1 Problem Sheet 0

Problems on Measure and Integration

Let $(E, \mathcal{B}, \mu)$ be a measure space. For $p \geq 1$ denote by L^p the equivalence class of functions $f : E \to \mathbf{R}$ with

$$L_p(X, \mathcal{B}, \mu) := \left\{ f : E \to \mathbf{R} : \int |f(x)|^p \mu(dx) < \infty \right\}.$$

This is a Banach space with norm $\|f\|_{L_p} = \left(\int |f(x)|^p \mu(dx) \right)^{\frac{1}{p}}$. A family of functions $f_n : (E, \mathcal{B}) \to \mathbf{R}$ is said to be uniformly integrable (u.i.) if for any $\epsilon > 0$ there is $A > 0$ such that if $a > A$ then $\int_{|f_n(x)|>a} |f_n(x)|\mu(dx) < \epsilon$ for all n.

A sequence of functions f_n is said to converge in measure if $\lim_{n\to\infty} \mu(|f_n - f| > \epsilon) = 0$ for any $\epsilon > 0$. For a finite measure almost sure convergence implies convergence in measure.

1. (a) Prove Markov's inequality for a non-negative function:

   $$\mu(f > a) \leq \frac{1}{a} \int f(x) d\mu(x)$$

   and Chebyshev's Inequality:

   $$\mu\left(\left| f(x) - \int_E f(x)\mu(dx) \right| \geq a \right) \leq \frac{1}{a^2} \int_E \left(f(x) - \int_E f(x)\mu(dx) \right)^2 \mu(dx).$$

   (b) Let $f : (E, \mathcal{B}, \mu) \to \mathbf{R}$ be an integrable function show that $\mu(x : |f(x)| > a)$ decays at least linearly. What do you deduce if $f \in L^p$ for some $p = 2, 3, \dots,$?

   (c) Show that if for some $p > 1$, $\|f_n\|_{L^p} \leq C$ for all n then $\{f_n\}$ is uniformly integrable.

2. Show that if $\{X_t, t \in I\}$ is a family of uniformly integrable random variables, it is L^1 bounded (i.e. $\sup_t \int |X_t| d\mu < \infty$).

3. Consider the family of functions $f_n : [0, 1] \to \mathbf{R}$,

   $$f_n(x) = \begin{cases} c_n, & x \in [0, \frac{1}{n}] \\ 0, & x \in (\frac{1}{n}, 1]. \end{cases}$$

   Indicate conditions on c_n so that $\{f_n\}$ is uniformly integrable.

4. Let $f_n, f \in L^1$. Show that $\{f_n\}$ is uniformly integrable and $f_n \to f$ in measure implies that f_n converges to f in L^1 and
$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

5. For finite measures show that $L^p \subset L^q$ if $1 \le q < p$.

6. Construct Lebesgue integrals. Give an example of a right continuous increasing function $F : [0,\infty) \to [0,1]$ with $F(0) = 0$. Construct the Riemann Stieljes measure on $[0,\infty)$ that is associated to F. Let $G(x) = F(x) + 1$. Relate μ_F to μ_G. In your case interpret the integral $\int f(x) d\mu_F(x)$.

Problems on Conditional Expectation

1. Show that if (X, Y) is a 2-dimensional Gaussian r.v. then X is independent of Y if and only if $Cov(X, Y) = 0$.

2. Show that if X and Y are independent random variables on $(\Omega, \mathcal{F}, P)$ and $f, g : \mathbf{R} \to \mathbf{R}$ Borel measurable with $\mathbf{E}|f(X)g(Y)| < \infty$ then $\mathbf{E}[f(X)g(Y)] = \mathbf{E}f(X)\mathbf{E}g(Y)$.

3. Let L^2 denote the equivalent classes of L^2 functions $f : \Omega \to \mathbf{R}$. It is a Banach space with norm $\sqrt{\mathbf{E}|f|^2}$. Let $\mathcal{G}$ be a sub-σ algebra of $\mathcal{F}$ show that there is a unique measurable function $\bar{f} : (\Omega, \mathcal{G}) \to \mathbf{E}$ which minimizes the distance $d(f, g) := \sqrt{\mathbf{E}|f - g|^2}$ among $\mathcal{G}$-measurable functions and prove that $\bar{f} = \mathbf{E}\{f|\mathcal{G}\}$ a.s.

4. Let $X, Y : \Omega \to \mathbf{R}$ be two random variables with a joint density f. For any $A \in \mathcal{B}(\mathbf{R}^2)$, $P(\{\omega : (X(\omega, Y(\omega)) \in A\}) = \int_A f(x, y) dx dy$.
   (a) Show that for $B \in \mathcal{B}(\mathbf{R})$,
   $$P(Y \in B | X = x) = \int_B \frac{f(x, y)}{f(x)} dy,$$
   a.s. with respect to the distribution μ_X.
   (b) Show that if $Y \in L^1$,
   $$\mathbf{E}(Y | X = x) = \int y \frac{f(x, y)}{f(x)} dy, a.s..$$

5. Let $\Omega = [-\frac{1}{2}, \frac{1}{2}]$, $\mathcal{F} = \mathcal{B}([-\frac{1}{2}, \frac{1}{2}])$ and P the Lebesgue measure.

(a) Let $A_1 = \{x : x \geq 0\}$ and $A_2 = \{x : x < 0\}$. Let $\mathcal{G} = \{A_1, A_2, \emptyset, \Omega\}$. Let $B \in \mathcal{B}(\mathbf{R})$. Give a formula for $P\{B|\mathcal{G}\}$.

(b) Let $X(z) = z^2$. Show that

$$\mathbf{E}(\mathbf{1}_A|X)(z) = \frac{1}{2}\mathbf{1}_A(z) + \frac{1}{2}\mathbf{1}_A(-z).$$

(c) Let Y be an integrable variable, show that $\mathbf{E}\{Y|X\}(z) = \frac{1}{2}Y(z) + \frac{1}{2}Y(-z)$ a.s. Find a version of $\mathbf{E}(Y|X = z)$.

6. (a) Let $X_0, X_1, \dots, X_n$ be mean zero random variables with non-degenrate Gaussian distribution. Show that $\mathbf{E}(X_0|X_1, \dots, X_n\} = \sum_{j=1}^n a_j X_j$. Determine a_j in terms of the covariance matrix: $c_{ij} = cov(X_i, X_j)$.

   (b) More generally let X be a $\mathbf{R}^d$ valued and Y a $\mathbf{R}^k$ valued random variable such that they are jointly Gaussian. Write its covariance matrix in the block matrix form:

$$C = \left(\begin{array}{cc} C_{11} & C_{12} \\ C_{21} & C_{22} \end{array} \right),$$

   Assume that it is positive definite. Then for all $B \in \mathcal{B}(\mathbf{R}^d)$, $P(X \in B|Y) = \hat{\mu}(B)$ for μ the Gaussian distribution with mean $\hat{X} = C_{12}C_{22}^{-1}Y$ and covariance $\hat{K} = C_{11} - C_{12}C_{22}^{-1}C_{21}$. Show that $cov(X - \hat{X}) = \hat{K}$.

   (c) In part (2) above remove the assumption that X, Y are mean zero variables. Show that the conditional probability distribution of X given Y is Gaussian with mean $\hat{X} = \mathbf{E}X + C_{12}C_{22}^{-1}(Y - \mathbf{E}Y)$ and covariance $\hat{K} = C_{11} - C_{12}C_{22}^{-1}C_{21}$.

