A Spectral Gap for the Brownian Bridge measure on hyperbolic spaces

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1. Introduction

Let N be a finite or infinite dimensional manifold, μ a probability measure, d an differential operator with domain a subspace of the the L^2 space of functions which restricted to differentiable functions is the usual differentiation operator. Let d^* be its L^2 adjoint with respect to this measure. We ask whether the operator $L = -d^*d$ has a spectral gap. If ∇ is the gradient operator associated to d through Riesz representation theorem, in the case that we have a Hilbert space structure, this is equivalent to a Poincaré inequality $\int_N (f-\bar{f})^2 \mu(dx) \leqslant \frac{1}{\lambda_1} \int_N |\nabla f|^2 \mu(dx)$, where f ranges through an admissible set of real valued functions on a space N. If N is a compact closed Riemannian manifold, dx the volume measure and ∇ the Riemannian gradient operator, the best constant in the Poincaré inequality is given by taking infimum of the Raleigh quotient $\frac{\int_N |df|^2 dx}{\int_N f^2 dx}$ over the set of non-constant smooth functions of zero mean and is the spectral gap for the Laplacian operator, its first non-trivial eigenvalue. The operator concerned is given by the gradient operator and depend on the measure μ. Poincaré inequality does not hold for $\mathbf{R}^n$ with Lebesgue measure, it does hold for the Gaussian measure. For the standard normalised Gaussian measure and ∇ the gradient operator in Malliavin calculus, the Poincaré constant is 1 and the corresponding eigenfunction of the Laplacian is the Hermitian polynomial $x/2$. If h is a smooth function μ a measure which is absolutely continuous with respect to the Lebesgue measure with density e^{-2h}, for any f in the domain of d,

$$\int_N |df|^2(x)\mu(dx) = -\int_N \langle f, \Delta f\rangle(x)\mu(dx) - 2\int_N f\langle df, dh\rangle\mu(dx).$$

The corresponding Poincaré inequality is then related to the Bismut-Witten Laplacian $\Delta^h := \Delta + 2L_{\nabla h}$ on $L^2(M, e^{-2h}dx)$, which is unitarily equivalent to $\Box^h = \Delta + (|dh|^2 + \Delta h)$ on $L^2(M, dx)$. The spectral property of Δ^h, hence the validity of the Poincaré inequality for μ is determined by the spectral property of the Schrödinger operator $\square^h$ on $L^2(M; dx)$.

Let M be a smooth finite dimensional Riemannian manifold which is stochastically complete. Fix a number $T > 0$. The path space on M based at x_0 is:

$$\mathcal{C}_{x_0}M = \{\sigma : [0,T] \to M, \sigma(0) = x_0 | \ \sigma \text{ is continuous}\}.$$

For $y_0 \in M$, we define the subspaces

$$\mathcal{C}_{x_0,y_0}M = \{\sigma \in \mathcal{C}_{x_0}M \mid \sigma(T) = y_0\}, L_{x_0}M = \{\sigma \in \mathcal{C}_{x_0}M \mid \sigma(T) = x_0\}.$$

In this article our state space would be the loop space $L_{x_0}M$ endowed with the Brownian Bridge measure. In the case of the Wiener space, the Brownian Bridge measure $\mu_{0,0}$ is the law of the Brownian bridge starting and ending at 0, one of whose realisation is $B_t - \frac{t}{T}B_T$. It can also be realised as solution to the time-inhomogeneous stochastic differential equation $dx_t = dB_t - \frac{x_t}{T-t}dt$. The Brownian bridge measure is a Radon Gaussian measure and Gaussian measure theory applies to give the required the Poincaré inequality. In fact the stronger Logarithmic Sobolev inequality holds:

$$\int f^2 \log \frac{f^2}{\mathbf{E}|f|^2} \mu(dx) \leqslant 2 \int |\nabla f|^2 \mu(dx).$$

This however may not hold in general. In fact as noted by L. Gross[?] Poincaré inequalities do not hold on the Lie group S^1 due to the lack of connectedness of the loop space. Following that A. Eberle[?] gave an example of a compact simply connected Riemannian manifold on which the Poincaré inequality does not hold .

There are two standard arguments to prove the spectral gap theorem. The first argument applies to a compact state manifold N where $\inf_{f \in H^1, |f|_{L^2}=1, \int f=0} \int_M |\nabla f|^2 dx$ is attained, by a non-constant function, due to the Rellich-Kondrachov compact embedding theorem of $H^{1,q}$ into L^p. The other approach is the dynamic one which applies to Gaussian measures, due to the commutation relation. Namely we consider a Markov process with semigroup P_t, a finite invariant measure $e^{2h}dx$ (finiteness holds if $\inf_{|v|=1}\{Ric_x(v,v) - 2Hess_x(h)(v,v)\}$ is strongly stochastically positive[?]), and generator $\frac{1}{2}\Delta + \nabla h$. Suppose that $|dP_t f| \le \frac{1}{\rho}|df|$, then

$$\begin{aligned}
&\int_M (f-\bar{f})^2 d\mu = \int_M \left(f^2 - \bar{f}^2\right) \, d\mu = \lim_{t\to\infty} \int_M (f^2 - (P_t f)^2)(x) d\mu(x) \\
&= -\lim_{t\to\infty} \int_M \int_0^t \frac{\partial}{\partial s}(P_s f)^2 ds \, d\mu = \lim_{t\to\infty} \int_0^t \int_M (dP_s f)^2 \, d\mu \, ds \\
&= \int_0^\infty \int_M (dP_s f)^2 \, d\mu \, ds \le \frac{1}{\rho} \int_M |df|^2(x) \, d\mu.
\end{aligned}$$

