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Formulae for the Derivatives of Heat Semigroups

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Formulae for the derivatives of solutions of diffusion equations are derived which clearly exhibit, and allow estimation of, the equations' smoothing properties. These also give formulae for the logarithmic gradient of the corresponding heat kernels, extending and giving a very elementary proof of Bismut's well known formula. Corresponding formulae are derived for solutions of heat equations for differential forms and their exterior derivatives. © 1994 Academic Press, Inc.

1. INTRODUCTION

Let M be a smooth manifold. Consider first a non-degenerate stochastic differential equation,

$$dx_t = X(x_t) \circ dB_t + A(x_t)\, dt, \tag{1}$$

on M with smooth coefficients A, X, where $\{B_t : t \geqslant 0\}$ is a R^m-valued Brownian motion on a filtered probability space $\{\Omega, \mathscr{F}, \mathscr{F}_t, P\}$. Let P_t be the associated sub-Markovian semigroup and $\mathscr{A}$ the infinitesimal generator, a second-order elliptic operator. In [6] a formula for the derivative $d(P_t f)_{x_0}(v_0)$, of $P_t f$ at x_0 in direction v_0 of the form

$$d(P_t f)_{x_0}(v_0) = \frac{1}{t}\mathbf{E} f(x_t) \int_0^t \langle v_s, X(x_s)\, dB_s \rangle \tag{2}$$

was given, where $v_.$ is a certain stochastic process starting at v_0. The process $v_.$ could be given either by the derivative flow of (1) or in terms of a naturaly related curvature. In the latter case and when $\mathscr{A} = \frac{1}{2}\Delta_M$ for some Riemannian structure, the formula reduces to one obtained by Bismut in [1] leading to his well-known formula for $\nabla \log p_t(x, y)$, the gradient of the logarithm of the fundamental solution to the heat equation on a Riemannian manifold. Bismut's proof is in terms of a Malliavin calculus while the

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proofs suggested in [6] following the approach of Elliott and Kohlman [4] are very elementary. However, there the results were actually given for a compact manifold as a special case of a more general result which needed some differential geometric apparatus. Here we show that the formula holds in a more general context, extend it to higher derivatives, and give similar formulae for differential forms of all orders extracted from [13]. In particular, we have a simple proof of the formulae for somewhat more general stochastic differential equations.

One importance of these formulae is that they demonstrate the smoothing effect of P_t, showing clearly what happens at $t=0$. To bring out the simplicity we first give proofs of the basic results for Itô equations on R^n.

There are extensions to infinite-dimensional systems with applications to smoothing and the strong Feller property for infinite-dimensional Kolmogorov equations in [3, 17]. There are also applications to non-linear reaction-diffusion equations [15]. For other generalizations of Bismut's formula in a geometric context see [16]. The work of Krylov [11] in this general area must also be mentioned although the approach and aims are rather different.

Throughout this article, we use BC^r for the space of bounded C^r functions with their first r derivatives bounded (using a given Riemannian metric on the manifold).

2. Formulae with Simple Proof for R^n

For $M=R^n$, we can take the Itô form of (1),

$$dx_t = X(x_t)\,dB_t + Z(x_t)\,dt, \tag{3}$$

where $X\colon R^n \to L(R^m, R^n)$ and $Z\colon R^n \to R^n$ are C^∞ with derivatives $DX\colon R^n \to L(R^n, L(R^m, R^n))$ and $DZ\colon R^n \to L(R^n, R^n)$, etc. There is the derivative equation

$$dv_t = DX(x_t)(v_t)\,dB_t + DZ(x_t)(v_t)\,dt \tag{4}$$

whose solution $v_t = DF_t(x_0)(v_0)$ starting from v_0 is the derivative (in probability) of F_t at x_0 in the direction v_0. Here $\{F_t(-), t \geqslant 0\}$ is a solution flow to (3), so that $x_t = F_t(x_0)$, for $x_0 \in R^n$. We do not need to assume the existence of a sample smooth version of $F_t\colon M \times \Omega \to M$.