7. Let X_i and Y be real valued random variables. Let $\phi : \mathbf{R} \to \mathbf{R}$ be Borel measurable. Suppose that all the terms involved are integrable then

   (a)

$$\mathbf{E}\{\phi(X_1)Y|X_1\} = \phi(X_1)\mathbf{E}\{Y|X_1\}.$$

   (b) If $\sigma(X_1, Y)$ is independent of X_2 then

$$\mathbf{E}(Y|X_1, X_2) = \mathbf{E}(Y|X_1).$$

8. Let $X_1, X_2, \dots$ be independent identically distributed integrable random variables. Let $S_n = X_1 + X_2 + \dots + X_n$. Prove that

$$E(X_1|S_n, S_{n+1}, \dots) = \frac{S_n}{n}, \qquad a.s.$$

2 Hints on Preliminaries

Answer to Problem 2 on integration is in notes.

On Conditional Expectations:

Problem 2. Use Monotone Class argument

Problem 3. Orthorgonal projection and compute $\mathbf{E}[f-\bar{f}+f-g]^2$.

Problem 5.

$$\frac{1}{2}Y(\sqrt{z})+\frac{1}{2}Y(-\sqrt{z})$$

Problem 6a) Write down the joint distribution $f(x_0,x_1,\dots,x_n)$, compute the inverse of the covariance matrix.

$$C:=\begin{bmatrix} C_1 & B^T \\ B & C_2 \end{bmatrix}^{-1}=\begin{bmatrix} I & 0 \\ -C_2^{-1}B & I \end{bmatrix}\begin{bmatrix} (C_0-B^TC_2B)^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix}\begin{bmatrix} I & -B^TC_2^{-1} \\ 0 & I \end{bmatrix}.$$

The quadratic form is

$$\begin{aligned} Q(x,y) &:= \left\langle \begin{bmatrix} x \\ y \end{bmatrix}, C^{-1}\begin{bmatrix} x \\ y \end{bmatrix}\right\rangle \\ &= \left\langle x-B^TC_2^{-1}y,(C_0-B^TC_2^{-1}B)^{-1}(x-B^TC_2^{-1}y)\right\rangle-\langle y,C_2^{-1}y\rangle. \end{aligned}$$

Integrate x out in $\int xg(y)e^{-\frac{1}{2}Q(x,y)}$ and $\int \psi(y)g(y)e^{-\frac{1}{2}Q(x,y)}$ where $\psi(y)$ is the conditional expectation of x_0 with respect to $y=(x_1,\dots,x_n)$. Compare terms to see that

$$\psi(y)=B^TC_2^{-1}y.$$

and $\psi(Y)$ is a he Gaussian r.v. $B^TC_2^{-1}Y$.

Problem 6b) Define $\tilde{X}=X-\hat{X}$. Then $(\tilde{X},Y)$ is Gaussian with $cov(\tilde{X},Y)=0$. And

$$\begin{aligned} \mathbf{E}(\mathbf{1}_{X\in A}|Y) &= \mathbf{E}(\mathbf{1}_{(\tilde{X}+C_{12}C_{22}^{-1}Y)\in A}|Y) \\ &= \mathbf{E}(\mathbf{1}_{(\tilde{X}+C_{12}C_{22}^{-1}y))\in A}|_{y=Y} \\ &= \int \mathbf{1}_{(x+\hat{X})\in A}d\mu_{\tilde{X}}(x) \end{aligned}$$

Hence the conditional measure is that of $\tilde{X}$ shifted by $\hat{X}$, which is Gaussian $N(\hat{X},cov(\tilde{X}))$.

Problems 9. Use symmetry.

Problem Sheet One

Exercise 1

- Let $m \in \mathbf{R}^n$ and G a positive definite symmetric matrix. Compute $\int f(x)dx$ and $\int xf(x)dx$ where $f(x) = Ce^{-\frac{1}{2}\langle G(x-m), x-m\rangle}$.
- Define $p_t(m,x) = \frac{1}{(2\pi t)^{\frac{n}{2}}} e^{\frac{-\|x-m\|^2}{2t}}$. If X is a random variable with distribution $p_t(m,x)$, explain that $P(\|X-m\| > x) \sim C_1 e^{-C_2 x}$ for large x.

Exercise 2 Suppose that B_t is a one dimensional process on $(\Omega, \mathcal{F}, P)$ with finite dimensional distribution given below, for $0 < t_1 < \cdots < t_k$, $A_k \in \mathcal{B}(\mathbf{R})$,

$$\begin{aligned} & P(B_{t_1} \in A_1, \dots, B_{t_k} \in A_k) \\ = & \int_{A_1} \dots \int_{A_k} p_{t_1}(0, y_1) p_{t_2 - t_1}(y_1, y_2) \dots p_{t_k - t_{k-1}}(y_{k-1}, y_k) dy_k dy_{k-1} \dots dy_1. \end{aligned}$$

(a) Show that $\mathbf{E} B_s B_t = \min(s,t)$; (b) Show that B_t has independent increments; (c) Give the distribution of $B_t - B_s$ and show that $t \mapsto B_t$ is almost surely continuous.

Exercise 3 Let $X_t = B_t - tB_1, 0 \le t \le 1$. Show that $\mathbf{E}(X_s X_t) = s(1-t)$ for $s \le t$. Explain why X_t is a Gaussian process. A sample continuous Gaussian process with $X_0 = X_1 = 0$ and covariance $\mathbf{E}(X_s X_t) = (s \wedge t)(1 - (s \vee t))$ is a Brownian bridge from 0 to 0. Compute the density of $P(B_t \in A | B_1 = 0)$ and that of $\mathbf{E}\{B_{t_1} \in A_1, \dots, B_{t_n} \in A_n | B_1 = 0\}$ in terms of the heat kernel.

Note that $P(B_t \in A | \sigma(B_1)) = \phi(B_1)$ for some $\phi : \mathbf{R} \to \mathbf{R}$ Borel measurable. By $P(B_t \in A | B_1 = y)$ we mean $\phi(y)$.

Exercise 4 Let $b : \mathbf{R} \to \mathbf{R}$ be a Lipschitz continuous function and x_t solves $\dot{x}_t = b(x_t)$. Let W_t be a standard Brownian motion. Suppose $x_t^\epsilon : \Omega \to \mathbf{R}$ satisfies

$$x_t^\epsilon(\omega) = x_0 + \int_0^t b(x_s^\epsilon(\omega)) ds + \epsilon W_t(\omega).$$

Show that x_t^ϵ converges to x_t in probability as $\epsilon \to 0$.

Hint: Exercise 2: A process is a Gaussian process if its finite dimensional distributions are Gaussian. If $\sum a_k B_{t_k}$ is a Gaussian r.v. for all $a_k \in \mathbf{R}$, then $(B_{t_1}, \dots B_{t_n})$ is a (multi-variate) Gaussian random variable. The random variable with components the increments $(B_{t_k} - B_{t_{k-1}}, k = 1m \dots, n)$ is a linear transformation of a Gaussian random variable and is hence Gaussian. For Gaussian random variables pairwise independence implies independence and two Gaussian variables are independent if they are uncorrelated.

Problem Sheet 2

Part 1.

Exercise 5 A stochastic process X_t on $(\Omega, \mathcal{F}, \mathcal{F}_t)$ is progressively measurable if for each t, $(s, \omega) \mapsto X_s(\omega)$ is measurable as a map $([0,t] \times \Omega, \mathcal{B}([0,t]) \otimes \mathcal{F}_t))$ to $\mathbf{R}$. Show that right continuous (left continuous) adapted stochastic processes are progressively measurable.

Suppose that X_t is adapted to $\mathcal{F}_t$. Let $0 \le t_0 \le t_1 \le \cdots \le t_n \le t$. Are the following processes are progressively measurable?

$$X_t^{(n)}(\omega) = X_0 \mathbf{1}_{\{0\}}(t) + \sum_{i=0}^{n-1} X_{t_i}(\omega) \mathbf{1}_{(t_i, t_{i+1}]}(t), \quad Z_t^{(n)}(\omega) = \sum_{i=0}^{n-1} X_{t_i}(\omega) \mathbf{1}_{[t_i, t_{i+1})}(t),$$

$$\tilde{Y}_t^{(n)} = \sum_{i=0}^{n-1} X_{t_{i+1}} \mathbf{1}_{[t_i, t_{i+1})}(t).$$

Exercise 6 Let μ be a probability measure on $\mathbf{R}$. Define the product measure ν on $\mathcal{B}(\mathbf{R} \times \mathbf{R})$ by

$$\nu(A_1 \times A_2) = \mu(A_1) \times \mu(A_2), \qquad A_i \in \mathcal{B}(\mathbf{R}).$$

Let $\pi_i : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be the projections. Then π_1, π_2, as real valued random variables on $(\mathbf{R}^2, \mathcal{B}(\mathbf{R}^2), \nu)$, are independent.

