathcal{H}^1_\sigma$ agrees with H^1 as a linear space but has a different Hilbert space structure.

Let us take a short hand notation $\bar{f}$ for the conditional expectation of a random variable f with respect to $\sigma\{x_s, 0 \le s \le T\}$. So (2) reads:

$$\mathcal{H}^q_\sigma = \left\{\overline{\wedge^q T\mathcal{I}}_\sigma(h) \ \mid\ h \in \wedge^q L^{2,1}_0(R^m)\right\} \tag{3}$$

If Ric, R^2 and $\mathcal{R}$ are respectively the Ricci curvature, Weitzenbock curvature on two-vectors and the curvature tensor let $W_t : T_{x_0}M \to T_{x_t}M$ and $W^{(2)}_t : \wedge^2 T_{x_0}M \to \wedge^2 T_{x_t}M$ be respectively solutions to the covariant differential equations along the paths of $\{x_s : 0 \le s \le T\}$

$$\begin{cases} \frac{D}{dt}W_t(v) &= -\frac{1}{2}Ric^{\#}(W_t(v)) \\ W_0(v) &= v, \qquad v \in T_{x_0}M \end{cases}$$

and

$$\begin{cases} \frac{D}{dt}W^{(2)}_t(u) &= -\frac{1}{2}R^2(W^{(2)}_t(u)) \\ W^{(2)}_0(u) &= u, \qquad u \in \wedge^2 T_{x_0}M. \end{cases}$$

Here $\frac{D}{dt} = //_t \frac{d}{dt} //_t^{-1}$ in the first instance and $\frac{D}{dt} = \wedge^2 //_t \frac{d}{dt} \wedge^2 //_t^{-1}$ in the second.

In [2] it was shown that

$$\mathcal{H}^2_\sigma = \{(I + Q_\sigma)(v) \;\mid\; v \in \wedge^2 \mathcal{H}^1_\sigma\}$$

where I is the identity map and

$$Q_\sigma(v)_{s,t} = (1 \otimes W^s_t)\, W^{(2)}_s \int_0^s (W^{(2)}_r)^{-1} R_{\sigma_r}(v_{r,r})\, dr. \tag{4}$$

Here as below we shall liberally identify 2-vector fields on $C_{x_0}M$ with continuous maps from $[0,T]^2 \to \otimes^2 TM$. Similarly vectors in $\mathcal{H}^3_\sigma$ can be obtained from vectors in $\wedge^3 \mathcal{H}^1_\sigma$ by an explicit map involving Q_σ and the Weitzenbock curvature for three vectors. In particular $\mathcal{H}^1_\sigma, \mathcal{H}^2_\sigma, \mathcal{H}^3_\sigma$ are independent of the map X which induces them. They depend only on the Riemannian structure of M. Furthermore for $f : \Omega \to \wedge^2 L_0^{2,1}(R^m)$ (satisfying certain mild condition), not necessarily adapted to $\sigma\{x_s : 0 \leq s \leq T\}$, $\overline{\wedge^2 T\mathcal{I}(f)}$ is an L^2 section of $\sqcup_\sigma \mathcal{H}^2_\sigma$ by concrete calculations. As a consequence $\overline{\wedge^2 T\mathcal{I}(-)}$ sends elements of L^2 k-particle spaces to L^2 sections of $\sqcup_\sigma \mathcal{H}^2_\sigma$. Let $L^2\Gamma\mathcal{H}^q$ and $L^2\Gamma\mathcal{H}^{q*}$ stand respectively for L^2 sections of $\sqcup\mathcal{H}^q_\sigma$ and $\sqcup\mathcal{H}^{q*}_\sigma$.

Example. Consider the symmetric space (K, H, σ). Set $M = K/H$ and consider an ad_K-invariant inner product on the Lie algebra $\underline{\mathfrak{k}}$ of the Lie groups K invariant under σ. Identify $\underline{\mathfrak{k}}$ with R^m for $m = dim\underline{\mathfrak{k}}$. Define X by the derivative of the action of K on M. It can be shown that $\overline{\wedge^2 TI(-)}$ induces a continuous linear map from

$$L^2(C_0(R^m); \wedge^2 L_0^{2,1}(\underline{\mathfrak{k}})) \to L^2\Gamma\mathcal{H}^2.$$

The pull back map I^* extends to a continuous map

$$\mathcal{I}^* : L^2\Gamma\mathcal{H}^{2^*} \to L^2(C_0(R^m); \wedge^2 L_0^{2,1}(\underline{\mathfrak{k}})^*).$$

The exterior differential operator. Let d^q be the exterior differential operator on smooth differential q forms. For $q = 1, 2$ let $\mathrm{Dom}(d^q) \subset L^2\Gamma\mathcal{H}^{q*}$ be the space of smooth cylindrical differential q-forms on $C_{x_0}M$. Consider d as an operator from $Dom(d^q)$ to $L^2\Gamma\mathcal{H}^{q+1^*}$. By a cylindrical differential q-form on $C_{x_0}M$ we mean a differential q-form of the following form: for some $0 \leq t_1 \leq t_2 \leq \ldots \leq t_q \leq T$,

$$\phi(V) = \theta(V_{t_1,\ldots,t_q}), \qquad V \in \wedge^q TC_{x_0}M$$

where θ is a C^r q-form on the q-fold product $\overbrace{M \times \ldots \times M}^{q}$. It can be shown that the domain of d^{1^*} contains all smooth cylindrical forms and that smooth cylindrical 2-forms are dense in $L^2\Gamma\mathcal{H}^{2^*}$ just as cylindrical 1-forms are dense in $L^2\Gamma\mathcal{H}^{1^*}$.

Finally we state the decomposition theorem. From the above discussion we see that d is a densely defined operator.

Theorem 0.1 *[2] Let d be the exterior differential operator on smooth differential forms. The operator d from Dom(d) in $L^2\Gamma(\mathcal{H}^{2^*})$ to $L^2\Gamma(\mathcal{H}^{3^*})$ is closable with closure a densely defined operator $\bar{d}$,*

$$\bar{d} : Dom(\bar{d}) \subset L^2\Gamma(\mathcal{H}^{2^*}) \to L^2\Gamma(\mathcal{H}^{3^*}_{.})$$

Theorem 0.2 *[2]*

$$L^2\Gamma\mathcal{H}^{2^*} = (\ker d \cap ker d^*) \oplus \overline{Im(d)} \oplus \overline{Im(d^*)}.$$

The role played by Q can be seen by the following identity:

$$div(Q(v^1 \wedge v^2)) = -T(v^1, v^2) \tag{5}$$

for suitbale adapted v^1 and v^2 in $\Gamma\mathcal{H}^1$. Here T is the torsion for the "damped Markovian" connection used by Cruzeiro and Fang in [11]. This connection can be identified with the connection on $\mathcal{H}^1$ induced by $\overline{T\mathcal{I}}$ in the sense of [10]. One also notes that the part of $[v^1, v^2]$ which is not in $\Gamma\mathcal{H}^1$ is contained in $T(v^1, v^2)$. Note that (4) fits in with an observation of Cruzeiro-Fang [11] that $div(T(v^1, v^2)) = 0$ for a certain class of adapted vector fields v^1, v^2.

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