For $\phi\colon R^n \to L(R^n; R)$, define $\delta P_t(\phi)\colon R^n \to L(R^n, R)$ by

$$(\delta P_t(\phi))_{x_0}(v_0) = E\phi_{x_t}(v_t) \tag{5}$$

whenever the right-hand side exists. Here $\phi_x(v)=\phi(x)(v)$. In particular, this can be applied to $\phi_x=(df)_x := Df(x)$ where $f\colon R^n \to R$ has bounded derivative. Formal differentiation under the expectation suggests

$$d(P_t f)_{x_0}(v_0)=(\delta P_t(df))_{x_0}(v_0).$$

This is well known when X and Z have bounded first derivatives. It cannot hold for $f\equiv 1$ when (3) is not complete (i.e., explosive). In fact, we deal only with complete systems: we are almost forced to do this since for δP_t to have a reasonable domain of definition some integrability conditions on $DF_t(x_0)$ are needed and it is shown in [13] that non-explosion follows, for a wide class of symmetrizable diffusions, from $dP_t f=\delta P_t(df)$ for all $f\in C_K^\infty$ together with $E\chi_{t<\xi}\,|DF_t(x_0)|<\infty$ for all $x_0\in M$, $t>0$. Here ξ is the explosion time. Precise conditions for $d(P_t f)=(\delta P_t)(df)$ are given in the Appendix below.

Our basic result is the following. It originally appeared in this form in [13].

THEOREM 2.1. *Let* (3) *be complete and non-degenerate, so there is a right inverse map* $Y(x)$ *to* $X(x)$ *for each* x *in* R^n, *smooth in* x. *Let* $f\colon R^n\to R$ *be* BC^1 *with* $\delta P_t(df)=d(P_t f)$ *almost surely* (*w.r.t. Lebesgue measure*) *for* $t\geqslant 0$. *Then for almost all* $x_0\in R^n$ *and* $t>0$,

$$d(P_t f)(x_0)(v_0)=\frac{1}{t}\,Ef(x_t)\int_0^t \langle Y(x_s)(v_s), dB_s\rangle_{R^m}, \qquad v_0\in R^n, \tag{6}$$

provided $\int_0^t \langle Y(x_s)(v_s), dB_s\rangle_{R^m}$, $t\geqslant 0$, *is a martingale.*

Proof. Let $T>0$. Parabolic regularity ensures that Itô's formula can be applied to $(t,x)\mapsto P_{T-t}f(x)$, $0\leqslant t\leqslant T$ to yield

$$P_{T-t}f(x_t)=P_T f(x_0)+\int_0^t d(P_{T-s}f)_{x_s}(X(x_s)\,dB_s) \tag{7}$$

for $t\in[0,T)$. Taking the limit as $t\to T$, we have

$$f(x_T)=P_T f(x_0)+\int_0^T d(P_{T-s}f)_{x_s}(X(x_s)\,dB_s).$$

Multiplying through by our martingale and then taking expectations using the fact that f is bounded, we obtain

$$\begin{aligned} Ef(x_T)\int_0^T \langle Y(x_s)\, v_s, dB_s\rangle &= E\int_0^T d(P_{T-s}f)_{x_s}(v_s)\, ds \\ &= E\int_0^T ((\delta P_{T-s})(df))_{x_s}(v_s)\, ds \\ &= \int_0^T ((\delta P_s)((\delta P_{T-s})(df)))_{x_0}(v_0)\, ds \\ &= \int_0^T (\delta P_T(df))_{x_0}(v_0)\, ds = T\,\delta P_T(df)_{x_0}(v_0) \end{aligned}$$

by the equivalence of the lax of x_s with the Lebesgue measure and the semigroup property of δP_t. ∎

Remarks. 1. The proof shows that under our conditions equality in (6) holds for each $x_0 \in M$ if and only if $\delta P_t(df) = d(P_t f)$ at each point. This is true provided $x \mapsto E\,|DF_t(x)|$ is continuous. The same holds for various variations of Theorem 2.1 which follow.

