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LACK OF STRONG COMPLETENESS FOR STOCHASTIC FLOWS

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It is well known that a stochastic differential equation (SDE) on a Euclidean space driven by a Brownian motion with Lipschitz coefficients generates a stochastic flow of homeomorphisms. When the coefficients are only locally Lipschitz, then a maximal continuous flow still exists but explosion in finite time may occur. If, in addition, the coefficients grow at most linearly, then this flow has the property that for each fixed initial condition x, the solution exists for all times almost surely. If the exceptional set of measure zero can be chosen independently of x, then the maximal flow is called *strongly complete*. The question, whether an SDE with locally Lipschitz continuous coefficients satisfying a linear growth condition is strongly complete was open for many years. In this paper, we construct a two-dimensional SDE with coefficients which are even bounded (and smooth) and which is *not* strongly complete thus answering the question in the negative.

1. Introduction. We will assume throughout that $(\Omega, \mathcal{F}, \mathbf{P})$ is a given probability space. Let us consider the following stochastic differential equation (SDE) on $\mathbf{R}^d$:

$$dX_t = \sum_{i=1}^{n} \sigma_i(X_t)\, dB_t^i + \sigma_0(X_t)\, dt, \tag{1.1}$$

where $B^1, \dots, B^n$ are independent standard Wiener processes defined on $(\Omega, \mathcal{F}, \mathbf{P})$, and the σ_i are locally Lipschitz continuous vector fields, and hence the SDE has a unique local solution for each initial condition $X_0 = x$.

It is well known that such SDEs with global Lipschitz coefficients do not only possess a unique global solution for each fixed initial condition but also a version of the global solution which is continuous in the initial data [2]. This global solution generates in fact a *stochastic flow of homeomorphisms* [10, 13]. It is also well known that for a unique global strong solution to exist, it suffices that the coefficients of the SDE satisfy a suitable local regularity condition and a growth condition at infinity, for example, a local Lipschitz condition and a linear growth condition. Local Lipschitz continuity guarantees local existence and uniqueness of solutions as well as continuous dependence of the local flow on initial conditions

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while the linear growth condition (which can in fact be weakened a bit by allowing additional logarithmic terms) allows us to pass from the existence of a local solution to that of a global solution by a Gronwall's lemma procedure. SDEs which have a global strong solution for each initial condition are said to be *complete* or *weakly complete*. A complete SDE need not have a continuous modification of the solution as a function of time and the initial data. This marks a departure of the theory of stochastic flows from that of deterministic ordinary differential equations. However there is so far only a pitifully small number of examples of complete stochastic differential equations whose solutions do not admit a continuous modification as a function of time and initial data. Not a single such example has coefficients which are locally Lipschitz and of linear growth (in spite of a remark in [8] stating the contrary). The basic example is $dx_t = dW_t$ on $\mathbf{R}^2 \setminus \{0\}$ for a two-dimensional Wiener process W. It was first given in [7]. It is clear that $x + W_t$ does not explode in $\mathbf{R}^2 \setminus \{0\}$ for each individual x, as a Brownian motion does not see single points. The unique maximal flow is given by $\{x + W_t(\omega), x \in \mathbf{R}^2 \setminus \{0\}\}$ (up to explosion), and it explodes for any given ω. This SDE is equivalent to the following SDE on $\mathbf{R}^2$, through the transformation $z \mapsto \frac{1}{z}$ in the complex plane representation,

$$dx_t = (y_t^2 - x_t^2)\, dB_t^1 - 2x_t y_t\, dB_t^2,$$
$$dy_t = -2x_t y_t\, dB_t^1 + (x_t^2 - y_t^2)\, dB_t^2,$$

where B^1, B^2 are independent standard Brownian motions. See also [3] for an example of a strongly complete SDE with the same infinitesimal generator and [14] for showing that this same SDE, that is, $dx_t = dW_t$ on $\mathbf{R}^n \setminus \{0\}$, is strongly $n-2$-complete but not strongly $n-1$ complete.

Our aim here is to construct stochastic differential equations which are complete but not *strongly complete*, that is, which do not admit a continuous modification. In the examples, the lack of strong completeness is achieved by rapidly oscillating vector fields. The example which we will present in the next section shows that even under the additional constraint that the equation has no drift and the diffusion coefficient is bounded and C^∞, there may not exist a global solution flow. Even more, in our example the SDE is driven by a single one-dimensional Brownian motion. The key idea is to construct rapidly oscillating vector fields in such a way that the flow behaves almost as if it were driven by an infinite number of independent Brownian motions. We will provide more detailed heuristics at the beginning of Section 2. Note that such examples are clearly impossible for scalar equations, so the dimension of the state space of the SDE has to be at least 2. Our examples are in $\mathbf{R}^2$. This also answers a 29-year-old conjecture (see page 200, Chapter VIII, Section 2D in [6]), whether the existence of a uniform cover for the SDE implies that the SDE is strongly complete; see Remark 2.8 for details.

1.1. *Preliminaries and a survey of positive results*. We review briefly results for strong completeness of the SDE (1.1). The existence of a local (continuous) solution flow is well known (see, e.g., [4, 10]). The following lemma on the existence of a maximal (continuous) flow generated by the SDE (1.1) is taken from [13] (Theorem 4.7.1) (see also [4]).

LEMMA AND DEFINITION 1.1 (Maximal flow). Suppose that the vector fields σ_i are locally Lipschitz continuous. Then there exist a function $\tau : \mathbf{R}^d \times \Omega \to (0, \infty]$ and a map $\phi : \{(t, x, \omega) : x \in \mathbf{R}^d, \omega \in \Omega, t \in [0, \tau(x, \omega))\} \to \mathbf{R}^d$ such that the following hold:

1. For each $x \in \mathbf{R}^d$, $\phi_t(x, .)$ solves (1.1) with initial condition x on $[0, \tau(x, \omega))$;
2. $\phi_t(x, \omega) : \{(t, x) : t < \tau(x, \omega)\} \to \mathbf{R}^d$ is a continuous function of (t, x);
3. for each x, $\limsup_{t \to \tau(x,\omega)} |\phi_t(x, \omega)| = \infty$ on $\{\omega : \tau(x, \omega) < \infty\}$.