Exercise 7 Define $P_t f(x) = \frac{1}{\sqrt{2\pi t}} \int e^{\frac{-(y-x)^2}{2t}} f(y) dy$ for f bounded measurable. We say $f \in BC^2$ if f and its first two derivatives are bounded.

- Show that P_t has the semigroup property: $P_{t+s} f = P_t P_s f$ and observe that $P_t f \ge 0$ if $f \ge 0$.
- If f is BC^2,
  $$\lim_{t \to 0} \frac{P_t f(x) - f(x)}{t} = \frac{1}{2} f''(x).$$
  The linear operator $\mathcal{A}$ defined by $\mathcal{A} f := \lim_{t \to 0} \frac{P_t f(x) - f(x)}{t}$ whenever the limit exists is the generator of P_t.
- Show that $\frac{d}{dx}(P_t f)(x) = \frac{1}{t} \mathbf{E}[f(x + B_t) B_t]$ where $B_t \sim N(0,t)$. Can you show this holds when f is not differentiable ?

Exercise 8 A zero mean Gaussian process B^H_t is a fractional Brownian motion of Hurst parameter H, $H \in (0,1)$, if its covariance is

$$\mathbf{E}(B^H_t B^H_s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Then $\mathbf{E}|B^H_t - B^H_s|^p = C|t-s|^{pH}$. It is Brownian motion when $H = 1/2$. (Otherwise this process is not even a semi-martingale).

Show that B^H_t has Hölder continuous paths of order $\alpha < H$.

Exercise 9 Let W_t be a standard Brownian motion and

$$y^\epsilon_t = y_0 + \epsilon \int_0^t b(y^\epsilon_s)ds + \sqrt{\epsilon} W_t.$$

Assume that b is bounded, as $\epsilon \to 0$, show that y^ϵ_t on any finite time interval converges uniformly in time on any finite time interval $[0,t]$.

$$\mathbf{E} \sup_{0 \le s \le t} (y^\epsilon_s - y_0) \to 0.$$

Set $z^\epsilon_t := y^\epsilon_{t/\epsilon}$. Show that $z^\epsilon_t = z_0 + \int_0^t b(z^\epsilon_s)ds + \bar{W}_t$ where $\bar{W}_t$ is a Brownian motion. [Hint: use $P(\sup_{s \le t} B_s \ge a) = 2P(B_t \ge a)$.]

Exercise 10 Let $(W, \mathcal{B}, P)$ be the Wiener space and let $\mathcal{F}_t$ be the natural filtration of the coordinate process π_t. Show that π_t is a Markov process with respect to its natural filtration. This means for any bounded Borel measurable function f,

$$\mathbf{E}\{f(\pi_t)|\sigma(\pi_r : 0 \le r \le s)\} = \mathbf{E}\{f(\pi_t)|\sigma(\pi_s)\}.$$

Part 2.

Exercise 11 Let $X_1, X_2, \dots$ be independent random variables with $\mathbf{E}X_i = 0$. Let $\mathcal{F}_n = \sigma\{X_1, \dots, X_n\}$. Let $S_n = X_1 + \dots + X_n$. Show that for $j = 1, 2, \dots$, $\mathbf{E}\{S_{n+j}|\mathcal{F}_n\} = S_n$.

Exercise 12 Let $X_1, X_2, \dots$ be independent random variables with $\mathbf{E}X_i = 1$. Let $\mathcal{F}_n = \sigma\{X_1, \dots, X_n\}$. Let $M_n = \Pi^n_{k=1} X_k$. Show that for all $n, j = 1, 2, \dots$, $\mathbf{E}\{M_{n+j}|\mathcal{F}_n\} = M_n$.

Exercise 13 Let $X : \Omega \to \mathbf{R}$ be integrable. Let $(\mathcal{F}_t, t \ge 0)$ be a filtration. Define $X_t = \mathbf{E}\{X|\mathcal{F}_t\}$, $t \ge 0$. Show that $\mathbf{E}\{X_t|\mathcal{F}_s\} = X_s$ given $t > s \ge 0$.

Exercise 14 Let $X_1, X_2, \ldots$ be independent random variables with $\mathbf{E}X_i = 0$. Let $\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$. Let $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$. Let $C_n = f_n(X_1, \ldots, X_{n-1})$ for some Borel function $f_n : \mathbf{R}^n \to \mathbf{R}$. Define

$$I(C, X)_n = \sum_{1 \le k \le n} C_k (S_k - S_{k-1}).$$

This is called the martingale transform. Compute $\mathbf{E}\{I(C,X)_n - I(C,X)_{n-1} | \mathcal{F}_{n-1}\}$.

Exercise 15 Let X_n be a sequence of random variables bounded in L^1. Suppose that for all $a < b$,

$$P(\omega : \liminf_{n\to\infty} X_n(\omega) < a < b < \limsup_{n\to\infty} X_n(\omega)\}) = 0.$$

Show that $\lim_{n\to\infty} X_n(\omega)$ exists almost surely and the limit is almost surely finite.

Hint: Exercise 5. Recall that the tensor σ-algebra is the smallest such that each projection is measurable.

Exercise 7. Let $z = \frac{y-x}{\sqrt{t}}$ then $P_t f(x) = \int \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} f(x + t\sqrt{z}) dz$. Taylor expand $f(x + t\sqrt{z})$ at x and observe that $\int z e^{-\frac{z^2}{2}} dz = 0$.

Exercise 10. The class of functions $\pi_{i=1}^n g_i(B_{s_i})$, where g_i are Borel measurable and $0 \le s_0 < s_1 < \ldots s_n = s < t$, are sufficient for determining conditional expectation with respect to $\mathcal{F}_s^B$. Show that

$$\mathbf{E} f(B_t) \pi_{i=1}^n g_i(B_{s_i}) = \mathbf{E}\, \mathbf{E}\{f(B_t) | \sigma(B_s)\} \pi_{i=1}^n g_i(x_{B_i}).$$

For example consider

$$\mathbf{E} f(B_t) g_2(B_s) g_1(B_{s_1}) = \mathbf{E}\mathbf{E}\{f(B_t) g_2(B_s) g_1(B_{s_1}) | \sigma(B_{s_1}) \vee \sigma(B_s - B_{s_1})\}$$

and use the independent increments property: For $0 \le s_0 < s_1 < \cdots < s_n$, the increments $\{(B_{s_{i+1}} - B_{s_i})\}_{i=0}^n$ are independent random variables.

Problem Sheet 3

Let $(\Omega, \mathcal{F}, \mathcal{F}_t)$ be a filtered probability space. Part 1. All processes in this part are real valued.

Exercise 16 If M_t is an L^2 bounded martingale show that for $s < t$, $\mathbf{E}(M_t - M_s)^2 = \mathbf{E}M_t^2 - \mathbf{E}M_s^2$.

Exercise 17 Let ϕ be a convex function. Show that

(a) If X_t is a sub-martingale and ϕ is increasing then $\phi(X_t)$ is a sub-martingale.

(b) If X_t is martingale then $\phi(X_t)$ is a sub-martingale. Show that $|X_t|$ is a sub-martingale.

Exercise 18 Let $\{X_n, n = 0, 1, 2 \dots\}$ be an $\mathcal{F}_n$-adapted stochastic process with $X_n \in L^1$ and $X_0 = 0$. Define $G_n = \mathbf{E}\{X_{n+1} - X_n | \mathcal{F}_n\}$, $n \geq 1$.