2. The martingale hypothesis is satisfied if

$$\int_0^t E\,|Y(x_s)(v_s)|^2\, ds < \infty$$

for all t. In turn this is implied by the uniform ellipticity condition $|Y(x)(w)|^2 \leqslant (1/\delta)\,|w|^2$ for all $x, w \in R^n$, for some $\delta > 0$, together with

$$\int_0^t E\,|v_s|^2\, ds < \infty, \qquad t \geqslant 0. \tag{8}$$

Under these conditions, (6) yields

$$\sup_{x\in R^n} |d(P_t f)_x| \leqslant \frac{1}{t} \sup_{x\in R^n} |f(x)|\, \frac{1}{\delta} \sup_{x\in R^n} \sqrt{\int_0^t E\,|DF_s(x)|^2\, ds}.$$

In particular, if X, Z have bounded first derivatives, then Gronwall's inequality together with (4) yields a constant α with

$$\sup_{x\in R^n} |d(P_t f)_x| \leqslant \frac{1}{\delta}\frac{1}{\alpha t}\sqrt{e^{\alpha t} - 1}\, \sup_{x\in R^n} |f(x)|. \tag{9}$$

For Sobolev norm estimates see (33) below.

COROLLARY 2.2. *Let* (1) *be complete and uniformly elliptic. Then* (6) *holds for all* f *in* BC^1 *provided that* $H_2(x)(v, v)$ *is bounded above; i.e.,* $H_2(x)(v, v) \leqslant c\,|v|^2$. *Here* H_2 *is defined by*

$$H_2(x)(v, v) = 2\langle DZ(x)(v), v\rangle + \sum_1^m |DX^i(x)(v)|^2 + \sum_1^m \frac{1}{|v|^2} \langle DX^i(x)(v), v\rangle^2.$$

Proof. 1. By Lemma A2 in the Appendix, we have $\int_0^t E\,|v_s|^2\,ds$ finite for each $t > 0$ while Theorem A5 and its remark give us the a.s. differentiability required.

2. The case in which there is a zero-order term and the coefficients are time dependent can be dealt with in the same way: Let $\{\mathscr{A}_t : t \geqslant 0\}$ be second-order elliptic operators on R^n with

$$\mathscr{A}_t(f)(x) = \tfrac{1}{2}\,\text{trace}\, D^2 f(x)(X_t(x)(-), X_t(x)(-)) + Df(x)(Z_t(x)) + V_t(x) f(x)$$

for X_t, Z_t as X, Z before, for each $t > 0$ continuous in t together with their spatial derivatives, and with $V_\cdot(\cdot)\colon [0, \infty) \times R^n \to R$ continuous and bounded above on each $[0, T] \times R^n$. For each $T > 0$ and $x_0 \in R^n$ let $\{x_t^T : 0 \leqslant t \leqslant T\}$ be the solution of

$$dx_t^T = X_{T-t}(x_t^T)\,dB_t + Z_{T-t}(x_t^T)\,dt,$$

with $x_0^T = x_0$ (assuming no explosion) and set

$$\alpha_t^T(x_0) = e^{\int_0^t V_{T-s}(x_s^T)\,ds}, \qquad 0 \leqslant t \leqslant T.$$

Also write $x_t^T(\omega) = F_t^T(x_0, \omega)$. Now suppose $u_\cdot(\cdot) : [0, \infty) \times R^n \to R$ satisfies

$$\frac{\partial u_t}{\partial t} = \mathscr{A}_t u_t, \qquad t > 0, \tag{10}$$

and is $C^{1,2}$ and bounded on each $[0, T] \times R^n$. Then, as before, we can apply Itô's formula to $\{u_{T-t}(x) : 0 \leqslant t \leqslant T\}$ to see that $\{u_{T-t}(x_t^T) : 0 \leqslant t \leqslant T\}$ is a martingale and $u_t(x) = E\alpha_t^t(x_0)\,u_0(x_t^t)$; e.g., see [9]. If we also assume

(i) V_t is C^1 for each t and continous and bounded above on each $[0, T] \times R^n$,

(ii) we can differentiate under the expectation to have for almost all $x_0 \in R^n$,

$$Du_t(x_0)(v_0) = E\left(\alpha_t^t(x_0)\, Du_0(x_t^t)(v_t^t) + \alpha_t^t(x_0)\, u_0(x_t^t) \int_0^t DV_{t-s}(x_s^t)\, v_s^t\, ds\right)$$

where v_s^t solves

$$dv_s^t = DX_{t-s}(x_s^t)(v_s^t)\, dB_s + DZ_{t-s}(x_s^t)(v_s^t)\, ds$$
$$v_0^t = v_0, \qquad 0 \leqslant s \leqslant t.$$

(iii) For $Y_t(x)$, a right inverse for $X_t(x)$, assume $\int_0^t \langle Y_{T-s}(x_s^T)(v_s^T), dB_s \rangle$, $0 \leqslant t \leqslant T$, is a martingale.