The map ϕ is called a *maximal* (*local*) *flow*. The pair (ϕ, τ) is unique up to a null set. If, for each $x \in \mathbf{R}^d$, we have $\tau(x, \omega) = \infty$ almost surely, then we call the SDE (or the maximal flow) *complete* or *weakly complete*. If, moreover, there exists a set Ω_0 such that $\tau(x, \omega) = \infty$ for all $x \in \mathbf{R}^d$ and all $\omega \in \Omega_0$, then the SDE or the maximal flow are called *strongly complete* [or a *global* (*solution*) *flow*].

Usually, flows are assumed to have two time parameters (an additional one for the starting time) and to satisfy a corresponding composition property, but in this paper we will not dwell on this.

While slight regularity ensures the existence of a maximal flow, global existence is guaranteed by suitable bounds on the growth of the vector fields (and/or their derivatives). The best known and used results on the existence of a global solution flow are globally Lipschitz vector fields for SDEs on $\mathbf{R}^d$ in Itô form, given in [2] or [13], Theorem 4.5.1, and SDEs on compact manifolds [1]. It becomes apparent in [14] that the existence of a global solution flow of an SDE in Stratonovich form is related to the growth on the vector fields and their first derivatives. Similar conditions imply the existence of a flow of homeomorphisms/diffeomorphisms as it is well known that if the SDE (1.1) and its adjoint

$$dX_t = \sum_{i=1}^{n} \sigma_i(X_t)\, dB_t^i - \sigma_0(X_t)\, dt + \sum_{i=1}^{n} D\sigma_i(\sigma_i)\, dt \tag{1.2}$$

are both strongly complete, then the solution generates a flow of homeomorphisms (see [4, 11, 12, 14]). The best growth condition so far for strong completeness of an SDE is probably that given in [14], whose main theorems, Theorems 4.1 and 5.1, are intended to treat SDEs on general manifolds: if the vector fields have linear growth, and their derivatives have logarithmic growth, then (1.1) is strongly complete (Theorem 6.2) (see also [8]). A similar result on strong completeness holds allowing the derivative of the vector fields to grow at the rate of $|x|^\varepsilon$ ([14], Corollary 6.3). Sufficient conditions for strong completeness for stochastic delay differential equations can be found in [15].

Finally let us provide some intuition behind the global existence or nonexistence of stochastic flows. We state an elementary criterion for strong completeness with converse whose essence will be used in the proof for the claim in the example we will construct.

REMARK 1.2. Let $U_1 \subset U_2 \subset U_3 \subset \cdots$ be an exhausting sequence of bounded open subsets of $\mathbf{R}^d$. Take $\phi_t(x, \omega)$ to be the maximal flow and let K be a compact set and $\tau_n^K := \inf\{t > 0 : \phi_t(K) \nsubseteq U_n\}$. Define $\tau^K = \lim_{n\to\infty} \tau_n^K$. If, for two sequences $\{a_n\}$ and $\{b_n\}$ with $\sum_n a_n = \infty$ and $\sum_n b_n < \infty$,

$$\mathbf{P}\{\tau_n^K - \tau_{n-1}^K \leq a_n, \tau_{n-1}^K < \infty\} \leq b_n,$$

then the first Borel–Cantelli lemma implies that τ^K is almost surely infinite, and if this property holds for every compact set K, then the flow is strongly complete.

Conversely, let $\{a_n\}$ and $\{b_n\}$ be two summable sequences, and assume that there exists some compact subset K and finite random times T_j such that $\tau^K \leq \sum_j T_j$ and

$$\mathbf{P}\{T_n \geq a_n\} \leq b_n \qquad \text{for all } n.$$

Then, again by the first Borel–Cantelli lemma, $\tau^K < \infty$ almost surely, so the flow is *not* strongly complete in this case.

2. Negative results. Below, we will construct an example of an SDE in the plane of the form

$$\begin{aligned} dX(t) &= \sigma(X(t), Y(t))\, dW(t), \\ dY(t) &= 0, \end{aligned} \tag{2.1}$$

which is *not* strongly complete and where $\sigma : \mathbf{R}^2 \to (0, \infty)$ is bounded, bounded away from 0 and C^∞.

Before going into details, let us explain the idea of the construction. From (2.1) it is clear that in our example trajectories move on straight lines parallel to the first coordinate axis. If the equation was driven by a family of Brownian motions (rather than a single one) which are indexed by $y \in \mathbf{R}$ and are independent for different values of y, then clearly the supremum over all solutions at time 1 (say) with initial conditions of the form $(0, y)$, $0 \leq y \leq 1$ would be infinite. Such a modification would of course contradict our assumptions but we can (and will) try to approximate this behavior using an equation of type (2.1) with carefully chosen σ (satisfying all properties stated above). Our σ will exhibit increasingly heavy oscillations when $x \to \infty$ with different frequencies for different values of y. Thus we can make sure that for different values of y, the solutions behave (for large x) almost as if they were driven by independent Brownian motions, in spite of the fact that they are all driven by the same Brownian motion. If we manage to construct σ such that approximate independence sets in sufficiently quickly, then we can hope to observe exploding solutions, that is, lack of strong completeness. In fact it will turn out that in our example, solutions for different values of y will not be asymptotically independent but that solutions can be asymptotically written as a sum of two Brownian motions: one which is the same for all y and another one which is independent for different y. This property suffices to show that strong completeness does not hold.

2.1. *A bunch of lemmas*. Lemma 2.2 below is the key to the construction of our example. While known results in homogenization theory state convergence in law of the solutions of a sequence of SDEs like (2.2) to a Brownian motion (with a certain *effective* diffusion constant) we are not aware that the asymptotics of the joint laws of the solutions has been investigated in the literature. The proof of Lemma 2.2 will use the following lemma.

