Curvature and topology: spectral positivity

K. D. Elworthy
Mathematics Institute
University of Warwick
Coventry CV4 7AL,U.K.

X.-M. Li
Mathematics Institute
University of Warwick
Coventry CV4 7AL,U.K.

Steven Rosenberg
Department of Mathematics
Boston University
Boston, MA 02215, USA
sr@math.bu.edu

1 Introduction

Let M be a complete smooth Riemannian manifold. Its Laplace-Beltrami operator Δ is well known e.g. [Che73] to be essentially self-adjoint on $L^2(M;\mathbb{R})$ with core the space $C_0^\infty(M)$ of C^∞ functions on M with compact support. We will identify it with its closure on L^2 and take it to be non-negative (using the opposite sign convention to that usually used by probabilists). A function $f\colon M \to \mathbb{R}$ will be said to be *spectrally positive* if there exists $c > 0$ such that $\Delta + f \geq c$ as an unbounded operator on $L^2(M)$; this is equivalent to

$$\int_M \phi(x)\left(\Delta\phi(x) + f(x)\phi(x)\right) dx \geq c|\phi|^2_{L^2} \tag{1}$$

for all $\phi \in C_0^\infty(M)$. It has become apparent from the references [ER88a], [ER91], [Li92b], [Li93], [ER93], that many vanishing and related theorems which were originally formulated in terms of positivity of certain curvature expressions can be formulated in terms of the spectral positivity of these expressions; these apply in particular to the classical Bochner theorems for vanishing of de Rham cohomology, to Myers' theorem on the finiteness of $\pi_1(M)$ and to the homotopy vanishing theorems of Lawson and Simon [LS73] and others [HW86], [Ohn86], together with extensions of these. Here we gather these results together and comment on them, following a brief discussions of spectral positivity. We will be mainly considering compact manifolds. For noncompact M the relevant concept often appears to be *stochastic positivity*:

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log \mathbb{E} e^{-\frac{1}{2}\int_0^t f(x_s)ds} < 0,$$

and *strong stochastic positivity:* for all compact set K in M

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \sup_{x\in K} \log \mathbb{E} e^{-\frac{1}{2}\int_0^t f(x_s)ds} < 0,$$

where $\{x_s : 0 \leq s < \infty\}$ is a Brownian motion on M, or $M \cup \infty$ if necessary, starting from some point $x_0 \in M$, i.e. a sample continuous Markov process on M with differential generator $-\frac{1}{2}\Delta$ and associated semigroup P_t, and $\mathbb{E}$ denotes expectation.

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2 Spectral positivity, stochastic positivity, point positivity, and not much negativity

A. The classical Bochner theorems may be phrased in terms of a continuous function f, defined via the curvature of M, being nonnegative everywhere and strictly positive at some point of M; see §4 below. This condition we will call *point positivity*. For compact M, since $\Delta + f$ has discrete spectrum, point positivity implies spectral positivity. Here is a compendium of results of this nature:

Proposition 2.1 *For M compact and f continuous:*

(i) spectral positivity is equivalent to stochastic positivity,

(ii) if f is spectrally positive, then $\int_M f$ is positive.

For general M:

(iii) stochastic positivity of f on M implies that of the lift of f on any Riemannian cover of M, as does strong stochastic positivity.

(iv) strong stochastic positivity implies spectral positivity.

(v) spectral positivity implies that for all compact subsets K, U of M

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \sup_{x\in K} \log \mathbb{E} \chi_U(x_t) e^{-\frac{1}{2}\int_0^t f(x_s)ds} < 0.$$

(vi) The set of (strongly) stochastic positive functions is convex and if f lies in it so does αf for $0 < \alpha \leq 1$. The same holds for spectral positivity.

Proof: Set

$$\mu_f(x_0) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log \mathbb{E} e^{-\frac{1}{2}\int_0^t f(x_s)ds}.$$

Let λ_0 be the smallest eigenvalue of $\frac{1}{2}(\Delta + f)$. Then as is known, for compact manifolds,

$$\mu_f = -\lambda_0$$

(cf. [ER88a]). This implies (i). To see $\int_M f > 0$, take $\phi \equiv 1$ in (1). Part (iii) is clear since the projection onto M of the distribution of the horizontal lift $\{\tilde{x}_t : t \geq 0\}$ of $\{x_t : t \geq 0\}$ are the distributions of $\{x_t\}$ for all x_0 [Elw82]. For (iv) see [Gav79], and for (v) see [ER88a]. Finally (vi) is seen by Hölder's inequality.