Then for each $0 < t \leqslant T$,

$$Du_t(x_0)(v_0) = \frac{1}{t} Eu_0(x_t^t)\, e^{\int_0^t V_{t-s}(x_s^T)\, ds} \int_0^t \langle Y_{t-s}(x_s^t)(v_s^t), dB_s \rangle$$
$$+ \frac{1}{t} Eu_0(x_t^t)\, {}^{\int_0^t V_{t-s}(x_s^T)\, ds} \int_0^t \int_0^s DV_{t-r}(x_r^t)(v_r^t)\, dr\, ds. \quad (11)$$

The only real additional ingredients in the proof are the almost sure identities

$$F_r^{T-s}(F_s^T(x_0, \omega), \theta_s(\omega)) = F_{s+r}^T(x_0, \omega), \qquad (x_0, \omega) \in M,$$

and

$$\alpha_s^T(x_0, \omega)\, \alpha_{T-s}^{T-s}(F_s^T(x_0, \omega), \theta_s(\omega)) = \alpha_T^T(x_0, \omega),$$

where $\theta_s : \Omega \to \Omega$ is the shift, e.g., using the canonical representation of $\{B_t : t \geqslant 0\}$.

Note that for X, Z with first two derivatives bounded and f in BC^2, we can differentiate twice under the integral sign [8] to see directly that $P_{T-t} f(x)$ is sufficiently regular to prove (6). This gives (6) without using elliptic regularity results and from this (e.g., via (9)) we can approximate to obtain the smoothing property directly (see [3] for this approach in infinite dimensions). ∎

For further smoothing, we can use the next result (c is a constant).

THEOREM 2.3. *Assume that Eq. (3) is complete and has uniform ellipticity: X has a right inverse Y, which is bounded on R^n. Suppose also*

1. *For each* $x_0, u_0 \in R^n$ *and each* $T>0$,

$$\int_0^T E\,|DF_s(x_0)(u_0)|^2\,ds \leqslant c\,|u_0|^2. \tag{12}$$

2. *For each* $t>0$,

$$\sup_{0\leqslant s\leqslant t}\ \sup_{y_0\in R^n}\,(E\,|D^2F_s(y_0)(u_0, v_0)|) \leqslant c\,|u_0|\,|v_0|$$

and

$$\sup_{0\leqslant s\leqslant t}\ \sup_{y_0\in R^n}\,(E\,|DF_s(y_0)|) \leqslant c.$$

Let f *be in* BC^2 *and such that* $d(P_t f)_{x_0} = \delta P_t(df)_{x_0}$ *for almost all* $x_0 \in R^n$ *and that we can differentiate* $P_t f$ *under the expectation to give, for almost all* x_0,

$$\begin{aligned} D^2P_t f(x_0)(u_0)(v_0) &= ED^2 f(x_t)(DF_t(x_0)\,u_0, DF_t(x_0)\,v_0) \\ &\quad + EDf(x_t)(D^2F_t(x_0)(u_0, v_0)) \end{aligned} \tag{13}$$

for each $t\geqslant 0$. *Then for almost all* x_0 *in* R^n *and all* $t>0$,

$$\begin{aligned} &D^2P_t f(x_0)(u_0, v_0) \\ &\quad = \frac{4}{t^2} E\left\{ f(x_t) \int_{t/2}^{t} \langle Y(x_s)\,v_s,\ dB_s\rangle \int_0^{t/2} \langle Y(x_s)\,u_s, dB_s\rangle \right\} \\ &\quad -\frac{2}{t} E\int_0^{t/2} D(P_{t-s}f)(x_s)(DX(x_s)(v_s)(Y(x_s)\,u_s))\,ds \\ &\quad +\frac{2}{t} E\int_0^{t/2} D(P_{t-s}f)(x_s)(D^2F_s(x_0)(u_0, v_0))\,ds. \end{aligned} \tag{14}$$