LEMMA 2.1. *Let $X^\varepsilon = (X_1^\varepsilon, X_2^\varepsilon, \ldots)$, $\varepsilon > 0$, be a family of continuous local martingales starting at* 0. *Let $B_1, B_2, \ldots$ be independent standard Brownian motions, $\alpha_{ij} \in \mathbf{R}$, $i, j \in \mathbf{N}$ such that $\sum_j \alpha_{ij}^2 < \infty$ for all $i \in \mathbf{N}$, $V_i := \sum_{j=1}^\infty \alpha_{ij} B_j$, $i \in \mathbf{N}$ and $V = (V_1, V_2, \ldots)$. If the quadratic variation $[X_k^\varepsilon, X_l^\varepsilon]_t$ converges in law to $[V_k, V_l]_t = t \sum_{j=1}^\infty \alpha_{kj}\alpha_{lj}$ for all $k, l \in \mathbf{N}$, $t \geq 0$, then X^ε converges to V weakly as $\varepsilon \to 0$, that is, for each $n \in \mathbf{N}$, $(X_1^\varepsilon, \ldots, X_n^\varepsilon)$ converges in law to $(V_1, \ldots, V_n)$ with respect to the uniform topology on compact intervals.*

PROOF. This follows from the much more general Theorem VIII.2.17 in [9] which is formulated with respect to the Skorohod topology, but since all processes in the lemma have continuous paths we also have convergence with respect to the uniform topology. □

LEMMA 2.2. *Let $H_i : \mathbf{R} \to [0, \infty)$, $i = 1, 2$, be Lipschitz continuous with period* 1, *and assume that H_1 is nonconstant and $H_1^2(x) + H_2^2(x) > 0$ for all x. Let W_i, $i = 1, 2$ be independent standard one-dimensional Brownian motions and $\varepsilon > 0$. Consider the SDE*

$$
\begin{aligned}
&dX^\varepsilon(t) = H_1\left(\frac{1}{\varepsilon} X^\varepsilon(t)\right) dW_1(t) + H_2\left(\frac{1}{\varepsilon} X^\varepsilon(t)\right) dW_2(t), \\
&X^\varepsilon(0) = x.
\end{aligned}
\tag{2.2}
$$

There exist $\hat{\alpha}, \hat{\beta} > 0$ (not depending on the initial condition x) such that the following holds: if (ε_n) is a sequence of positive reals satisfying $\varepsilon_{n+1}/\varepsilon_n \to 0$ as $n \to \infty$, then $(X^{\varepsilon_n} - x, X^{\varepsilon_{n+1}} - x, \ldots)$ converges weakly to $(\hat{\alpha} B_0 + \hat{\beta} B_1, \hat{\alpha} B_0 + \hat{\beta} B_2, \ldots)$ as $n \to \infty$, where $B_0, B_1, \ldots$ are independent standard Brownian motions.

PROOF. By the previous lemma, it suffices to show that there exist $\hat{\alpha}, \hat{\beta} > 0$ such that $[X^\varepsilon - x]_t$ and $[X^\varepsilon - x, X^{\tilde{\varepsilon}} - x]_t$ converge to $(\hat{\alpha}^2 + \hat{\beta}^2)t$, respectively, $\hat{\alpha}^2 t$ in law for each $t \ge 0$ as $\varepsilon \to 0$ and $\tilde{\varepsilon} \to 0$ such that $\tilde{\varepsilon}/\varepsilon \to 0$. Set $z^\varepsilon(t) = \frac{1}{\varepsilon} X^\varepsilon(t\varepsilon^2)$ and let $W_i^\varepsilon(t) = \frac{1}{\varepsilon} W_i(t\varepsilon^2)$, $i = 1, 2$, be the rescaled Brownian motions. Then $z^\varepsilon(t)$ satisfies

$$dz^\varepsilon(t) = H_1(z^\varepsilon(t))\, dW_1^\varepsilon(t) + H_2(z^\varepsilon(t))\, dW_2^\varepsilon(t).$$

The projection to [0, 1] is an ergodic Markov process with invariant measure μ

$$\mu(dy) = \frac{1}{v} \frac{dy}{H_1^2(y) + H_2^2(y)}$$

for $v = \int_0^1 \frac{1}{H_1^2(y) + H_2^2(y)}\, dy$ the normalising constant. If f is a continuous periodic function with period 1, denote by $\bar{f}$ its average

$$\bar{f} = \int_0^1 f(x)\, d\mu(x).$$

Then by the ergodic theorem for z^ε, for each fixed $t \ge 0$,

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_0^t f\left(\frac{1}{\varepsilon} X^\varepsilon(s)\right) ds &= \lim_{\varepsilon\to 0} \int_0^t f\left(z^\varepsilon\left(\frac{s}{\varepsilon^2}\right)\right) ds \\ &= \lim_{\varepsilon\to 0} \varepsilon^2 \int_0^{t/\varepsilon^2} f(z^\varepsilon(r))\, dr \qquad (2.3) \\ &= t\bar{f}. \end{aligned}$$

The convergence is in L^p for every $p > 0$. This applies in particular to H_1 and H_2. The zero mean martingale diffusion process $X^\varepsilon(\cdot) - x$ has quadratic variation

$$[X^\varepsilon - x]_t = \int_0^t \left[H_1\left(\frac{1}{\varepsilon} X^\varepsilon(s)\right)\right]^2 ds + \int_0^t \left[H_2\left(\frac{1}{\varepsilon} X^\varepsilon(s)\right)\right]^2 ds,$$

which, due to (2.3), converges in L^1 to $\beta_1^2 t$, where

$$\beta_1 := \sqrt{\left(\int_0^1 (H_1^2(x) + H_2^2(x))\, d\mu(x)\right)}.$$

Next, we show that

$$[X^\varepsilon - x, X^{\tilde{\varepsilon}} - x]_t \to \hat{\alpha}^2 t \qquad \text{for } \hat{\alpha} := (\bar{H}_1^2 + \bar{H}_2^2)^{1/2} \text{ as } \varepsilon, \tilde{\varepsilon}/\varepsilon \to 0. \qquad (2.4)$$