Note that spectral positivity for noncompact manifolds is far from positivity since $\lambda_0(M)$, the bottom of the L^2 spectrum of Δ on M, may be negative. For examples of manifolds of constant negative curvature with spectrally positive Ricci curvature see [ER88a]. For a different concept of 'almost non-negative' curvature see [FY92] and the references there.

B. We now consider analytic/geometric conditions on a function f guaranteeing spectral positivity. We assume that there exist $A, B > 0$ such that for all $\phi \in C_0^\infty(M^n)$ the following Sobolev inequality holds:

$$||\phi||^2_{2n/(n-2)} \leq A||\nabla\phi||_2^2 + B||\phi||_2^2.$$

Such an inequality exists if M is compact or has positive injectivity radius. Let $W^n = (W^n, g)$ denote a generic Riemannian n-manifold. If M is compact, and

$$\begin{aligned} M \in \mathcal{N} &= \mathcal{N}(n, K, D, V) \\ &= \{W^n : \text{Ric} \geq K, \text{diam } W \leq D, \text{vol}(W) \geq V\}, \end{aligned}$$

then we may take $A = A(\mathcal{N}), B = B(\mathcal{N})$ [Gal88]. For any $\phi : M \to \mathbb{R}$, set $\phi_-(x) = \min\{\phi(x), 0\}$ and $\phi_+(x) = \phi(x) - \phi_-(x)$. The following result shows spectral positivity for functions which are positive on "most" of M. Note that (ii),(iii), are stronger than the more intuitive (i).

Theorem 2.2 *(i) Assume M is compact and choose K, D, V such that $M \in \mathcal{N}(n, K, D, V)$. Choose $a, b > 0$. Then there exists $\epsilon = \epsilon(\mathcal{N}, a, b)$ such that if $f \geq -b$ everywhere and $f \geq a$ except on a set U with $\text{vol}(U)/\text{vol}(M) < \epsilon$, then f is spectrally positive.*

(ii) If there exists $w > 0$ such that

$$||(f-w)_-||_{n/2} < \min\{A^{-1}, wB^{-1}\},$$

then f is spectrally positive.

(iii) Given $q > n$, there exists $\epsilon(n,q) > 0$ such that if

$$||(f-w)_-||_1 < \epsilon(n,q)w[(A||f_-||_{q/2})^{\frac{q}{q-n}} + B]^{-n/2},$$

then f is spectrally positive.

The proof of (i) is in [ER91], and (ii),(iii) are in [RY93].

3 Finiteness of volume and fundamental group

A. Myers' theorem asserts that a manifold with Ricci curvature bounded below by a positive constant is compact and has finite fundamental group; the second assertion follows immediately from the first by passing to the universal cover. There have been various extensions showing that compact manifolds with mostly positive curvature, in various senses, have finite fundamental group [Wu91b],[ER], [RY93]. These are implied by the stochastic positivity criterion:

Theorem 3.1 *[Li93] If the Ricci curvature of M is strongly stochastically positive and bounded below, then M has finite volume and finite fundamental group.*

Here we set $\rho(x) = \min_{|v|\leq 1}\{\mathrm{Ric_x(v,v)}\}$ and say that the Ricci curvature is strongly stochastically positive if ρ is. The proof that M has finite volume is essentially that of Bakry [Bak86]. A more precise criterion is that

$$\sup_{x_0 \in K} \int_0^\infty \mathbb{E}\left[\mathrm{e}^{-\frac{1}{2}\int_0^t \rho(x_s)ds}\right] dt < \infty$$

for each compact set K, which follows from strong stochastic positivity. Again the finiteness of $\pi_1(M)$ comes from considering the universal covering of M via Proposition 2.1 (iii). From this and Proposition 2.1 (i) we obtain:

Corollary 3.2 *[Li93] Any compact Riemannian manifold with spectrally positive Ricci curvature has finite fundamental group.*

Thus by Theorem 2.2, a compact manifold with "mostly positive" Ricci curvature has finite fundamental group. A purely geometric proof of this, relying only on the Bishop-Gromov comparison theorem, is in [RY93] (although the precise meaning of mostly positive differs).