If also $\int_0^{t/2} \langle DY(x_s)(DF(x_0)\,u_0)(DF(x_0)\,v_0), dB_s\rangle$ *is a martingale, then*

$$\begin{aligned} &D^2P_t f(x_0)(u_0, v_0) \\ &\quad = \frac{4}{t^2} E\left\{ f(x_t) \int_{t/2}^{t} \langle Y(x_s)\,v_s, dB_s\rangle \int_0^{t/2} \langle Y(x_s)\,u_s, dB_s\rangle \right\} \\ &\quad +\frac{2}{t} E\left\{ f(x_t) \int_0^{t/2} \langle DY(x_s)(u_s)(v_s), dB_s\rangle \right\} \\ &\quad +\frac{2}{t} E\left\{ f(x_t) \int_0^{t/2} \langle Y(x_s)\,D^2F_s(x_0)(u_0, v_0), dB_s\rangle \right\}. \end{aligned} \tag{15}$$

Proof. Since $d(P_{T-t}f)$ is smooth and satisfies the relevant parabolic equation, by Itô's formula (e.g., [6, Cor. 3E1]), if $0 \leqslant t < T$,

$$d(P_{T-t}\,df)_{x_t}(v_t) = d(P_T f)_{x_0}(v_0) + \int_0^t \nabla(d(P_{T-s}f)_{x_s})(X(x_s)\,dB_s)(v_s)$$
$$+ \int_0^t (d(P_{T-s}f)_{x_s}(DX(x_s)(v_s)\,dB_s)$$

giving

$$(df)_{x_T}(v_T) = d(P_T f)_{x_0}(v_0) + \int_0^T D^2(P_{T-s}f)(x_s)(X(x_s)\,dB_s)(v_s)$$
$$+ \int_0^T D(P_{T-s}f)(x_s)(DX(x_s)(v_s)\,dB_s).$$

Using the uniform ellipticity and Hypothesis 1 (i.e., Eq. (12)), this gives

$$E\left\{(df)_{x_T}(v_T)\int_0^T \langle Y(x_s)\,u_s, dB_s\rangle\right\}$$
$$= E\int_0^T D^2(P_{T-s}f)(x_s)(u_s)(v_s)\,ds$$
$$+ E\int_0^T D(P_{T-s}f)(x_s)(DX(x_s)(v_s)(Y(x_s)\,u_s))\,ds.$$

Thus, by (13) and using the two hypotheses to justify changing the order of integration,

$$T[D^2P_T f(x_0)(u_0, v_0)]$$
$$= E\left\{Df(x_T)(v_T)\int_0^T \langle Y(x_s)\,u_s, dB_s\rangle\right\}$$
$$- E\left\{\int_0^T D(P_{T-s}f)(x_s)(DX(x_s)(v_s)\,(Y(x_s)\,u_s))\,ds\right\}$$
$$+ E\int_0^T \{(P_{T-s}f)(x_s)(D^2F_s(x_0)(u_0, v_0))\}\,ds.$$

Now let $T = t/2$, and replace f by $P_{t/2}f$. Note that by Theorem 2.1 and the Markov property (or cocycle property of flows)

$$DP_{t/2}f(x_{t/2})(v_{t/2}) = \frac{2}{t}E\left\{f(x_t)\int_{t/2}^t \langle Y(x_s)\,v_s, dB_s\rangle \,|\, x_s : 0 \leqslant s \leqslant t/2\right\}.$$

We see

$$
\begin{aligned}
&D^2P_tf(x_0)(u_0, v_0)\\
&\quad = \frac{4}{t^2} E\left\{f(x_t)\int_{t/2}^{t} \langle Y(x_s)\, v_s, dB_s\rangle \int_0^{t/2} \langle Y(x_s)\, u_s, dB_s\rangle\right\}\\
&\qquad -\frac{2}{t} E\int_0^{t/2} D(P_{t/2-s}f)(x_s)(DX(x_s)(v_s)(Y(x_s)u_s))\, ds\\
&\qquad +\frac{2}{t} E\int_0^{t/2} D(P_{t/2-s}f)(x_s)(D^2F_s(x_0)(u_0, v_0))\, ds,
\end{aligned}
$$

giving (14). Now apply Itô's formula to $\{P_{t-s}f(x_s) : 0 \leqslant s < t\}$ at $s = t/2$ to obtain

$$
P_{t/2}f(x_{t/2}) = P_tf(x_0) + \int_0^{t/2} D(P_{t/2-s}f)(x_s)(X(x_s)\, dB_s). \tag{16}
$$