We have

$$\begin{aligned} [X^\varepsilon - x, X^{\tilde{\varepsilon}} - x]_t = &\int_0^t H_1\left(\frac{1}{\varepsilon} X^\varepsilon(s)\right) H_1\left(\frac{1}{\tilde{\varepsilon}} X^{\tilde{\varepsilon}}(s)\right) ds \\ &+ \int_0^t H_2\left(\frac{1}{\varepsilon} X^\varepsilon(s)\right) H_2\left(\frac{1}{\tilde{\varepsilon}} X^{\tilde{\varepsilon}}(s)\right) ds. \end{aligned}$$

Let $f:\mathbf{R}\to\mathbf{R}$ be continuous and periodic with period 1 and $g(s):=f(s)-\bar{f}$. Then

$$\begin{aligned}
\int_0^t & f\Big(\frac{1}{\varepsilon}X^{\varepsilon}(s)\Big)f\Big(\frac{1}{\tilde{\varepsilon}}X^{\tilde{\varepsilon}}(s)\Big)\,ds\\
&=\int_0^t g\Big(\frac{1}{\varepsilon}X^{\varepsilon}(s)\Big)g\Big(\frac{1}{\tilde{\varepsilon}}X^{\tilde{\varepsilon}}(s)\Big)\,ds\\
&\qquad+\bar{f}^2t+\bar{f}\int_0^t g\Big(\frac{1}{\varepsilon}X^{\varepsilon}(s)\Big)\,ds+\bar{f}\int_0^t g\Big(\frac{1}{\tilde{\varepsilon}}X^{\tilde{\varepsilon}}(s)\Big)\,ds.
\end{aligned}$$

The sum of the last three terms converges to $\bar{f}^2t$ by (2.3), so in order to prove (2.4), it suffices to show that the first term converges to zero in probability. Let $N:=\lceil 1/(\varepsilon\tilde{\varepsilon})\rceil$, $b_i:=it/N$, $i=0,1,\dots,N$, and $C:=\sup_{x\in[0,1]}|g(x)|$. Then

$$\begin{aligned}
\bigg|&\int_0^t g\Big(\frac{1}{\varepsilon}X^{\varepsilon}(s)\Big)g\Big(\frac{1}{\tilde{\varepsilon}}X^{\tilde{\varepsilon}}(s)\Big)\,ds\bigg|\\
&=\bigg|\sum_{i=0}^{N-1}\int_{b_i}^{b_{i+1}} g\big(z^{\varepsilon}(s/\varepsilon^2)\big)g\big(z^{\tilde{\varepsilon}}(s/\tilde{\varepsilon}^2)\big)\,ds\bigg|\\
&=\tilde{\varepsilon}^2\bigg|\sum_{i=0}^{N-1}\int_{b_i/\tilde{\varepsilon}^2}^{b_{i+1}/\tilde{\varepsilon}^2} g\big(z^{\varepsilon}(s\tilde{\varepsilon}^2/\varepsilon^2)\big)g\big(z^{\tilde{\varepsilon}}(s)\big)\,ds\bigg|\\
&\le\tilde{\varepsilon}^2\sum_{i=0}^{N-1}\bigg|g\Big(z^{\varepsilon}\Big(\frac{b_i}{\varepsilon^2}\Big)\Big)\bigg|\bigg|\int_{b_i/\tilde{\varepsilon}^2}^{b_{i+1}/\tilde{\varepsilon}^2} g\big(z^{\tilde{\varepsilon}}(s)\big)\,ds\bigg|\\
&\qquad+\tilde{\varepsilon}^2\sum_{i=0}^{N-1}\int_{b_i/\tilde{\varepsilon}^2}^{b_{i+1}/\tilde{\varepsilon}^2}\bigg|g\Big(z^{\varepsilon}\Big(\frac{\tilde{\varepsilon}^2s}{\varepsilon^2}\Big)\Big)-g\Big(z^{\varepsilon}\Big(\frac{b_i}{\varepsilon^2}\Big)\Big)\bigg|\big|g\big(z^{\tilde{\varepsilon}}(s)\big)\big|\,ds\\
&\le\tilde{\varepsilon}^2C\sum_{i=0}^{N-1}\bigg|\int_{b_i/\tilde{\varepsilon}^2}^{b_{i+1}/\tilde{\varepsilon}^2} g\big(z^{\tilde{\varepsilon}}(s)\big)\,ds\bigg|\\
&\qquad+\tilde{\varepsilon}^2C\sum_{i=0}^{N-1}\int_{b_i/\tilde{\varepsilon}^2}^{b_{i+1}/\tilde{\varepsilon}^2}\bigg|g\Big(z^{\varepsilon}\Big(\frac{\tilde{\varepsilon}^2s}{\varepsilon^2}\Big)\Big)-g\Big(z^{\varepsilon}\Big(\frac{b_i}{\varepsilon^2}\Big)\Big)\bigg|\,ds.
\end{aligned}$$

The expected value of the first term converges to 0 as $\varepsilon\to 0$ by the ergodic theorem since $\tilde{\varepsilon}/\varepsilon\to 0$, and $\mathbf{E}|g(z^{\varepsilon}(\frac{\tilde{\varepsilon}^2s}{\varepsilon^2}))-g(z^{\varepsilon}(\frac{b_i}{\varepsilon^2}))|$ converges to zero as $\varepsilon\to 0$ uniformly for all i, $s\in[b_i\tilde{\varepsilon}^{-2},b_{i+1}\tilde{\varepsilon}^{-2}]$ since z^{ε} has uniformly bounded diffusion coefficients. This proves (2.4).