When M is noncompact but complete with Ricci curvature satisfying

$$\mathrm{Ric}_x > -\frac{n}{n-1}\frac{1}{r^2(x)}, \quad \text{when } r > r_0$$

for some positive constant r_0, it has infinite volume [CGT82]. See also [Wu91a]. Here r denotes the distance function in M from a fixed point.

Thus there is an extension of Myers' compactness result:

Corollary 3.3 *[Li93] If M is a complete Riemannian manifold with $\mathrm{Ric}_x > -\frac{n}{n-1}\frac{1}{r^2(x)}$, for r big, then M is compact if the Ricci curvature is strongly stochastically positive.*

However we do not have a diameter estimate relevant to the situation (see [CGT82]). Possilbly strong stochastic positivity of the Ricci curvature will imply compactness. However spectral positivity does not imply either compactness of M or finitude of $\pi_1(M)$ as can be seen from remark 4 §3 of [ER88a]

4 Bochner type theorems

A. Let $\mathcal{R}^p(x) : \Lambda^p T_x^* M \to \Lambda^p T_x^* M$ be the p-th Weitzenböck curvature tensor of M^n at x, i.e. the term which appears in the Weitzenböck formula

$$\Delta^p \phi = -\mathrm{trace}\nabla^2\phi + \mathcal{R}^p\phi$$

for the Laplacian $\Delta^p\phi$ of a p-form ϕ, and let $\underline{\mathcal{R}}^p(x)$ be its lower bound at $x \in M$. Thus $\underline{\mathcal{R}}^1(x) = \underline{\mathrm{Ric}}(x) = \rho(x)$. The semigroup domination

$$|\mathrm{e}^{-\frac{t}{2}\Delta^p}\phi| \leq \mathrm{e}^{-\frac{t}{2}(\Delta + \underline{\mathcal{R}}^p)}|\phi| \tag{2}$$

at each point of M, which goes back to Meritet [Mer79] (cf. [DL82]) leads quickly to cohomology vanishing results given spectral positivity of $\underline{\mathcal{R}}^p$ [Mal76], [Mer79], [ER88a], [ER91], see also [Mal74]. Indeed, the spectral positivity of $\underline{\mathcal{R}}^p$ implies that the right hand side of (2) decays as $t \to \infty$, while the left hand side is constant on the kernel of Δ^p.

Theorem 4.1 *[ER88a] If $\underline{\mathcal{R}}^p$ is spectrally positive on M, then the p-th deRham L^2 cohomology group of M vanishes, and there are no nonzero L^2 harmonic p-forms on M. If M is compact, and $\underline{\mathcal{R}}^p$ is spectrally positive, then $H^p(M;\mathbb{R})=0$.*

Since pointwise positivity implies spectral positivity, the last statement generalizes the usual Bochner vanishing theorem: $\underline{\mathcal{R}}^p > 0 \Rightarrow H^p(M;\mathbb{R}) = 0$. For manifolds with boundary see [Mer79].

A more direct approach to vanishing theorems was given by Berard [Bér90]. Let E be a hermitian vector bundle over M with a compatible connection ∇ and a symmetric endomorphism $V : E \to E$. An application of Kato's inequality shows that $\underline{V}$ spectrally positive implies $\mathrm{Ker}(\nabla^*\nabla + V) = 0$ on $L^2(E)$. Letting $E = \Lambda^p T^* M$ and $V = \mathcal{R}^p$ reproduces the last theorem.

B. The topology of M near infinity was investigated from the point of view of spectral positivity in [ER88b], [ER91]. For example:

Theorem 4.2 *[ER88b] If $\underline{Ric}$ is spectrally positive then M has at most one infinite volume end, and the dimension of the first cohomology group with compact supports, $H^1_c(M,\mathbb{R})$, is one less than the number of finite volume ends (or zero if there are none).*

C. Let $\tilde{M}$ be the universal cover of M furnished with the Riemannian metric induced from its projection on M. By investigating the effects of stochastic positivity of $\underline{\mathcal{R}}^p$ on $\tilde{M}$ with the use of *bounded* simplicial chains, and applying Proposition 2.1 (iii) it is possible to obtain results about homology and homotopy.