Equation (15) follows on multiplying (16) by $\int_0^{t/2} \langle DY(x_s)(u_s)(v_s), dB_s\rangle$ and also by $\int_0^{t/2} \langle Y(x_s)\, D^2F_s(x_0)(u_0, v_0), dB_s\rangle$ and taking expectations to replace the second and third terms in the right-hand side of (16), using the identity

$$
DX(x)(u)(Y(x)v) + X(x)\, DY(x)(u)(v) = 0. \tag{17}
$$

Remarks. (A) Formula (14) combined with Theorem 2.1 has some advantage over (15) for estimation since the derivative of Y does not appear.

(B) Formula (15) can be obtained by applying Theorem 2.1, with t replaced by $t/2$, to $P_{t/2}f$ and then differentiating under the expectation and stochastic integral sign, assuming this is legitimate, then using the Markov property to replace the $P_{t/2}f(x_{t/2})$ by $f(x_t)$.

(C) Hypotheses 1 and 2 of the theorem and the conditions on the function f are satisfied if $|DX|$, $|D^2X|$, $|DA|$, and $|D^2A|$ are bounded. See Lemma A2, theorem A5, and Proposition A8 in the Appendix. Furthermore, the martingale condition needed for (15) also holds if DY is bounded as a bilinear map.

3. Formulae with Simple Proof for M

For a general smooth manifold M, we return to the Stratonovich equation (1). We continue to assume non-explosion and non-degeneracy. Thus now $X(x)$ is a surjective linear map of R^m onto the tangent space T_xM to

M at x and A is a smooth vector field on M. Write $X^i(x) = X(x)(e_i)$ for $e_1, ..., e_m$ an orthonormal basis for R^m. Thus (1) becomes

$$dx_t = \sum_1^m X^i(x_t) \circ dB_t^i + A(x_t)\, dt. \tag{18}$$

Here $\{B_t^i, t \geq 0\}$ are independent one-dimensional Brownian motions. The generator $\mathscr{A}$, being elliptic, can be written $\mathscr{A} = \frac{1}{2}\Delta + Z$ where Δ denotes the Laplace–Beltrami operator for an induced Riemannian metric on M and Z is a smooth vector field on M. Using this metric and the Levi–Civita connection,

$$Z = A^X = \tfrac{1}{2} \sum_1^m \nabla X^i(X^i(x)) + A. \tag{19}$$

The derivative equation extending (4) is most concisely expressed as a covariant equation

$$dv_t = \nabla X(v_t) \circ dB_t + \nabla A(v_t) dt. \tag{20}$$

By definition, this means

$$d\tilde{v}_t = //_t^{-1} \nabla X(//_t \tilde{v}_t) \circ dB_t + //_t^{-1} \nabla A(//_t \tilde{v}_t)\, dt \tag{21}$$

for $\tilde{v}_t = //_t^{-1} v_t$ with $//_t : T_{x_0}M \to T_{x_t}M$ parallel translation along the paths of $\{x_t : t \geq 0\}$.

Recall that covariant differentiation gives linear maps

$$\nabla A \colon T_x M \to T_x M, \qquad x \in M,$$

$$\nabla X \colon T_x M \to L(R^m; T_x M) \qquad x \in M,$$

and

$$\nabla^2 A \colon T_x M \to L(T_x M; T_x M) \qquad x \in M,$$

sometimes considered as a bilinear map by

$$\nabla^2 A(u, v) = \nabla^2 A(u)(v) \quad \text{etc.}$$

For the (measurable) stochastic flow $\{F_t(x) : t \geq 0,\ x \in M\}$ to (1), the derivative in probability now becomes a linear map between tangent spaces written

$$T_{x_0} F_t \colon T_{x_0} M \to T_{x_t} M, \qquad x_0 \in M,$$

or

$$TF_t \colon TM \to TM,$$

and $v_t = T_{x_0} F_t(v_0)$, the derivative at x_0 in the direction v_0.