(a) Let $A_0 = A_1 = 0$ and $A_n = \sum_{j=1}^{n-1} G_j$, $n \geq 2$. Show that $A_n \in \mathcal{F}_{n-1}$ (previsible).

(b) Let $M_0 = 0$ and $M_n = X_n - A_n$ show that M_n is a martingale. Then $X_n = M_n + A_n$. This is the analogue of the Doob-Meyer decomposition for continuous time processes.

(c) If X_n has another decomposition $X_n = \tilde{M}_n + \tilde{A}_n$, where $\{\tilde{M}_n\}$ is a martingale with $\tilde{M}_0 = 0$, $\tilde{A}_n$ is process with $\tilde{A}_n \in \mathcal{F}_{n-1}$ and $\tilde{A}_0 = \tilde{A}_1 = 0$. Show that $M_n = \tilde{M}_n$ and $A_n = \tilde{A}_n$ a.s. If X_n is a sub-martingale show that A_n is an increasing process.

(d) Let (M_n) be a martingale with $\mathbf{E}M_n^2 < \infty$ and $M_0 = 0$ show that there is an increasing process A_n such that $M_n^2 = N_n + A_n$ where N_n is a martingale and A_n is an increasing process. Show that $A_n - A_{n-1} = \mathbf{E}\{(M_n - M_{n-1})^2 | \mathcal{F}_{n-1}\}$.

Note that A_n is the discrete analogue for the martingale bracket or quadratic variation of (M_n).

Exercise 19 If B_t is a standard Brownian motion show that

(a) For any $s \geq 0$, $B_t - B_s$ is independent of $\mathcal{F}_s$ where

$$\mathcal{F}_s = \sigma\{B_r : 0 \leq r \leq s\}.$$

(b) if $a \neq 0$ is a real number $\frac{1}{\sqrt{a}} B_{at}$ is a Brownian motion;

(c) For any $t_0 \geq 0$, $B_{t_0+t} - B_{t_0}$ is a standard Brownian motion;

(d) B_t, $B_t^2 - t$ and $\exp(B_t - t/2)$ are martingales;

(e) For any $0 \leq s < t$, $\mathbf{E}\{(B_t - B_s)^2 | \mathcal{F}_s\} = t - s$;

(f) Defien a process W_t by $W_0 = 0$ and $W_t = tB_{\frac{1}{t}}$ when $t > 0$. Show that W_t is a Brownian motion.

(g) $\lim_{t\to\infty} \frac{B_t}{t} = 0$.

Exercise 20 (a) Suppose that X_t is continuous bounded and $\mathcal{F}_t$-adapted. Let $0 \leq t_0 < t_1 < \cdots < t_n \leq t$. Let

$$X_t^{(n)}(\omega) = X_0 \mathbf{1}_{\{0\}}(t) + \sum_{i=0}^{n-1} X_{t_i}(\omega) \mathbf{1}_{(t_i, t_{i+1}]}(t).$$

Show that as $\max_{0 \leq i < n}(t_{i+1} - t_i) \to 0$, $\mathbf{E} \int_0^t |X_s^{(n)} - X_s|^2 ds \to 0$.

(b) Let B_t be a one dimensional Brownian motion. Compute

$$\mathbf{E} \sum_{i=0}^{n-1} X_{t_i}(\omega)[B(t_{i+1}) - B(t_i)] \quad \text{and} \quad \mathbf{E} \left(\sum_{i=0}^{n-1} X_{t_i}(\omega)[B(t_{i+1}) - B(t_i)] \right)^2.$$

Exercise 21 Let (B_t) be a 1-dimensional Brownian motion. Is the Brownian Bridge $(B_t - tB_1, 0 \leq t \leq 1)$ a martingale? Is the Ornstein-Uhlenbeck process $e^{-t} B_{e^{2t}}$ a martingale?

Exercise 22 Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space. Let Q be a probability measure such that $Q << P$. Let $f = \frac{dQ}{dP}$. Let Q_t and P_t are restrictions of Q, P to $\mathcal{F}_t$. Let $f_t = \mathbf{E}\{f | \mathcal{F}_t\}$. Show that f_t is a L^1 bounded martingale and $\frac{dQ_t}{dP_t} = f_t$.

Part 2.

Exercise 23 Let $(x_t, t \geq 0)$ be a sample continuous real valued process. Let $\mathcal{F}_t := \sigma(x_s : 0 \leq s \leq t)$ be its natural filtration. Let $a > 0$, Show that

$$D(\omega) := \inf_{t \geq 0} \{t : |x_t(\omega)| \geq a\}$$

is a $\mathcal{F}_t$ stopping time.

Exercise 24 If T_n is a sequence of stopping times such that $T = \lim_{n\to\infty} T_n$ exists almost surely. Show that T is a stopping time.

Exercise 25 If S is a stopping time define

$$\mathcal{F}_S = \{A \in \mathcal{F}_\infty : A \cap \{S \le t\} \in \mathcal{F}_t, \forall t \ge 0\}.$$

Show that for all $t \ge 0$, $\{S < t\}$, $\{S > t\}$ and $\{S = t\}$ are in $\mathcal{F}_S$. Let T be a stopping time. Show that $\{S < T\}$, $\{S > T\}$ and $\{S = T\}$ are in $\mathcal{F}_S$.

Hint: Exercise 18. The solution is shorter than the question and the implication is longer than the question! For part C) induct on n. $A_1 = \tilde{A}_1 = 0$ implies that $M_1 = \tilde{M}_1$. Try to prove $\tilde{A}_1 = \tilde{A}_2$ by the martingale property of M_n and that A_n is previsible.

Exercise 19. (a) follows from that B_t has independent increments: $\{(B_{t_{j+1}} - B_{t_j}), j = 1, 2, \ldots, n\}$ are independent for any $0 \le t_1 < t_2 < \cdots < t_n$.

A Brownian motion is characterised being a Gaussian process with covariance $\mathbf{E}(B_s B_t) = \min(s,t)$. See hint on this on the previous exercise sheet. For (f) observe that the probability that $\lim_{t\to 0, t>0} W_t = 0$ is the same as $\lim_{t\to 0, t>0} B_t = 0$. Apply (f) to obtain (g).

Exercise 20 (b): use exercise 19 (a), and (e).

Exercise 24. Write limits as supremums and infimums.

Problem Sheet 4: Martingales

Exercise 26 Let $a > 0, p > 1$.

(a) Let (X_n) be a sub-martingale (or a martingale). Prove the maximal inequality: letting $A = \{\max_{1\le k\le n} X_k \ge a\}$,

$$P\left(\max_{0\le k\le n} X_k \ge a\right) \le \frac{1}{a}\mathbf{E}[X_n \mathbf{1}_A].$$

(b) Let $Y : \Omega \to \mathbf{R}_+$ be a random variable show that for any constant $C > 0$,

$$\mathbf{E}(Y \wedge C)^p = \int_0^C p t^{p-1} P(Y \ge t) dt.$$

(c) Suppose that $|X_n|$ is a sub-martingale. Let $X^* = \sup_{0\le k\le n} |X_k|$ Show that

$$\mathbf{E}(X^* \wedge C)^p \le \frac{p}{p-1}\mathbf{E}\left(|X_n|(X^* \wedge C)^{p-1}\right).$$

(d) Suppose that X_n is a martingale or a positive sub-martingale. Let $X^* = \sup_{0\le k\le n} |X_k|$. Show that $\mathbf{E}(X^* \wedge C)^p \le \left(\frac{p}{p-1}\right)^p \mathbf{E}|X_n|^p]$ and that

$$\mathbf{E}\sup_{0\le k\le n} |X_k|^p \le \left(\frac{p}{p-1}\right)^p \mathbf{E}|X_n|^p.$$

Exercise 27 Let I be an interval of $\mathbf{R}_+$ and $(X_t, t \in I)$ a right continuous martingale or a right continuous positive submartingale. Let $\|X\|_{L_p} := (\mathbf{E}|X|^p)^{\frac{1}{p}}$.