Theorem 4.3 *[ER91] Suppose M is compact and that $2 \le p \le \dim M - 2$. Then if $\underline{\mathcal{R}}^p$ is spectrally positive and $H_{p-1}(\tilde{M};\mathbb{R}) = 0$, we have $H_p(\tilde{M};\mathbb{R}) = 0$. In particular $\underline{\mathcal{R}}^2$ spectrally positive implies $H_2(\tilde{M};\mathbb{R}) = 0$.*

These results were not known previously even in the pointwise positive case.

Corollary 4.4 *Suppose M is compact.*

(i) If $\underline{\mathcal{R}}^2$ is spectrally positive, then $\pi_2(M)$ is a torsion group and the orders of its elements are bounded.

(ii) If dim $= 3$ and $\underline{\mathrm{Ric}}$ is spectrally positive, then M is covered by a homotopy sphere.

(iii) If dim M=4 and $\underline{\mathrm{Ric}}$ and $\underline{\mathcal{R}}^2$ are spectrally positive, then M is covered by S^4.

(iv) If $\underline{\mathcal{R}}^p$ is spectrally positive for $1 \le p \le \dim M - 1$, then $\tilde{M}$ as well as every finite cover of M are real homology spheres.

(i) and (iii) are topological consequences of Theorem 4.3, and (iii) is a consequence of Corollary 3.2. Finally, (iv) for $\tilde{M}$ follows from Theorem 4.3, and for finite covers from Theorem 4.1 and Proposition 2.1.

D. Other geometric results given by Bochner type arguments admit similar generalizations from pointwise positivity to spectral positivity via either semigroup domination or Bérard's technique. The following results are taken from [Ros90]:

Theorem 4.5 *(i) Let M be a compact spin manifold with spectrally positive scalar curvature. Then the $\hat{A}$-genus of M vanishes.*

(ii) Let $\dim M = n$, *and let $\underline{K}(x)$ be the minimum of the sectional curvatures of M at x. Assume M is a minimal hypersurface of S^{n+1}. If $2n\underline{K}$ is spectrally positive, then M is an equator.*

(iii) Let M be an Einstein manifold with sectional curvatures at most one. Pick $F \in (1 - \frac{4}{n}, 1]$ if n is even and $F \in (1 - \frac{4n}{n^2-1}, 1]$ if n is odd. If $\underline{K} - F$ is spectrally positive, then M is isolated in the space of Einstein metrics.

(iv) If $-\underline{\mathrm{Ric}}$ is spectrally positive, then M admits no nontrivial one-parameter family of isometries.

The pointwise positivity results corresponding to this theorem are due to Lichnerowicz (for (i)), Berger and Ebin (for (ii) and (iii)), and Bochner and Yano (for (iv)); cf. the references in [Ros90].

E. There are results concerning endomorphisms V which are "not too negative" (as opposed to the "mostly positive" condition for spectral positivity) [Bér86], [BB90]. The motivation is again the proof of Bochner's theorem, which easily extends to show that $\underline{V} \geq 0$ implies $\dim \mathrm{Ker}(\nabla^*\nabla + V) \leq \dim E$, for M compact. For any bundle endomorphism V, set $V_{\min} = \min\{\underline{V}(x) : x \in M\}$.

Theorem 4.6 *(i) [Bér86] Let* $\dim M = n$ *and* $\operatorname{diam} M = D$. *Given $a \in \mathbb{R}^+$ with $\mathrm{Ric}_{\min} D^2 \geq -a^2$, there exists $b = b(n,a) > 0$ such that $V_{\min} D^2 \geq -b$ implies* $\dim \mathrm{Ker}(\nabla^*\nabla + V) \leq \dim E$.