In particular there is a Poincaré inequality if the Bakry-Emery condition $Ric_x - 2Hess_x(h) > \rho > 0$ holds. The dynamic argument can be modified leading to the beautiful Clark-Ocone formula approach,[?,?] However none of these approaches seems to work well for the Brownian bridge measure on a general path space. The main problem comes down to estimates on the heat kernel. However we do have one positive result for non-flat spaces: if M is the hyperbolic space we have indeed a spectral gap.[?] But it remains an open question whether the Logarithmic Sobolev inequality holds.

2. The Spectral Gap Theorem

A. Denote by Cyl the set of smooth sylindrical functions on $C_{x_0}M$,

$$Cyl_t = \{F|F(\sigma) = f(\sigma_{s_1}, \dots, \sigma_{s_k}), f \in C_K^\infty(M^k), 0 < s_1 < \dots < s_k \leqslant t < T\}.$$

The Brownian bridge measure μ_{x_0,y_0} is defined through integration of $F \in Cyl$,

$$p_T(x_0, y_0) \int_{\mathcal{C}_{x_0}M} f(\sigma_{s_1}, \dots, \sigma_{s_n}) d\mu_{x_0,x_1}(\sigma)$$
$$= \int_{M^n} f(x_1, \dots, x_n) p_{s_1}(x_0, x_1) \dots p_{s_n - s_{n-1}}(x_{n-1}, x_n) p_{T-s_n}(x_n, y_0) \Pi_{i=1}^n dx_i.$$

This cylindrical measure extends to a real measure if for some constants $\beta > 0$ and $\delta > 0$,

$$\int\int d(y,z)^\beta \frac{p_s(x_0,y)p_{t-s}(y,z)p_{T-t}(z,y_0)}{P_T(x_0,y_0)} dydz \leqslant C|t-s|^{1+\delta}. \qquad (1)$$

Throughout this article we shall assume the hear kernel satisfies the above inequality[?] and a number of assumptions, all of which hold true on the hyperbolic space,[?] we shall assume. See[?] for detail.

We now define the gradient operator. Take the levi-Civita connection ∇, whose parallel translation along a path σ is denoted by $/\!\!/_\cdot$. Define the tangent sub-space $H_\sigma = \{/\!\!/_s k_s : k \in L_0^{2,1}(T_{x_0}M)\}$, to $T_\sigma \mathcal{C}_{x_0}M$ which we call the Bismut tangent space with Hilbert space norm induced from the Cameron Martin space. We identify $T_{x_0}M$ with $\mathbf{R}^n$. Let μ be a probability measure on $C_{x_0}M$ including measures which concentrates on a subspace e.g. the loop space. The differential operator d, which restricts to differential functions is the usual exterior derivative from the space of L^2 functions to the space of L^2 H-valued 1-forms, is closable whenever Driver's integration by parts formula holds,[?] which we assume hold true. We define $\mathbb{D}^{1,2} \equiv \mathbb{D}^{1,2}(C_{x_0}M)$ to be the closure of smooth cylindrical function Cyl_t,

$t < T$ under this graph norm:

$$\sqrt{\int_{C_{x_0}M} |\nabla f|^2_{H_\sigma}(\sigma)\mu(d\sigma) + \int f^2(\sigma)d\mu(\sigma)}$$

and this is the domain of the corresponding gradient defined by: $df(V) = \langle \nabla f, V\rangle_H$.

B. In Aida[?] it was shown that for M the standard hyperbolic space, of constant negative curvature -1.

$$\int_{C_{x_0}H^n} f^2 \log \frac{f^2}{\log |f|^2_{L^2}} d\mu_{x_0,y_0}(\gamma) \leqslant \int_{C_{x_0}H^n} C(\gamma)|\nabla f|^2 d\mu_{x_0,y_0}(\gamma) \qquad (2)$$

for $C(\gamma) = C_1(n)+C_2(n)\sup_{0\leqslant t\leqslant 1} d^2(\gamma_t, y_0)$. To obtain thisPoincaré inequality with modified Dirichlet form, he first proved an integration by parts formula from which a Clark-Ocone formula of the form:

$$\mathbf{E}^{\mu_{x_0,y_0}}\{F|\mathcal{G}_t\} = \mathbf{E}^{\mu_{x_0,y_0}}F + \int_0^t \langle H_s(\gamma), dW_s\rangle,$$