- Prove the maximal inequality: for $p \ge 1$ and $a > 0$,

$$a^p P(\sup_{t\in I} |X_t| \ge a) \le \sup_{t\in I} \mathbf{E}(|X_t|^p).$$

- Prove Doob's L^p inequality: for $p > 1$,

$$\left\|\sup_{t\in I} |X_t|\right\|_{L^p} \le \left(\frac{p}{p-1}\right) \sup_t \|X_t\|_{L^p}.$$

Exercise 28 Let M_t be a continuous integrable and adapted stochastic process.

(a) Show that if for all bounded stopping times T, $\mathbf{E}M_T = \mathbf{E}M_0$ then (M_t) is a martingale.

(b) Let X_t be a $\mathcal{F}_t$-martingale and $\mathcal{G}_t$-adapted where $\mathcal{G}_t \subset \mathcal{F}_t$. Show that M_t is a $\mathcal{G}_t$-martingale.

(c) Let T be a stopping time, M_t a martingale. Show that M^T is both a $\mathcal{F}_t$ and $\mathcal{F}_{t\wedge T}$ martingale.

Exercise 29 Let M_t be a bounded martingale, show that

(a) Show that if $a < b \le c < d$ then

$$\mathbf{E}(M_d - M_c)(M_b - M_a) = 0.$$

(b) If $a < b < c$, $\mathbf{E} M_a M_b (M_b - M_a)(M_c - M_b) = 0$. Show that

$$\mathbf{E}\left[M_a(M_c - M_a) - M_a(M_b - M_a) - M_b(M_c - M_b)\right]^2 \le \mathbf{E}(M_a - M_b)^2(M_c^2 - M_b^2),$$

Also $\mathbf{E}\left[M_a(M_c - M_a) - M_a(M_b - M_a) - M_b(M_c - M_b)\right]^2 \le \frac{1}{2}\delta\mathbf{E}[M_c^2 - M_a^2]$ where $\delta = \max\{(M_b - M_a)^2, (M_c - M_b)^2\}$.

Exercise 30 Let $\Delta^n : 0 \le t_1^n = \frac{1}{2^n} < t_2^n = \frac{2}{2^n} < \dots$ be a dyadic partition. Let M_t be a bounded sample continuous martingale with $M_0 = 0$.

(a) Define

$$Y_t^n = \sum_{j=0}^{\infty} M_{t_j^n}(M_{t\wedge t_{j+1}^n} - M_{t\wedge t_j^n}).$$

In another word, if $t \in (t_N^n, t_{N+1}^n]$,

$$Y_t^n = \sum_{j=0}^{N-1} M_{t_j^n}(M_{t\wedge t_{j+1}^n} - M_{t\wedge t_j^n}) + M_{t_N^n}(M_t - M_{t_N^n}).$$

Show that for each n, Y_t^n is a martingale.

(b) Show that Y_t^n converges to a process, Y_t in probability.

(c) Show that Y_t is a martingale, which is later seen as the stochastic integral

$$\int_0^t M_s dM_s.$$

(d) Prove the summation by parts formula:

$$M_t^2 = M_0^2 + 2Y_t^n + \sum_{j=0}^{\infty}(M_{t\wedge t_{j+1}} - M_{t\wedge t_j})^2.$$

(e) Let

$$Z_t^n = \sum_{j=0}^{\infty} (M_{t\wedge t_{j+1}} - M_{t\wedge t_j})^2.$$

Show that Z_t^n converges in probability to an increasing process A_t. The process A_t is often denoted as $\langle M, M\rangle_t$ and is called the quadratic variation process of M_t. we have the following special case of Itô's formula:

$$M_t^2 = M_0^2 + 2\int_0^t M_s dM_s + \langle M, M\rangle_t.$$

Exercise 31 Let $\Delta_n : 0 \le t_1 \le t_2 \le \cdots \le t_{N(n)} = t$, be a a sequence of partitions of $[0, t]$ with $|\Delta_n| \to 0$. Show that the following convergence holds in probability:

$$\lim_{n\to\infty} \sum_{j=0}^{N(n)-1} (B_{t_{j+1}} - B_{t_j})^2 = t.$$

Exercise 32 Let M, N be bounded martingales with $M_0 = N_0 = 0$. Define

$$\langle M, N\rangle_t := \frac{1}{2}\langle M+N, M+N\rangle_t - \frac{1}{2}\langle M, M\rangle_t - \frac{1}{2}\langle N, N\rangle_t.$$

Show that for the dyadic partition,

$$\sum_{j=0}^{\infty} (M_{t\wedge t_{j+1}} - M_{t\wedge t_j})(N_{t\wedge t_{j+1}} - N_{t\wedge t_j})$$

converges in probability to $\langle M, N\rangle$ and $M_t N_t - \langle M, N\rangle_t$ is a martingale.

If M_t and N_t are furthermore independent their quadratic variation $\langle M, N\rangle_t$ vanishes.

Problem Sheet 5: Stochastic Integrals

Exercise 33 If $B_t = (B_t^2, \dots, B_t^n)$ is a d-dimensional BM show that $\langle B^i, B^j \rangle_t = \delta_{ij} t$ and

$$\|B_t\|^2 = 2\sum_{i=1}^n \int_0^t B_s^i dB_s^i + nt,$$

where $\|B_t\|^2 = \sum_i |B_t^i|^2$.

Exercise 34 Let B_t be 1-dimensional Brownian Motion,

1. Compute $\mathbf{E} \int_0^T B_s^2 dB_s$.
2. Show that $\langle B_t, \int_0^t B_s^3 dB_s \rangle = \int_0^t B_s^3 ds$.
3. Is $\int_0^t e^{e^{B_s}} dB_s$ a local martingale? a true martingale?
4. Simplify $\int_0^t (2B_s + 1) d \left(\int_0^s B_r d(B_r + r) \right)$.
5. Prove that if H and K are continuous bounded and adapted semi-martingales,

$$\langle HK, B \rangle_t = \int_0^t H_r d\langle K, B \rangle_r + \int_0^t K_r d\langle H, B \rangle_r .$$

Exercise 35 Let $f, g : \mathbf{R} \to \mathbf{R}$ be C^2 functions, B_t one dimensional Brownian motion. Write the following as an integral with respect to the Lebesque measure.

1. $\langle f(B_t + t), g(B_t) \rangle$ where f and g are smooth functions.
2. $\langle \exp(M - \frac{1}{2}\langle M \rangle), \exp(N - \frac{1}{2}\langle N \rangle) \rangle$ where M_t and N_t are continuous local martingales.

Exercise 36 Interpret $\int_0^t s \, dB_s$ by a Lebesque integral.

Exercise 37 Let $\sigma : \mathbf{R} \to \mathbf{R}$ and $b : \mathbf{R} \to \mathbf{R}$ be Borel measurable functions. Suppose that x_t is an adapted time continuous stochastic process such that the following identity holds:

$$x_t = x_0 + \int_0^t \sigma(x_s) dB_s + \int_0^t b(x_s) ds.$$

If f is C^2 define

$$\mathcal{A} f = \frac{1}{2} \sigma^2(x) \frac{\partial^2 f}{\partial x^2} + b(x) \frac{\partial}{\partial x} f.$$

Show that $f(x_t) - f(x_0) - \int_0^t \mathcal{A} f(x_s) ds$ is a local martingale.

Exercise 38 Show that a positive local martingale is a super-martingale.

Exercise 39 Let $B_t, t \le T$ be a Brownian motion with $B_0 = 0$ and $\mathcal{F}_t$ its natural filtration augmented with null sets. It is know that $\mathcal{F}_0$ consists of sets of null or full measure and $\mathcal{F}_t$ is right continuous. Hence if $M_0 \in \mathcal{F}_0$ it is necessary that $M_0 = C$ some constant C almost surely.