(ii) [BB90] Assume $n \geq 3$. There exists $F = F(n, a, ||\underline{\mathrm{Ric}}_+||_\infty, ||\underline{\mathrm{Ric}}_+||_1)$ such that

$$\dim H^1(M;\mathbb{R}) \leq \frac{F}{\mathrm{vol(M)}} \int_M (\underline{\mathrm{Ric}}_-(x))^{n/2}\, dx.$$

5 Homotopy and homology vanishing for submanifolds of R^n

A. The method used here comes from the study of moment stability for stochastic flows of stochastic differential equations. Such an equation on M is written

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt \tag{3}$$

where X is a section of $\mathrm{Hom}(\underline{\mathbb{R}}^m, TM)$ for some m (so that $X(x) : \mathbb{R}^m \to T_xM$ is a linear map for each x in M) and A is a vector field on M. Here $\underline{\mathbb{R}}^m$ is the trivial $\mathbb{R}^m$ bundle over M, and $B_t : \Omega \to \mathbb{R}^m, t \geq 0$, is a Brownian motion on $\mathbb{R}^m$ with probability space $\{\Omega, \mathcal{F}, P\}$, say. A solution to (3) is a map $x: [0, \infty) \times \Omega \to M$ measurable in Ω and almost surely continuous in t (for non-compact M there may be a finite lifetime). Under suitable regularity conditions, such a solution $F_t(x_0, \omega)$ exists for all initial points x_0 in M and for compact manifolds it can be chosen to be continuous in $[0, \infty) \times M$ for almost all $\omega \in \Omega$, [Kun90], [Elw82], [Elw88], [IW89], and $F_t(-, \omega) : M \to M$ will be a C^∞ diffeomorphism.

The derivative map $TF_t(-, \omega) : TM \to TM$ and induced maps $\Lambda^q TF_t(-, \omega): \Lambda^q TM \to \Lambda^q TM$, $q = 1, \ldots, n$ thus exist. Given a Riemannian metric on M there are the moment exponents

$$\mu_K^q(p) = \overline{\lim_{t \to \infty}} \sup_{x_0 \in K} \frac{1}{t} \log \mathbb{E}|\Lambda^q T_{x_0} F_t|^p$$

$p \in \mathbb{R}$, $q = 1, 2, \ldots, n$, K a compact subset of M.

The s.d.e. (3) is *p-th moment stable* if $\mu_K^1(p) < 0$ for K an arbitrary point in M and *strongly p-th moment stable* if $\mu_K^1(p) < 0$ for each compact set K of M e.g. see [BS88], [Elw92].

B. For M compact it is easy to see that strong 1-moment stability implies that M is simply connected. Indeed, let $\sigma: S^1 \to M$ be a C^1 loop in M of minimal length $\ell(\sigma)$ in its homotopy class. Let $\sigma_t(\omega)(\theta) = F_t(\sigma(\theta), \omega)$ for $\theta \in S^1$ so $\sigma_t(\omega)$ is a loop homotopic to σ for $0 \leq t < \infty$, almost all ω. Then

$$\ell(\sigma) \leq \ell\left(\sigma_t(\omega)\right) = \int_{S^1} \left| TF_t(-, \omega)\left(\dot{\sigma}(\theta)\right)\right| d\theta$$

giving

$$\begin{aligned} \ell(\sigma) &\leq \mathbb{E}\ell(\sigma_t) \\ &= \int_{S^1} \mathbb{E}|TF_t(-, \omega)(\dot{\sigma}(\theta))| d\theta \end{aligned}$$

$$\leq \int_{S^1}\left(\sup_{\theta\in S^1}\mathbb{E}|T_{\sigma(\theta)}F_t|\right)|\dot\sigma(\theta)|d\theta$$
$$= \ell(\sigma)\left(\sup_{\theta\in S^1}\mathbb{E}|T_{\sigma(\theta)}F_t|\right)\to 0$$

as $t\to\infty$, given strong 1-moment stability. So σ must be constant.

There is a corresponding argument for the vanishing of $\pi_2(M)$ given strong 2-moment stability using the Sacks-Uhlenbeck theorem to obtain minimal area 2-spheres in M, but for higher homotopy groups it is not so clear what happens. However integral homology classes can be represented by minimal area currents in M when M is compact [Fed70], and essentially the same arguments [ER93] lead to the vanishing of $H_q(M;\mathbb{Z})$ given $\mu_K^q(1)<0$, and so given strong q-th moment stability.