All that remains to show is that $\hat{\beta}:=\sqrt{\beta_1^2-\hat{\alpha}^2}>0$ but this is true (by Jensen's inequality) since $\int H_2^2(x)\,d\mu(x)\ge\bar{H}_2^2$ and $\int H_1^2(x)\,d\mu(x)>\bar{H}_1^2$ since H_1 is nonconstant. Therefore the proof of the lemma is complete. □

We will need the following elementary lemmas.

LEMMA 2.3. *Let $W, B^1, B^2, \dots$ be independent standard Brownian motions and let $\hat{\alpha}, \hat{\beta}, a, \delta, S, T > 0$. Then*

$$\lim_{n\to\infty} \mathbf{P}\left(\bigcup_{i=1}^{n}\left(\left\{\sup_{0\le t\le S}(\hat{\beta}B_t^i + \hat{\alpha}W_t) \ge a\right\} \cap \left\{\inf_{0\le t\le T}(\hat{\beta}B_t^i + \hat{\alpha}W_t) \ge -\delta\right\}\right)\right) = 1.$$

PROOF. For $i \in \mathbf{N}$, let

$$A_i := \left\{\omega : \sup_{0\le t\le S}(\hat{\beta}B_t^i + \hat{\alpha}W_t) \ge a, \inf_{0\le t\le T}(\hat{\beta}B_t^i + \hat{\alpha}W_t) \ge -\delta\right\}.$$

Birkhoff's ergodic theorem implies that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{A_i} = \mathbf{P}(A_1|\sigma(W)) \qquad \text{a.s.},$$

which is strictly positive almost surely, so the assertion of the lemma follows. □

The following is a quantitative version of the Borel–Cantelli lemma which provides an upper bound and, as a corollary, a lower bound, for M events out of N events to happen simultaneously. This elementary lemma is most likely a known result. We give below a simple proof, which benefitted from discussion with M. Hairer.

LEMMA 2.4 (A quantitive Borel–Cantelli lemma). *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $\{A_i\}$, $1 \le i \le N$ events with $\mathbf{P}(A_i) = p_i$. Then*:

- *the probability that at least M of the events happen simultaneously is smaller or equal to $\sum_{i=1}^N p_i/M$;*
- *the probability that at least M of the events $\{A_i\}$ happen simultaneously is at least $\frac{\sum_{i=1}^N p_i - M + 1}{N - M + 1}$.*

PROOF. Let $Q_{M,N}$ be the set of ω which belong to at least M of the N events from $\{A_i\}$:

$$\begin{aligned}\mathbf{P}(Q_{M,N}) &= \mathbf{P}\{\omega : \#\{1 \le i \le N : \omega \in A_i\} \ge M\} \\ &= \mathbf{P}\left\{\omega : \sum_{i=1}^N \mathbf{1}_{A_i}(\omega) \ge M\right\} \le \frac{1}{M}\mathbf{E}\sum_{i=1}^N \mathbf{1}_{A_i} = \frac{1}{M}\sum_{i=1}^N p_i.\end{aligned}$$

For the corresponding lower bound denote by B_i the complement of A_i and $q_i := \mathbf{P}(B_i) = 1 - p_i$. Let $Q_{M,N}^c$ be the complement of $B_{M,N}$, which is the event that at most $M - 1$ of the events A_i happen or, equivalently, the set on which at least $N - M + 1$ events from the $\{B_i : 1 \le i \le N\}$ happen. It follows from the first part that

$$\mathbf{P}(Q^c_{M,N}) \le \frac{\sum_{i=1}^N q_i}{N - M + 1} = \frac{N - \sum_{i=1}^N p_i}{N - M + 1},$$

so that

$$\mathbf{P}(Q_{M,N}) \ge 1 - \frac{N - \sum_{i=1}^N p_i}{N - M + 1} = \frac{\sum_{i=1}^N p_i - M + 1}{N - M + 1},$$

as required. □

PROPOSITION 2.5. *Let $0 < \alpha \le \beta$. Then, for every $T > 0$, $\varepsilon > 0$, there exists a $\delta = \delta(T, \varepsilon) > 0$ such that for each $m \in \mathbf{N}$, there exists some $N \in \mathbf{N}$ such that the following holds: for every $0 < \tilde{\delta} \le \delta$ and every sequence $M_1, M_2, \dots$ of martingales with continuous paths on the same space $(\Omega, \mathcal{F}, \mathbf{P})$, starting at zero such that $\alpha \le \frac{d}{dt}\langle M_i \rangle_t \le \beta$ for all i and t, the stopping time*

$$\tau := \inf\{t > 0 : M_i(t) \ge \tilde{\delta} \text{ for at least } m \text{ different } i \in \{1, \dots, N\}\}$$

satisfies

$$\mathbf{P}\{\tau \le T\} \ge 1 - \varepsilon.$$

PROOF. Clearly, it suffices to prove the last statement for some fixed $\tilde{\delta} = \delta > 0$ because then it trivially also holds for each smaller $\tilde{\delta} > 0$. For $\delta > 0$, let $\lambda_i^\delta := \mu\{0 \le t \le T : M_i(t) \ge \delta\}$, where μ denotes normalized Lebesgue measure on $[0, T]$. We claim that there exist $\delta > 0$ and $u > 0$ such that for all $i \in \mathbf{N}$, we have

$$\mathbf{P}\{\lambda_i^\delta \ge u\} \ge 1 - \frac{\varepsilon}{2}. \tag{2.5}$$

Assume that this has been shown. For $k \ge 2$, let

$$\Omega_k := \{\lambda_i^\delta \ge u \text{ for at least } k \text{ different } i \in \{1, \dots, 2(k-1)\}\}.$$

Then, by Lemma 2.4,

$$\mathbf{P}(\Omega_k) \ge \frac{2(k-1)(1 - \varepsilon/2) - k + 1}{2(k-1) - k + 1} = 1 - \varepsilon.$$

Invoking Lemma 2.4 once more, we see that on Ω_k, there exists some $t \in [0, T]$ such that $M_i(t) \ge \delta$ for at least m different $i \in \{1, \dots, 2k - 2\}$ provided that the numerator $uk - m + 1$ in the formula in Lemma 2.4 is strictly positive. Letting $k := \lceil \frac{m}{u} \rceil$ and $N := 2k - 2$, the assertion of the proposition follows at once.