(a) Let D be the set of f in $L^2(\Omega, \mathcal{F}_T)$ with the property that there is $h \in L^2(B)$ such that

$$f = \mathbf{E} f + \int_0^T h_s dB_s.$$

Show that D is a closed subspace of $L^2(\Omega, \mathcal{F}_\infty)$.

(b) If Z_t is a semi-martingale let $X_t = e^{Z_t - \frac{1}{2}\langle Z, Z\rangle_t}$. Show that

$$X_t = 1 + \int_0^t X_s dZ_s.$$

(c) It is known that the family

$$\left\{ \sum_{i=1}^n a_i \exp\left(\int_0^T (h_i)(s) dB_s - \frac{1}{2}\int_0^T h_i^2(s) ds \right) : a_i \in \mathbf{R}, h_i \in L^2([0,T], \mathbf{R}), n = 1, 2, \dots, \right\}$$

is dense in $L^2(\Omega, \mathcal{F}_T, \mathbf{R}^d)$.

Show that for all $f \in L^2(\Omega, \mathcal{F}_T)$ there is a unique $h \in L^2(B)$ such that

$$f = \mathbf{E} f + \int_0^T h_s dB_s.$$

(d) Let M be an L^2 bounded continuous martingale show that there is a unique $h \in L^2(B)$ such that for all $t \le T$,

$$M_t = M_0 + \int_0^t h_s dB_s.$$

[Hint: Recall the correspondence between $L^2(\Omega, \mathcal{F}_T)$ and L^2 bounded continuous martingales.]

(e) If M_t is a continuous local martingale show that there is a unique h progressively measurable such that $\int_0^t h_s^2 ds < \infty$ and $M_t = M_0 + \int_0^t h_s dB_s$. Congratulations! You've proved the integral representation theorem for martingales.

Problem Sheet 6: Itô's formula, Martingale Inequalities

Exercise 40

1. Prove that if M_t is a positive local martingale then it is a supermartingale.
2. If N_t is a local martingale show that $e^{N_t - \frac{1}{2}\langle N,N\rangle_t}$ is a local martingale and $\mathbf{E} e^{N_t - \frac{1}{2}\langle N,N\rangle_t} \le 1$.
3. Prove that if $\mathbf{E}\exp\left(\frac{1}{2}+\epsilon)\langle N,N\rangle_t\right) < \infty$ then $e^{N_t - \frac{1}{2}\langle N,N\rangle_t}$ is a true martingale.

Exercise 41 Show that any positive local martingale N_t with $N_0 = 1$ can be written in the form of $N_t = \exp\left(M_t - \frac{1}{2}\langle M,M\rangle_t\right)$ where M_t is a local martingale.

Exercise 42 Suppose that S_i are stopping times with $0 \le S_1 \le S_2 \le t$ almost surely. Let f be an adapted continuous process with $\int_0^t \mathbf{E} f_r^2 dr < \infty$. Define

$$\int_{S_1}^{S_2} f_r dB_r = \int_0^{S_2} f_r dB_r - \int_0^{S_1} f_r dB_r.$$

(a) Show that $\mathbf{E}\int_{S_1}^{S_2} f_r dB_r = 0$ and $\mathbf{E}(\int_{S_1}^{S_2} f_r dB_r)^2 = \mathbf{E}\int_{S_1}^{S_2} (f_r)^2 dr$.

Hint: $\int_{S_1}^{S_2} f_r dB_r = \int_0^t \mathbf{1}_{S_1 < r \le S_2} f_r dB_r$.

(b) Show that $\mathbf{E}\{\int_{S_1}^{S_2} f_r dB_r | \mathcal{F}_{S_1}\} = 0$

(c) Show that

$$\mathbf{E}\left\{\left(\int_{S_1}^{S_2} f_r dB_r\right)^2 | \mathcal{F}_{S_1}\right\} = \mathbf{E}\left\{\int_{\tau_1}^{S_2} f_r^2 dr | \mathcal{F}_{S_1}\right\}.$$

(d) If $\int_0^\infty \mathbf{E} f_r^2 dr < \infty$, the conclusions above hold for unbounded S.T. $S_1 < S_2$.

Exercise 43 If f be an adapted continuous function with $\int_0^t \mathbf{E} f_r^2 dr < \infty$. Show that for any $N > 0, C > 0$

$$P\left(\sup_{0 \le s \le t} |\int_0^s f_r dB_r| > C\right) \le P\left(\int_0^t (f_r)^2 dr > N\right) + C_2 \frac{N}{C^2}$$

for some constant C_2.

Exercise 44 Let τ be a bounded stopping time. Let B_t be an $\mathcal{F}_t$ Brownian motion and let $\mathcal{G}_t = \mathcal{F}_{t+\tau}$. Show that $W_t := B_{t+\tau} - B_\tau$ is a standard $\mathcal{G}_t$-Brownian motion.

Exercise 45 (Burkholder-Davis-Gundy Inequality) For every $p > 0$, there exist universal constants c_p and C_p such that for all continuous real valued local martingales vanishing at 0,

$$c_p \mathbf{E}\langle M, M\rangle_T^{\frac{p}{2}} \le \mathbf{E}(\sup_{t<T} |M_t|)^p \le C_p \mathbf{E}\langle M, M\rangle_T^{\frac{p}{2}}$$

where T is a finite number, infinity or a stopping time.

(1) Show that for any bounded continuous process H and stopping time T,

$$c_p \mathbf{E}\left(\int_0^T H_s^2 d\langle M, M\rangle_s\right)^{\frac{p}{2}} \le \mathbf{E}\sup_{s\le T}\left|\int_0^s H_r dM_r\right|^p \le C_p \mathbf{E}\left(\int_0^T H_s^2 d\langle M, M\rangle_s\right)^{\frac{p}{2}}.$$

(2) For $p \ge 2$ prove the right hand side of the Burkholder-Davis-Gundy Inequality. [Hint: Apply Itô's formula]

(3) For $p \ge 4$, prove the left hand side of the Burkholder-Davis-Gundy Inequality. [Hint: Begins with $\langle M, M\rangle_t = M_t^2 - 2\int_0^t M_s dM_s$, followed by an application of the elementtary identity $|a+b|^p \le c(p)(|a|^p + |b|^p)$ for some constant $c(p)$ and an application of Kunita-Watanabe inequality.

Exercise 46 Let X be a continuous semi-martingale and f be a convex function.

(a) Prove by approximating f with smooth functions, that there exists a continuous increasing process A^f such that

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s)dX_s + A_t^f.$$

[Hint: First assuem that $|X_t| \le C$. Let $\rho : \mathbf{R} \to \mathbf{R}_+$ be a smooth function with compact support in $(-\infty, 0]$ with $\int_{\mathbf{R}} \rho(x)dx = 1$. Let $\rho_n(x) = n\rho(nx)$ and $f_n = \int_{-\infty}^0 f(x+y)\rho_n(y)dy$ the convolution function. Then $f_n \to f$, f'_n increases to f'_-.]

(b) If $f(x) = |x|$ then $f'_-(x) = \text{sgn}(x)$ where

$$\begin{aligned} \text{sgn}\,(x) &= 1, \quad x > 0 \\ &= -1, \quad x \le 0. \end{aligned}$$

Prove the following Tanaka's formula(e), for any $a \in \mathbf{R}$,

$$\begin{aligned}
(X_t - a)^+ &= (X_0 - a)^+ + \int_0^t \mathbf{1}_{\{X_s > a\}} dX_s + \frac{1}{2} L_t^a, \\
(X_t - a)^- &= (X_0 - a)^- - \int_0^t \mathbf{1}_{\{X_s \le a\}} dX_s + \frac{1}{2} L_t^a, \\
|X_t - a| &= |X_0 - a| + \int_0^t \operatorname{sgn}(X_s - a) dX_s + L_t^a.
\end{aligned}$$

Here L_t^a is an increasing continuous function and is called the local time of X_t at a.