where W_t is the martingale part of the anti-development of the Brownian bridge and

$$H(s,\gamma) = \mathbf{E}^{\mu_{x_0,y_0}}\{L(\gamma)\frac{d}{ds}\nabla F(\gamma)(s)|\mathcal{G}_s\}$$

almost surely with respect to the product measure $dt \otimes \mu_{x_0,y_0}$. Here $\mathcal{G}_t$ is the filtration generated by $\mathcal{F}_t$ and the end point of the Brownian bridge. Unlike the case for Gaussian measures or for the Brownian motion measures, the function L in the Clark-Ocone formula is not a deterministic function, which underlines why the Clark-Ocone approach itself is not good enough to give the required inequality.

Theorem 2.1. *Let $M = H^n$, the hyperbolic space of constant curvature -1. Then Poincaré inequality holds for the Brownian bridge measure μ_{x_0,x_0}.*

The proof of the theorem is based on Lemmas **??** and **??**, (**??**) and that

$$\int_{\mathcal{C}_{x_0}M} e^{Cd^2(\sigma,y_0)} d\mu_{x_0,y_0}(\sigma) < \infty.$$

C. The Laplace Beltrami operator on a complete Riemannian manifold may not have a spectral gap. But it has always a local spectral gap, by restriction to an exhausting relatively compact open sets U_n. The constant λ_1 may blow up as n goes to infinity. Once a blowing up rate for local Poincaré inequalities are obtained, we arrived at 'weak Poincaré inequalities' and in the case of Entropy the 'weak Logarithmic Sobolev inequalities':

$$\mathbf{Var}(f) \leqslant \alpha(s) \int |\nabla f|^2 d\mu + s|f|_\infty^2,$$

$$\mathbf{Ent}(f^2) \leqslant \beta(s) \int |\nabla f|^2 d\mu + s|f|_\infty^2.$$

Here $\mathbf{Var} f$ denoted the variance $\mathbf{E}(f - \mathbf{E}f)^2$ of a function f and $\mathbf{Ent}(f)$ its entropy $\mathbf{E}f \log \frac{f}{\mathbf{E}f}$ by $\mathbf{Ent}(f)$, α and β are non-decreasing functions from $(0, \infty)$ to $\mathbf{R}_+$. We first state the following estimate:

Lemma 2.1. *Let μ be any probability measure on $\mathcal{C}_{x_0}M$ with the property that there exists a positive function $u \in \mathbb{D}^{1,2}$ such that Aida's type inequality holds:*

$$\mathbf{Ent}(f^2) \leqslant \int u^2 |\nabla f|^2 d\mu, \qquad \forall f \in \mathbb{D}^{1,2} \cap L_\infty \tag{3}$$

Assume furthermore that $|\nabla u| \leqslant a$ and $\int e^{Cu^2} d\mu < \infty$ for some $C, a > 0$. Then for all functions f in $\mathbb{D}^{1,2} \cap L_\infty$

$$\mathbf{Ent}(f^2) \leqslant \beta(s) \int |\nabla f|^2 d\mu + s|f|_\infty^2, \tag{4}$$

where $\beta(s) = C|\log s|$ for $s < r_0$ where C and r_0 are constants.

D. Functional inequalities describes how the L^2 norms of a function is controlled by the homogeneous $H1$ norm and possibly the L_∞ norm in the case of weak type inequalities. They describe the concentration phenomenon of the measure. On the other hand concentration inequalities are related with isoperimetric inequalities. For example let $h = \inf_A \frac{\mu(\partial A)}{\min\{\mu(A), \mu(N/A)\}}$ where the infimum is taken over all open subsets of N. Then $h^2 \leqslant 4\lambda_1$ by Cheeger.[?] On the other hand if K is the lower bound of the Ricci curvature, $\lambda_1 \leqslant C(\sqrt{K}h + h^2)$,[?] See also,[??] and.[?]

For finite dimensional spaces it was shown in[?] and[?] one can pass from capacity type of inequalities to weak Logarithmic Sobolev inequalities and vice versa with great precision. Although they did not phrase the theorem in terms of Malliavin calculus, most of their results work in infinite dimensional spaces and in particular the following lemma will transforms our estimates on the blowing up rate of the logarithmic Sobolev inequality into a spectral gap result .

Lemma 2.2. *If for all f bounded measurable functions in $\mathbb{D}^{2,1}(C_{x_0}M)$, the weak logarithmic Sobolev inequality holds for $0 < s < r_0$, some given $r_0 > 0$,*

$$\mathbf{Ent}(f^2) \leqslant \beta(s) \int |\nabla f|^2 d\mu + s|f|_\infty^2$$

where $\beta(s) = C \log \frac{1}{s}$ *for some constant* $C > 0$*, Then Poincaré inequality*

$$\mathbf{Var}(f) \leqslant \alpha \int |\nabla f|^2 d\mu.$$

holds for some constant $\alpha > 0$.

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