C. For an isometrically immersed submanifold of $\mathbb{R}^m$ there is a natural choice of X: choose $X(x):\mathbb{R}^m\to T_xM$ to be the orthogonal projection for each x in M. If we take $A\equiv 0$, the solutions to (3) are Brownian motions on M [Elw82], [Elw92], and (3) is called a *gradient Brownian system*. When A is a gradient, conditions on X, A which imply strong moment stability of a given order can be obtained in the form of stochastic positivity type conditions [Li92b]. These involve the second fundamental form and Weitzenböck curvatures. Together with the considerations above they yield:

Theorem 5.1 *[ER93] (i) Let M be compact and isometrically immersed in $\mathbb{R}^m$ with second fundamental form α. If $\underline{\mathcal{R}}^q-\frac{1}{2}||\alpha||^2$ is spectrally positive, then $H_q(M;\mathbb{Z})=H_{n-q}(M;\mathbb{Z})=0$. Moreover if $\underline{Ric}-\frac{1}{2}||\alpha||^2$ is spectrally positive, then $\pi_1(M)=0$, and if $\underline{\mathcal{R}}^2-\frac{1}{2}||\alpha||^2$ is spectrally positive, then $\pi_2(M)=0$.*

(ii) Let $Z:\mathbb{R}^m\to\nu$ be the orthogonal projection of $\mathbb{R}^m$ into the normal bundle of M, and let $Z^i(x)=Z(x)(e_i)$, where $\{e_1,\dots,e_m\}$ is the standard basis of $\mathbb{R}^m$. Let A be the shape operator of M in $\mathbb{R}^m$, and let $\mathrm{Ric}^{\#}:T_xM\to T_xM$ *be the dual to the Ricci tensor acting on one-forms. Define $B:T_xM\to T_xM$ by $B(v)=-\sum_i A(A(v,Z^i),Z^i)+\mathrm{Ric}^{\#}$. If $\underline{B}$ is spectrally positive, then for any Riemannian manifold N and any Riemannian metric on M there are no nonconstant stable harmonic maps $f:N\to M$, and the homotopy class of any map $g:N\to M$ contains maps of arbitrarily small energy. The same holds for maps of M to N.*

This turns out to be an extension of results of Lawson & Simons [LS73] and Howard & Wei [HW86], see also Ohnita [Ohn86]. In fact the methods used are very similar. They use the deterministic flows $S_t^i:M\to M, t\in\mathbb{R}$, generated by the vector fields $X^i(x)\equiv X(x)e_i$ to show that there can be no nontrivial minimal currents in the dimension considered in Lawson and Simon's case, or no stable harmonic maps in Howard and Wei's case. Both [LS73], [HW86] require strict positivity conditions. The demonstration that Theorem 5.1 does generalize this work depends on the not well enough known fact [ER93] that the adjoint $(\mathcal{R}^q)^* : \Lambda^q TM \to \Lambda^q TM$ of the Weitzenböck curvature is given by:

$$(\mathcal{R}^q)^* V_0 = -\sum_{i=1}^{m} \frac{D^2}{\partial t^2} \Lambda^q (TS_t^i)(V_0). \tag{4}$$

The equality (4) together with Gauss's theorem also yields the apparently new (intrinsic!) equality

$$< (\mathcal{R}^q)^*(v_1 \wedge \ldots \wedge v_q), v_1 \wedge \ldots \wedge v_q >= \sum_{j=0}^{q} \sum_{l=q+1}^{\dim M} K(v_j, v_l)$$

where $< v_i, v_j >= \delta_{i,j}$, $v_i \in T_x M$, and $K(v_j, v_l)$ is the sectional curvature of the (v_j, v_l)-plane [ER93].

Moreover Theorem 5.1 could have been stated with $\underline{\mathcal{R}}^q$ replaced by the infimum $\underline{\check{\mathcal{R}}}^q$ of $\mathcal{R}^q$ over primitive vectors

$$\underline{\check{\mathcal{R}}}^q(x) = \inf\{< (\mathcal{R}^q)^*(v_1 \wedge \ldots \wedge v_q), v_1 \wedge \ldots \wedge v_q >:< v_i, v_j >= \delta_{i,j}, v_i \in T_x M\}.$$

Recall that $\underline{K}(x)$ is the minimum of the sectional curvatures at the point x. We then have

Corollary 5.2 *[ER93] If a compact manifold M isometrically immersed in $\mathbb{R}^m$ has $\underline{K} - \frac{||\alpha||^2}{2(\dim M - 1)}$ spectrally positive, then M is a homotopy sphere.*