Exercise 47 Let σ and b be smooth functions from $\mathbf{R}^d$ to $\mathbf{R}$ with (at most) linear growth:

$$|\sigma(x)| \le c(|1 + |x|), \qquad |b(x)| \le c(1 + |x|).$$

Let T_n an inccreasing sequence of stopping times.

(a) Let B_t be a one dimensional Brownian motion. Let $(x_t, t \ge 0)$ be a real valued adapted sample continuous process, s.t. for all t

$$x_t^{T_n} = x_0 + \int_0^{t \wedge T_n} \sigma(x_s) dB_s + \int_0^{t \wedge T_n} b(x_s) ds$$

Show that $\mathbf{E}(|x_t|^2) < \infty$.

(b) State and prove a multi-dimensional of the above statement. Keep the following notation: $x_t = (x_t^1, \dots, x_t^d)$ and

$$(x_t^j)^{T_n} = x_0^j + \sum_{k=1}^m \int_0^{t \wedge T_n} \sigma_k^j(x_s) dB_s^k + \int_0^{t \wedge T_n} b^j(x_s) ds.$$

(c) Can you modify the proof to show that $\mathbf{E}(\sup_{t \le T} |x_t|^2) < \infty$?

Hint: Gronwall's lemma says that if $\alpha_t \le C_1 + \int_0^t g(s) \alpha_s ds$ then

$$\alpha_t \le C_1 \exp\left(\int_0^t g(s) ds\right).$$

Problem Sheet 7: Stochastic Differential Equations

Let B_t be a one dimensional Brownian motion on a given filtered probability space. Let $\sigma, b : \mathbf{R} \to \mathbf{R}$ be locally bounded and Borel measurable.

Exercise 48 Write down a solution to $dx_t = x_t g(B_t) dB_t$ where $g : \mathbf{R} \to \mathbf{R}$ is a bounded Borel measurablke function. Verify you claim. Is this a strong solution? Does pathwise uniqueness hold? Show that if $|g^2(x)| \le C + Ce^x$ then $\int_0^t g(B_s) dB_s$ is a martingale.

Exercise 49 Black-Scholes equation. Let S_t be a stock. It is postulated that S_t is governed by

$$dS_t = \sigma(t) S(t) dB_t + b(t) S_t dt.$$

Here $\sigma, b : \mathbf{R}_+ \to \mathbf{R}$ are Borel measurable functions. Give an explicit solution.

Exercise 50 Suppose that σ and b are real valued Lipschitz continuous functions. Suppose that for all $t \ge 0$,

$$\begin{aligned} x_t &= x_0 + \int_0^t \sigma(x_s) dB_s + \int_0^t b(x_s) ds, \\ y_t &= x_0 + \int_0^t \sigma(y_s) dB_s + \int_0^t b(y_s) ds \end{aligned}$$

Prove that $\mathbf{E}(x_t - y_t)^2 = 0$.

Exercise 51 Let $\sigma : \mathbf{R} \to \mathbf{R}$ be BC^1 and $f : \mathbf{R} \to \mathbf{R}$ a solution of the ODE $\dot{f} = \sigma(f)$. For $g : \mathbf{R} \to \mathbf{R}$ locally bounded and Borel measurable, assume that

$$dy_t = dB_t + g(f(y_t)) dt$$

has a solution y_t. Let $b = \sigma g + \frac{1}{2} \dot{\sigma} \sigma$. Show that $f(y_t)$ solves

$$dx_t = \sigma(x_t) dB_t + b(x_t) dt$$

Exercise 52 (Transform a drift) Consider the SDE: $dx_t = \sigma(x_t) dB_t + b(x_t) dt$, where σ, b are continuous. Let $\mathcal{L} = \frac{1}{2} \sigma^2 \frac{d^2}{dx^2} + b(x) \frac{d}{dx}$. A function s is the scale function if $\mathcal{L} s = 0$. Assume that $\sigma > 0$. Then

$$\dot{s}(x) = e^{-\int_0^x \frac{2b(y)}{\sigma^2(y)} dy}.$$

Since $\dot{s} > 0$ the scale function is increasing whose inverse on its image is denoted by s^{-1}. Let $\tilde{\sigma}(y) = \sigma(s^{-1}(y))\dot{s}(s^{-1}(y))$ if y is in the image of s otherwise let $\tilde{\sigma}(y) = 0$. Define $y_t = s(x_t)$. Show that y_t solves:

$$dy_t = \tilde{\sigma}(y_t)dB_t.$$

Prove that if b is bounded, then pathwise uniqueness holds for the SDE $dx_t = dB_t + b(x_t)dt$.

Exercise 53 Consider the SDE, with Stratonovitch integration,

$$\begin{aligned} dx_t &= \frac{y_t}{r_t} \circ dB_t \\ dy_t &= \frac{-x_t}{r_t} \circ dB_t \end{aligned}$$

Show that $r_t = \sqrt{x_t^2 + y_t^2}$. Show that $r_t = 1$ for all time if $r_0 = 1$. Conclude that the SDE can be considered to be defined on the circle S^1.

Exercise 54 Consider $dx_t = \sigma(x_t)dB_t + b(x_t)dt$. Assume that σ and b are locally Lipschitz continuous and are of at most linear growth:

$$|\sigma(x)| \le C(1 + |x|), \qquad \langle x, b(x)\rangle \le C(1 + |x|^2).$$

Let x_t be a solution. Prove that there is no explosion.

Exercise 55 Consider

$$\begin{aligned} dx_t &= y_t dB_t^1 \\ dy_t &= y_t dB_t^2. \end{aligned}$$

Show that if $y_0 > 0$ then y_t is positive and hence the SDE can be considered to be defined on the upper half plane. Compute the infinitesimal generator L. This is known as the Brownian motion on the hyperbolic space (upper half plane model).

Exercise 56 Discuss the uniqueness and existence problem for the SDE

$$dx_t = \sin(x_t)dB_t^1 + \cos(x_t)dB_t^2.$$

Problem Sheet 8: SDEs

Let $\sigma_j, 1 \le j \le m$ and b be measurable locally bounded vector fields on $\mathbf{R}^n$ with components $b = (b^1, \dots, b^n)$ where $b^j : \mathbf{R}^n \to \mathbf{R}$ and $\sigma_j = (\sigma_j^1, \dots, \sigma_j^n)$. Write $|b|(x) = \sqrt{\sum (b^i)^2(x)}$. Let $B_t = (B_t^1, \dots, B_t^m)$ be a Brownian motion. We do not distinguish a row vector from a column vector. Let

$$\mathcal{L} = \frac{1}{2} \sum_{k=1}^{m} \sigma_k^i \sigma_k^j \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{l=1}^{n} b_l \frac{\partial}{\partial l}.$$

Exercise 57 For $\mathcal{L}$ given above, compute $\mathcal{L}(|x|^{2\alpha})$. Suppose that $|\sigma(x)|^2 \le c(1 + |x|^2)$ and $\langle b(x), x \rangle \le c(1 + |x|^2)$. Let $f(x) = |x|^2 + 1$, $g(x) = |x|^{-2}$. Prove that $\mathcal{L}f \le af$. How about $\mathcal{L}g(x)$ when $|x| > 1$? Here c, a are constants.

Exercise 58 Suppose that σ, b are locally Lipschitz continuous and have at most linear growth and let $F_t(x)$ be the solution to the SDE $E(\sigma, b)$. Show that for each $t > 0$, $\lim_{x \to \infty} F_t(x) = \infty$, with convergence in probability.

Exercise 59 A one dimensional continuous process $(x_t, 0 \le t \le 1)$ is said to be the Brownian bridge if it is a Gaussian process and such that $\mathbf{E} x_t = 0$ and $\mathbf{E}(x_t x_s) = s \wedge t - st$.