Analogous to the probability semigroup P_t, there is the following semigroup (formally) on differential forms:

$$\delta P_t \phi(v_1, ..., v_p) = E\phi(TF_t(v_1), ..., TF_t(v_p)). \tag{22}$$

Here ϕ is a p-form. If $\phi = df$ for some function f, then

$$\delta P_t(df)(v) = E\, df(TF_t(v)).$$

In [8], it was shown that $\delta P_t(df) = d(P_t f)$ if ∇X, ∇A, and $\nabla^2 X$ are bounded, and if the stochastic differential equation is strongly complete on R^n (or on a complete Riemannian manifold with bounded curvature). Theorems of this kind are since much improved partially due to the concept of strong 1-completeness [13]. See the Appendix for the definition of strong 1-completeness.

To differentiate $P_t f$ twice it is convenient to use the covariant derivative ∇TF_t which is bilinear:

$$\nabla T_{x_0} F_t \colon T_{x_0} M \times T_{x_0} M \to T_{x_t} M.$$

It can be defined by

$$\nabla T_{x_0} F_t(u_0, v_0) = \frac{D}{\partial s} T_{\sigma(s)} F_t(v(s))|_{s=0} \tag{23}$$

for σ a C^1 curve in M with $\sigma(0) = x_0$, $\dot{\sigma}(0) = u_0$, and for $v(s)$ the parallel translate of v_0 along σ to $\sigma(s)$, the derivative being a derivative in probability in general [8, p. 141].

The extensions of Theorems 2.1 and 2.3 can be written as follows and proved in essentially the same way; note that we can take $Y(x) = X(x)^*$.

THEOREM 3.1. *Let M be a complete Riemannian manifold and $\mathscr{A} = \frac{1}{2}\Delta + Z$. Assume* (1) *is complete. Let $f\colon M \to R$ be BC^1 with $\delta P_t(df) = d(P_t f)$ a.e. for $t \geqslant 0$. Then for almost all $x_0 \in M$,*

$$dP_t f(v_0) = \frac{1}{t} Ef(x_t) \int_0^t \langle v_s, X(x_s)\, dB_s \rangle_{x_s}, \qquad v_0 \in T_{x_0} M, \tag{24}$$

provided $\int_0^t \langle v_s, X(x_s)\, dB_s \rangle$ is a martingale. Furthermore, assume

1. *For each $T > 0$ and $x_0 \in M$,*

$$\int_0^T E\,|T_{x_0} F_s(u_0)|^2\, ds \leqslant c\, |u_0|^2, \qquad u_0 \in T_{x_0} M. \tag{25}$$

2. *For each* $T>0$,

$$\sup_{0\leqslant s\leqslant T}\ \sup_{y_0\in M}\ (E\,|\nabla T_{y_0}F_s(u_0,v_0)|)\leqslant c\,|u_0|\,|v_0|, \tag{26}$$

and

$$\sup_{0\leqslant s\leqslant T}\ \sup_{y_0\in M}\ (E\,|T_{y_0}F_s|)\leqslant c. \tag{27}$$

Let f be a BC^2 *function such that we can differentiate* $P_t f$ *under the expectation to give*

$$\nabla\,dP_t f(-)(-)=E\nabla\,df(TF_t(-),TF_t(-))+E\,df(\nabla TF_t(-,-)),\qquad a.e., \tag{28}$$

for each $t\geqslant 0$. *Then for almost all* $x_0\in M$, *all* u_0, v_0 *in* $T_{x_0}M$, *and* $t>0$,

$$\begin{aligned}\nabla d(P_t f)(u_0,v_0)&=\frac{4}{t^2}E\left\{f(x_t)\int_{t/2}^{t}\langle v_s, X(x_s)\,dB_s\rangle\int_0^{t/2}\langle u_s, X(x_s)\,dB_s\rangle\right\}\\&\quad+\frac{2}{t}E\left\{f(x_t)\left(\int_0^{t/2}\langle v_s,\nabla X(u_s)\,dB_s\rangle\right.\right.\\&\quad\left.\left.+\int_0^{t/2}\langle \nabla TF_s(u_0,v_0), X(x_s)\,dB_s\rangle\right)\right\}.\end{aligned}$$