D. When M is minimally immersed in S^{m-1} the contribution from the second fundamental form simplifies:

Theorem 5.3 *[ER93] Let M^n be minimally isometrically immersed in S^{m-1}. If $\underline{\check{\mathcal{R}}}^q - \frac{1}{2}q(n-q)$ is spectrally positive, then*

$$H_q(M; \mathbb{Z}) = H_{n-q}(M; \mathbb{Z}) = 0$$

In particular, if $\underline{K} - \frac{1}{2}$ is spectrally positive, then M is a homotopy sphere. Moreover if $\underline{Ric} - \frac{1}{2}(n-1)$ is spectrally positive, then $\pi_1(M) = 0$.

E. When M is not compact the situation is less clear, although the result for $\pi_1(M)$ does go over. Recall that $r(x)$ is the distance function [Li92a] [Li92b]:

Theorem 5.4 *Let M be a complete Riemannian manifold isometrically immersed in $\mathbb{R}^m$ with second fundamental form bounded pointwise in norm by $c[1+\ln(1+r(x))]^{\frac{1}{2}}$ for some $c \geq 0$. Then $\pi_1(M) = 0$ if ρ_1 has strong stochastic positivity for $\rho_1(x) = \inf_{|v|\leq 1}\{\mathrm{Ric}_x(v,v) - |\alpha_x(v,-)|^2_{H,S} + |\alpha_x(v,v)|^2\}$. Here $|-|_{H,S}$ denotes the Hilbert Schmidt norm.*

This theorem is a direct generalization of part (i) of theorem 5.1 since the strong stochastic positivity of ρ_1 is implied by that of $\underline{Ric} - \frac{1}{2}||\alpha||^2$ as shown in [ER93]. To compare with theorem 3.1, note that ρ is strong stochastically positive if ρ_1 is.

6 Spectral positivity in complex geometry

A. The complex analogue of Bochner's theorem is the Kodaira vanishing theorem for the complex Laplacian acting on sections of a holomorphic line bundle over a compact complex manifold. In general the complex Laplacians have a tractable Weitzenböck formula only if the manifold is Kähler, so this will be the standing assumption from here on. The results in this section are all from [Yin].

To set the notation, let M be a compact Kähler manifold of complex dimension n, F a holomorphic hermitian line bundle over M, ∇ a compatible connection on F with curvature form Ω, and $\tau_1 = \tau_1(x) \leq \ldots \leq \tau_n = \tau_n(x)$ the eigenvalues of Ω_x considered as an endomorphism of the antiholomorphic tangent space $T_x^{(1,0)}M$. (To be precise, the eigenvalues are computed with respect to an orthonormal frame field of type $(1,0)$ near x.) In our notation, $\tau_1 = \underline{\Omega}$. Let $H^{(0,q)}(M;F)$ be the cohomology groups of F-valued $(0,q)$-forms on M; by the Dolbeault theorem, these groups are isomorphic to $H^q(M,\mathcal{O}(F))$, the cohomology groups for the sheaf of holomorphic sections of F.

Theorem 6.1 *If*

$$\tau_1 + \ldots + \tau_q - \tau_{q+1} - \ldots - \tau_n + q\underline{\mathrm{Ric}}$$

is spectrally positive, then $H^{(0,q)}(M;F) = 0$. In particular, if $\underline{\mathrm{Ric}}$ is spectrally positive, then $H^{(0,q)}(M;\mathbb{C}) = 0$ for all $q > 0$.

As indicated, the difficulty in proving this result is in establishing the correct Weitzenböck formula.

Now let K_M^* be the dual of the canonical bundle over M endowed with the connection induced from the canonical connection on M compatible with the complex structure. Coupling this connection with ∇ gives a connection on $F \otimes K_M^*$ with curvature denoted by Θ. The classical Kodaira vanishing theorem gives $H^{(0,q)}(M;F) = 0$ for $q > 0$ provided $\underline{\Theta}$ is point positive. (In this case, we just say that Θ is point positive.) This can be generalized as in Theorem 2.2(i):

Theorem 6.2 *Let $\mathcal{P} = \mathcal{P}(n, V, D, K_1, K_2)$ be the collection of pairs (M, F), where*

(i) M is a compact Kähler manifold of complex dimension n with $\mathrm{vol}(M) \geq V$, $\mathrm{diam}(M) \leq D$, $\mathrm{Ric} \geq K_1$;

(ii) F is a hermitian line bundle over M with curvature Ω satisfying $|\underline{\Omega}| \leq K_2$.