1. Prove that if B_t is a Brownian motion, $x_t = B_t - tB_1$ is a Brownian bridge. Is x_t adapted to the natural filtration $\mathcal{F}_t^B$ of B_t?
2. Consider $dx_t = dB_t + \frac{y - x_t}{1-t} dt$. Find a solution to this SDE.

   [Hint: Try $x_t = (1-t)x_0 + ty + (1-t)\int_0^t \frac{dB_s}{1-s}$.]
3. Prove that $\lim_{t \uparrow 1} x_t = x$ in L^2.

Exercise 60 Let B^1 and B^2 be independent Brownian motions. In each case below compute the infinitesimal generator $\mathcal{L}$ and discuss whether the SDE explodes:

1. $$\begin{aligned} dx_t &= (y_t^2 - x_t^2) dB_t^1 - 2x_t y_t dB_t^2 \\ dy_t &= 2x_t y_t dB_t^1 + (y_t^2 - x_t^2) dB_t^2. \end{aligned}$$

2. $$\begin{aligned} dx_t &= (x_t^2 + y_t^2) dB_t^1 \\ dy_t &= (x_t^2 + y_t^2) dB_t^2 \end{aligned}$$

Exercise 61 Transform the following SDE into Itô form.

$$\begin{aligned} dx_t &= (y_t^2 - x_t^2) \circ dB_t^1 - 2x_t y_t \circ dB_t^2 \\ dy_t &= 2x_t y_t \circ dB_t^1 + (y_t^2 - x_t^2) \circ dB_t^2. \end{aligned}$$

Exercise 62 Write down the infinitesimal generator of

$$\begin{aligned} dx_t &= -(x_t^2 + y_t^2) x_t \circ dB_t^1 - (x_t^2 + y_t^2) y_t \circ dB_t^2 \\ dy_t &= (x_t^2 + y_t^2) y_t \circ dB_t^1 - (x_t^2 + y_t^2) x_t \circ dB_t^2. \end{aligned}$$

Exercise 63 Let D be a bounded domain of $\mathbf{R}^d$ with smooth boundary. Suppose that there is a C^2 solution to the Dirichlet problem: $\Delta u = 0$ on D and $u = f$ on the boundary ∂D of D. Let τ^D be the first exit time from D of the solution $F_t(x), x \in D$ of an SDE whose generator is $\frac{1}{2}\Delta$. Prove that $u(x) = \mathbf{E} f(F_\tau(x))$ and

$$\mathbf{E} f^2(F_\tau(x)) = u^2(x) + 2\mathbf{E} \int_0^\tau |\nabla u|^2 (F_s(x)) ds.$$

In the following if A is a matrix A^T stands for its transpose.

Exercise 64 Suppose that σ, b are smooth and σ has compact support. Let x_t be solution to $dx_t = \sigma(x_t) dB_t + b(x_t) dt$ with $x_0 \in \mathbf{R}^d$.

1. Let $\mu_t = \mathbf{E} x_t$. Show that $\mu_t = \mu_0 + \int_0^t \mathbf{E}(b(x_s)) ds$.
2. Let $C(t) = (x_t - \mu_t)(x_t - \mu_t)^T$. This is a $n \times n$ matrix with entries $C_{i,j}(t) = (x_t^i - \mu_t^i)(x_t^j - \mu_t^j)$. Write down a formula for $C(t)$.
3. Let $R(t) = \mathbf{E}(x_t - \mu_t)(x_t - \mu_t)^T$ be the covariance matrix. Let C be an $d \times d$ matrix and define $b(x) = Cx$. Show that

$$R_t = R_0 + \int_0^t C_s R_s ds + \int_0^t R_s C_s^T ds + \int_0^t \mathbf{E} \sigma(x_s) \sigma(x_s)^T ds.$$

Exercise 65 Show that if (X_t) is a Markov process then it is a Markov process with respect to its own filtration.

Problem Sheet 9: Girsanov Transform

Let $\sigma_j, 1 \le j \le m$ and b be measurable locally bounded vector fields on $\mathbf{R}^n$ with components $b = (b^1, \dots, b^n)$ where $b^j : \mathbf{R}^n \to \mathbf{R}$ and $\sigma_j = (\sigma_j^1, \dots, \sigma_j^n)^T$. Write $|b|(x) = \sqrt{\sum (b^i)^2(x)}$. Let $B_t = (B_t^1, \dots, B_t^m)$ be a Brownian motion. Let

$$\mathcal{L} = \frac{1}{2} \sum_{k=1}^{m} \sigma_k^i \sigma_k^j \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{l=1}^{n} b_l \frac{\partial}{\partial l}.$$

Exercise 66 Let Z be a 1-dimensional Gaussian random variable on $(\Omega, \mathcal{F}, P)$ with distribution $N(a, \sigma^2)$. Take $u \in \mathbf{R}$ and define a probability measure Q by

$$\frac{dQ}{dP}(\omega) = e^{(\frac{u}{\sigma^2})(Z(\omega)-a)-\frac{1}{2}(\frac{u}{\sigma})^2}.$$

Show that the distribution of $Z - u$ under Q is the distribution of Z under P. [Hint: compute the characteristic function.]

Exercise 67 Let $(M_t, t \le T)$ be a martingale on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. If N_t is a bounded martingale, under which measure is $M_t - \langle M, N \rangle_t$ a local martingale? Prove your assertion.

Exercise 68 Take $u \in \mathbf{R}^n$ and B_t a Brownian motion on $(\Omega, \mathcal{F}_t, P)$. Under which measure is $B_t - tu$ a Brownian motion under Q? Is Q a probability measure?

Exercise 69 Let $T > 0$ and Q and P be two equivalent measures on $(\Omega, \mathcal{F}_T, \mathcal{F}_t)$ with $\frac{dQ}{dP} = e^{\int_0^T h_s dB_s - \frac{1}{2}\int_0^T h_s^2 ds}$ where $h : [0, T] \to \mathbf{R}$ is locally bounded. Let $N_t = \int_0^t h_s dB_s$. By abuse of notation, if M_t is a P local martingale, we say that $M_t - \langle N, M \rangle_t$ is its Girsanov transform. Is Q a probability measure? If $(B_t, t \le T)$ is a Brownian motion with respect to P compute the Girsanov transform of the folowing martignales: (a) B_t; (2) $B_t^2 - t$, (3) $\int_0^t h_s dB_s$.

Exercise 70 Let $m = d$. Suppose that x_t satisfies $x_t = x_0 + B_t + \int_0^t b(x_s) ds$ where $b : \mathbf{R}^d \to \mathbf{R}^d$ is bounded C^2. Let $N_t = \int_0^t \langle dB_s, b(x_s) \rangle$. Compute $\langle N, N \rangle_t$. Show that under Q where $\frac{dQ}{dP} = \exp(-N_t - \frac{1}{2}\langle N, N \rangle_t)$, x_t has distribution $N(x_0, tI)$.

Exercise 71 1. Let u_t be a $C^{2,1}$ bounded solution to $\frac{\partial u_t}{\partial t} = \mathcal{L} u_t$ where $\mathcal{L} = \frac{1}{2}\Delta + \sum_{j=1}^n b_j \frac{\partial}{\partial x_j}$. Prove that

$$u(t, x) = \mathbf{E} f(x + B_t) \exp\left(\int_0^t \langle b(x + B_s), dB_s \rangle - \frac{1}{2} \int_0^t |b|^2 (x + B_s) ds \right).$$

2. Let $g : \mathbf{R}^n \to \mathbf{R}$ and assume that $b = \nabla g$. Show that

$$u_t(x) = \mathbf{E} f(x+B_t) \exp\left(g(x+B_t) - g(x_0) - \int_0^t [\mathcal{L}g + \frac{1}{2}|\nabla g|^2](x+B_s)ds \right).$$