It remains to prove (2.5). Let $\delta > 0$ (we will fix the precise values later). For ease of notation, we drop the index i (observe that all estimates below are uniform in i). The martingale M can be represented as a time-changed Brownian motion $M(t) = W([M]_t)$. Since $\frac{d}{dt}[M]_t \in [\alpha, \beta]$, we obtain for $a, t > 0$

$$\begin{aligned}\mathbf{P}\Big(\sup_{s\in[0,t]} M(s) \ge a\Big) &= \mathbf{P}\Big(\sup_{0\le s\le t} W([M]_s) \ge a\Big)\\ &\ge \mathbf{P}\Big(\sup_{0\le s\le \alpha t} W(s) \ge a\Big) = 2\mathbf{P}\big(W(\alpha t) \ge a\big)\\ &= \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{\alpha t}}^{\infty} e^{-x^2/2}\,dx =: p_0(a,t)\end{aligned}$$

and, analogously,

$$\begin{aligned}\mathbf{P}\Big(\inf_{s\in[0,t]} M(s) \ge -a\Big) &\ge \mathbf{P}\Big(\sup_{0\le s\le t} W(\beta s) \le a\Big) = \frac{2}{\sqrt{2\pi}} \int_0^{a/\sqrt{\beta t}} e^{-x^2/2}\,dx\\ &=: q_0(a,t).\end{aligned}$$

Let $\tilde{\tau}$ be the first time that $M(t) \ge 2\delta$. Using the fact that $M(\tilde{\tau} \wedge c + t) - M(\tilde{\tau} \wedge c)$ also satisfies the assumptions of the proposition for each $c \ge 0$, we get for $u \in (0, \frac{1}{2}]$

$$\begin{aligned}&\mathbf{P}\{\lambda^\delta(\omega) \ge u\}\\ &\quad\ge \mathbf{P}\Big(\inf_{\tilde{\tau}\le t\le \tilde{\tau}+uT} M(t) \ge \delta \Big| \tilde{\tau} \le \frac{T}{2}\Big)\mathbf{P}\Big(\tilde{\tau} \le \frac{T}{2}\Big)\\ &\quad\ge q_0(\delta, uT)\, p_0\Big(2\delta, \frac{T}{2}\Big).\end{aligned}$$

Choosing first $\delta > 0$ so small that the second factor is close to 1 and then choosing $u > 0$ small enough, we can ensure that the product is at least $1 - \varepsilon/2$ proving (2.5), so the proof of the proposition is complete. □

2.2. *The examples.*

EXAMPLE 2.6. Consider the SDE (2.1). We will start by defining the coefficient σ restricted to $\mathbf{R} \times [0, 1]$. Fix a smooth nonconstant, strictly positive function H of period one. To construct the example, we subdivide the square $[n, n+1] \times [0, 1]$ into Z_n horizontal strips of width $1/Z_n$ each, with Z_n increasing sufficiently quickly and let σ be equal to H sped up by a factor depending on the particular strip. Thus, the probability that one of the solutions starting from (n, y), $y \in [0, 1]$ will reach the next level $(n + 1, y)$ within a very short time will increase with n allowing us to conclude that strong completeness fails. We now state the precise assumptions.

Let $H : \mathbf{R} \to [\frac{1}{2}, 1]$ be an infinitely differentiable nonconstant function with period 1. Assume that all its derivatives vanish at 0. Fix a sequence of positive integers a_i, $i = 0, 1, \dots$ such that $a_0 = 1$ and $\lim_{i\to\infty} a_{i+1}/a_i = \infty$. Assume that $N_0, N_1, N_2, \dots$ are positive even integers whose values we will fix later. Let $Z_n := \prod_{i=0}^n N_i$, $n \in \mathbf{N}_0$ and define

$$\sigma(x, y) = H(a_i x)$$

$$\text{if } i \in \left\{0, 1, \dots, \frac{N_n}{2} - 1\right\}, x \in [n, n+1], \tag{2.6}$$

$$y \in \left[\frac{kN_n + 2i}{Z_n}, \frac{kN_n + 2i + 1}{Z_n}\right], \qquad k = 0, \dots, Z_{n-1} - 1.$$

Further, let $\sigma(x, y) = H(x)$ for $x \le 0$, $y \in [0, 1]$. On the set where σ has been defined, it is clearly bounded, strictly positive, and C^∞ (since we assumed that all derivatives of H to vanish at zero). It is also clear that σ can be extended to a C^∞ function taking values in $[1/2, 1]$ on all of $\mathbf{R}^2$. We claim that the associated flow is not strongly complete in case the integers $N_0, N_1, \dots$ are chosen to increase sufficiently quickly.

Let ψst denote the x-component of the maximal flow ϕ of the SDE started at time s ($s \le t$). We define a sequence of stopping times τ_n, $n \in \mathbf{N}_0$ and intervals $I_n \subseteq [0, 1]$ as follows: $\tau_0 := 0$, $I_0 := [0, 1]$, $\tau_{n+1} := \inf\{t > \tau_n : \sup_{y \in I_n} \psi_{\tau_n t}(n, y) = n + 1\}$ and let $I_{n+1} \subseteq I_n$ be some interval of the form $[\frac{2k}{Z_n}, \frac{2k+1}{Z_n}]$ on which the supremum in the definition of τ_{n+1} is attained (note that the supremum is attained for *every* point in such an interval if it is attained for *some* point in the interval). Define $\tau := \inf\{t \ge 0 : \sup_{y \in [0,1]} \psi_{0t}(0, y) = \infty\}$. Then $\tau \le \lim_{n \to \infty} \tau_n$ and it suffices show that $\mathbf{P}\{\tau_{n+1} - \tau_n \ge 2^{-n}\}$ is summable over n to deduce that $\mathbf{P}\{\tau < \infty\} > 0$. Since we will show that even the conditional probabilities given $\mathcal{F}_{\tau_n}$ are (almost surely) summable, and since τ_n is almost surely finite, it follows even that $\tau < \infty$ almost surely.