Given $\alpha > 0$, there exists $a = a(\mathcal{P}, \alpha) > 0$ such that if $(M, F) \in \mathcal{P}$ and $\underline{\Theta} > \alpha$ except on a set U with $\mathrm{vol}(U) < a$, then $H^{(0,q)}(M;F) = 0$ for all $q > 0$.

The proofs of these theorems are along the lines of Bochner type arguments originally given by Kodaira, and do not follow from the more modern proofs in e.g., [GH78].

B. The Kodaira vanishing theorem is the main tool in the proof of the Kodaira embedding theorem, which states that a compact complex manifold M admits a holomorphic line bundle F with pointwise positive curvature if and only if M is projective algebraic (i.e. M admits a holomorphic embedding into $\mathbb{C}P^N$ for some N.) The "if" direction is immediate. The usual proof of the nontrivial implication involves studying the blowup of M at arbitrary points $p \in M$. Unfortunately, it seems very hard to relate spectral positivity on M to that on the blowup manifold $\tilde{M}_p$. Luckily, a Kähler manifold is projective algebraic if it is Moisezon. Thus it turns out that to prove the Kodaira embedding theorem for the Kähler manifold M, it suffices to find just one $p \in M$ such that

$$H^{(0,1)}(\tilde{M}_p; \pi^* F \otimes L_p^{-\mu}) = 0 \tag{5}$$

for $\mu = 1, 2$. Here $\pi : \tilde{M}_p \to M$ is the canonical projection, and L_p is the line bundle associated to the blowup at p. (The Kähler case is still of interest, as there exist non-projective algebraic complex tori.)

By Theorem 6.1, (5) will follow from a spectral positivity assumption on the bundle $\pi^* F \otimes L_p^{-\mu}$, but this assumption will involve the geometry of $\tilde{M}_p$ rather than that of M. A detailed analysis of the relationship between the geometry of M and $\tilde{M}_p$ combined with Theorem 2.2(i) yields a generalization of the Kodaira embedding theorem. First, let g_E denote the standard inner product on $\mathbb{R}^{2n}$, and let $g = g_{i\bar{j}}$ denoted the hermitian metric on M in a coordinate chart. We say that a holomorphic coordinate chart (U,ϕ) on M is of class $\mathcal{C}(c_1,c_2)$ if (i) $\phi(U)$ is the ball of radius one in $\mathbb{C}^n$, (ii) $||g_{i\bar{j}}||_{C^2} \le c_1$ on U, and (iii) $g \ge c_2 g_E$. Of course, every chart can be taken to lie in some $\mathcal{C}(c_1,c_2)$.

Theorem 6.3 *Let $\mathcal{M} = \mathcal{M}(n, D, V, K_1, c_1, c_2, K_2)$ be the collection of pairs (M, F), where*

(i) M is a compact Kähler manifold of complex dimension n with $\mathrm{vol}(M) \ge V$, $\mathrm{diam}(M) \le D, \mathrm{Ric} \ge K_1$*;*

(ii) there exists a holomorphic coordinate chart (U,ϕ) of class $\mathcal{C}(c_1,c_2)$ with $\mathrm{vol(U)} \le V/2$*;*

(iii) F is a holomorphic hermitian line bundle on M with $|\underline{\Omega}(F)| \le K_2$. Given $\alpha > 0$, there exists $c = c(\mathcal{M},\alpha)$ such that if $(M,F) \in \mathcal{M}$ and $\underline{\Omega}(F) > \alpha$ except on a set $A \subset M - U$ with $\mathrm{vol}(A) < c$*, then M is projective algebraic.*

If F has pointwise positive curvature, one picks an arbitrary chart U and finds (i) the corresponding c_1, c_2 for U, (ii) n, D, V, K for M, (iii) α such that $\Omega(F) > \alpha$ everywhere. (i) and (ii) determine c, and (iii) guarantees that the hypotheses are all satisfied with $A = \emptyset$. Thus Theorem 6.3 does in fact give the Kodaira embedding theorem in the Kähler case.

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