Fix $n \in \mathbf{N}$. We will show that we can choose $N_n \in \mathbf{N}$ in such a way that

$$\mathbf{P}\{\tau_{n+1} - \tau_n \ge 2^{-n} | \mathcal{F}_{\tau_n}\} \le 2^{-n}.$$

Let $\hat{y} \in I_n$, and let M_j^n solve the following SDE:

$$dM_j^n(t) = \xi_j^n(M_j^n(t))\, dW(t),$$
$$M_j^n(0) = n,$$

where

$$\xi_j^n(z) := \begin{cases} \sigma(z, \hat{y}), & \text{if } z \le n, \\ H(a_j z), & \text{if } z \ge n. \end{cases}$$

Observe that ξ_j^n does not depend on the particular choice of $\hat{y} \in I_n$ and that (up to a shift of the Wiener process W) $M_j^n(t)$, $j = 0, \dots, \frac{N_n}{2} - 1$ are the solutions of our SDE after τ_n and until τ_{n+1} on the intervals $[\frac{lN_n+2j}{Z_n}, \frac{lN_n+2j+1}{Z_n}]$, sub-intervals of I_n, where l is chosen such that $(lN_n + 1)/Z_n \in I_n$. We need to ensure that for N_n

large enough, one of the M_j^n will reach the next level $n+1$ within time 2^{-n} with probability at least $1-2^{-n}$. Unfortunately, we cannot apply the homogenization Lemma 2.2 directly to the M_j^n, since they all have the same diffusion coefficient for $z \le n$. Therefore, we will wait at most time $T_n = \frac{1}{2}2^{-n}$ and show that for N_n large, it is very likely, that many of the M_j^n have reached at least level $n+\delta_n$ for some (possibly very small) $\delta_n > 0$. We will then apply the homogenization lemma only to these M_j^n. Of course, this is possible only if the solution does not go back to level n before time τ_{n+1}. Lemma 2.3 ensures, that with high probability, we can find at least one of the remaining M_j^n for which this is true. We now provide the details of the argument.

Step 1: We apply Proposition 2.5 to the martingales $M_j^n - n$, $j = 0, 1, 2, \dots,$ with $T_n = \frac{1}{2}2^{-n}$, $\varepsilon_n = \frac{1}{4}2^{-n}$, $\alpha = 1/4$, $\beta = 1$ and obtain a number $\delta_n > 0$ which satisfies (2.5) in the proof of Proposition 2.5. We can assume that $\delta_n < 1$.

Step 2: Now, we define $\tilde{M}_j^n$, $j \in \mathbf{N}_0$, as the solution of the SDE

$$\begin{aligned} d\tilde{M}_j^n(t) &= H(a_j \tilde{M}_j^n(t))\, dW(t),\\ \tilde{M}_j^n(0) &= n + \delta_n. \end{aligned}$$

Applying Lemma 2.2 to $\tilde{M}_0^n, \tilde{M}_1^n, \dots$ with $x = n+\delta_n$, $H_1 = H$, $H_2 = 0$, we see that $(\tilde{M}_k^n - x, \tilde{M}_{k+1}^n - x, \dots)$ converges in law to $(\hat{\alpha} B_0 + \hat{\beta} B_1, \hat{\alpha} B_0 + \hat{\beta} B_2, \dots)$ as $k \to \infty$, where $\hat{\alpha}, \hat{\beta} > 0$ and $B_0, B_1, \dots$ are independent standard Wiener processes.

Step 3: Next, Lemma 2.3 says that there exists some $m_n \in \mathbf{N}$ such that

$$\begin{aligned} \mathbf{P}\Biggl(\bigcup_{i=1}^{m_n}\biggl(\Bigl\{\sup_{0\le t\le (1/2)2^{-n}} \bigl(\hat{\alpha} B_0(t) + \hat{\beta} B_i(t)\bigr) \ge 1\Bigr\} \\ \cap \Bigl\{\inf_{0\le t\le 2^{-n}} \bigl(\hat{\alpha} B_0(t) + \hat{\beta} B_i(t)\bigr) \ge -\frac{\delta_n}{2}\Bigr\}\biggr)\Biggr) \\ \ge 1 - \varepsilon_n. \end{aligned}$$

Step 4: Let $\tilde{N}_n$ be the number in the conclusion of Proposition 2.5 associated with $T_n, \varepsilon_n, \delta_n$ and m_n. Thanks to the convergence stated in Step 2, we can find some $k_n \in \mathbf{N}$ such that for any subset $J \subseteq \{k_n, k_n+1, \dots, k_n+\tilde{N}_n - 1\}$ of cardinality m_n, we have

$$\mathbf{P}\biggl(\bigcup_{i\in J}\Bigl(\Bigl\{\sup_{0\le t\le (1/2)2^{-n}} \tilde{M}_i^n \ge n+1\Bigr\} \cap \Bigl\{\inf_{0\le t\le 2^{-n}} \tilde{M}_i^n \ge -\delta_n\Bigr\}\Bigr)\biggr) \ge 1 - 2\varepsilon_n.$$

Step 5: Define $N_n := k_n + \tilde{N}_n$. Using the strong Markov property and the fact that ψ is order preserving (and hence a solution starting at $n+\delta_n$ can never pass a solution starting at a larger value at the same time), we obtain for our choice of N_n that

$$\mathbf{P}\{\tau_{n+1} - \tau_n \geq 2^{-n} | \mathcal{F}_{\tau_n}\} \leq 3\varepsilon_n < 2^{-n}$$

as desired, so the proof is complete.

Note that if the SDE in the above example is changed into Stratonovich, then the SDE is strongly complete. In fact, more generally, if the vector fields driving a Stratonovich equation are smooth and of linear growth, and all vector fields commute, then the SDE is strongly complete since the solution can be represented as a composition of solutions of ODEs.

To produce an example in Stratonovich form, we use two independent Brownian motions (the corresponding two vector fields necessarily do not commute).

EXAMPLE 2.7. Consider the SDE

$$\begin{aligned} dX(t) &= \sigma_1(X(t), Y(t))\, dW_1(t) + \sigma_2(X(t), Y(t))\, dW_2(t), \\ dY(t) &= 0, \end{aligned} \tag{2.7}$$

where W_1, W_2 are two independent standard one-dimensional Brownian motions. We will construct bounded and C^∞ functions σ_1 and σ_2 such that $\sigma_1^2(x, y) + \sigma_2^2(x, y) = 1$ for all x, y such that the associated flow is not strongly complete. Note that due to the condition $\sigma_1^2(x, y) + \sigma_2^2(x, y) = 1$, it does not matter if we interpret the stochastic differentials in the Itô or Stratonovich sense.

The construction of the example resembles that of the previous one closely; the only difference being that this time, we consider two nonconstant C^∞ functions H_1, H_2 taking values in $[1/2, 1]$ such that $H_1^2(z) + H_2^2(z) = 1$ and apply Lemma 2.2 with these functions H_1, H_2 rather than with a single function H as before.

We essentially showed that we can trace back to and construct a random initial point $x_0(\omega)$ which goes out fast enough to explode. This is true in general: suppose that there is a maximal flow $\{\phi_t(x, \omega), t < \tau(x, \omega)\}$ to the SDE. It is strongly complete if and only if for all measurable random points $x(\omega)$ on the state space, $\phi_t(x(\omega), \omega)$ exists almost surely for all t.

REMARK 2.8. We mentioned in the Introduction the following open question on strong completeness: suppose that the SDE $dX_t = \sum_{i=1}^n \sigma_i(X_t)\, dB_t^i + \sigma_0(X_t)\, dt$ has a uniform cover, is it strongly complete (cf. conjecture 2D(ii) in Chapter 8 of [6])? By a uniform cover we mean an atlas (U_α, ϕ_α) where $\phi_\alpha : U_\alpha \to \mathbf{R}^d$ are diffeomorphisms such that: (i) $\phi_\alpha(U_\alpha)$ contains the centred ball B_3 of radius 3; (ii) $\{\phi_\alpha^{-1}(B_1)\}$ is a cover of the manifold $\mathbf{R}^d$ and (iii) if $\tilde{V}^\alpha$ denotes the image of the vector field V under ϕ_α, then $\{\tilde{\sigma}_k^\alpha, k = 0, 1, \ldots, n,$

$\sum_{i=1}^{n}(D^2\phi_\alpha)_{\phi_\alpha^{-1}(\cdot)}(\sigma_i(\phi_\alpha^{-1}(\cdot)),\sigma_i(\phi_\alpha^{-1}(\cdot)))\}$ are uniformly bounded on B_2. If such a cover exists, then the exit time from B_2 of the solution of the corresponding SDE starting from $x \in B_1$ is stochastically bounded away from zero uniformly for all x, and hence the solution to the original SDE is weakly complete. It has been thought that such a uniform cover will give an estimate for the exit time of the local flow from B_1 to B_2. The example above shows that this cannot be expected in general. It is well known that an SDE with linear growth condition has a uniform cover ([5], page 146), by taking a countable dense set of points x_α in $\mathbf{R}^d \setminus \{0\}$ and the open sets $U_\alpha = \{|x - x_\alpha| < 3|x_\alpha|\}$, and diffeomorphisms $\phi_\alpha(x) := \frac{x - x_\alpha}{|x_\alpha|}$. To cover the point 0, throw in the trivial chart (U_0, ϕ_0) with $U_0 = B_1$ and $\phi_0(x) = x$. Our counter example has bounded coefficients and hence a uniform cover.

Incidentally, we also answered Conjecture 2D(i) from [6] in the negative. The conjecture states that if a stochastic differential equation is weakly complete, and if each corresponding ordinary differential equation corresponding to the linear approximation, (W_1^π, W_2^π), to (W_1, W_2) is complete, then the SDE is strongly complete. For the SDE in Example 2.7, the piecewise approximations are

$$W_i^\pi(t,\omega) = \frac{t_{j+1} - t}{t_{j+1} - t_j} W_i(t_j,\omega) + \frac{t - t_j}{t_{j+1} - t_j} W_i(t_{j+1},\omega), \qquad t_j \le t < t_{j+1}.$$

The approximating ordinary differential equations with parameter ω are

$$\dot{X}^\pi(t) = \sigma_1(X^\pi(t), Y(t)) \frac{d}{dt} W_1^\pi(t) + \sigma_2(X^\pi(t), Y(t)) \frac{d}{dt} W_2^\pi(t),$$

which restricted to $[t_j, t_{j+1})$ are simply

$$\begin{aligned}\dot{X}^\pi(t) &= \sigma_1(X^\pi(t), Y(t)) \frac{W_1(t_{j+1},\omega) - W_1(t_j,\omega)}{t_{j+1} - t_j} \\ &\quad + \sigma_2(X^\pi(t), Y(t)) \frac{W_2(t_{j+1},\omega) - W_2(t_j,\omega)}{t_{j+1} - t_j}\end{aligned}$$

and have global solutions.

REMARK 2.9. It remains an open question whether an SDE with globally Lipschitz diffusion coefficients and a drift which is locally Lipschitz and of linear growth admits a global solution flow.

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