From Theorem 3.1, formula (24) holds for all x if $H_2(x)(v,v)\leqslant c\,|v|^2$ for some constant c, by Lemma A2, Theorem A5, and its remark. Here

$$\begin{aligned}H_2(x)(v,v):=&-Ric_x(v,v)+2\langle\nabla Z(x)(v),v\rangle+\sum_1^m|\nabla X^i(x)|^2\\&+\sum_1^m\frac{1}{|v|^2}\langle\nabla X^i(x)(v),v\rangle^2.\end{aligned} \tag{29}$$

Suppose the first three derivatives of X and the first two of A are bounded; then all the conditions of the theorem hold. See Lemma A2, Proposition A6, and Proposition A8 for details.

Now let $p_t\colon M\times M\to R$, $t>0$, be the heat kernel (with respect to the Riemannian volume element) so that

$$P_t f(x)=\int_M p_t(x,y)f(y)\,dy. \tag{30}$$

There is the following Bismut-type formula (see [6] and Section 5A below).

COROLLARY 3.2. *Suppose $\delta P_t(df) = d(P_t f)$ for all f in C_K^∞ and for all $t > 0$. Then, for $t > 0$,*

$$\nabla \log p_t(\cdot, y)(x_0) = \frac{1}{t} E\left\{\int_0^t (TF_s)^* X(x_s)\, dB_s \,|\, x_t = y\right\} \tag{31}$$

for almost all $y \in M$ provided $\int_0^t \langle v_s, X(x_s)\, dB_s\rangle$ is a martingale. In particular, (31) *holds if H_2 defined in* (29) *is bounded above.*

Proof. The proof is just as for the compact case. Let $f \in C_K^\infty$. By the smoothness of $p_t(-, -)$ for $t > 0$, we can differentiate Eq. (30) to obtain

$$d(P_t f)(v_0) = \int_M \langle \nabla p_t(-, y), v_0 \rangle_{x_0} f(y)\, dy. \tag{32}$$

On the other hand, we may rewrite Eq. (24) as follows:

$$d(P_t f)(v_0) = \int_M p_t(x_0, y) f(y) \times E\left\{\frac{1}{t}\int_0^t \langle TF_s(v_0), X(x_s)\, dB_s\rangle \,|\, x_t = y\right\} dy.$$

Comparing the last two equations, we get

$$\nabla p_t(-, y)(x_0) = p_t(x_0, y)\, E\left\{\frac{1}{t}\int_0^t TF_s^*(X\, dB_s) \,|\, x_t = y\right\}. \quad \blacksquare$$

Equality in (31) for all y follows from the continuity of the right-hand side in y: for this see [1], the Appendix to [16], or [19].

Let $h: M \to R$ be a smooth function. There is a corresponding Sobolev space $W^{p,1} = \{f: M \to R \text{ s.t. } f, \nabla f \in L^p(M, e^{2h}\, dx)\}$ for $1 \leqslant p \leqslant \infty$ with norm $|f|_{L^{p,1}} = |f|_{L^p} + |\nabla f|_{L^p}$. Here dx is the Riemannian volume measure.

COROLLARY 3.3. *Suppose $\mathcal{A} = \frac{1}{2}\Delta + \nabla h$ for smooth h and that*

$$k^2 =: \sup_{x \in M} E\int_0^t |T_x F_s|^2\, ds < \infty.$$

Then (24) *holds almost everywhere for any $f \in L^p$, $1 < p \leqslant \infty$, and, for $t > 0$, P_t gives a continuous map*

$$P_t: L^p(M, e^{2h}\, dx) \to W^{p,1}(M, e^{2h}\, dx), \qquad 1 < p \leqslant \infty$$

with

$$|(P_t f)|_{L^{p,1}} \leqslant \left(1+\frac{k_p}{t}\right)|f|_{L^p}, \tag{33}$$

where $k_p = k$ *for* $2 \leqslant p \leqslant \infty$, *and* $k_p = c_p k^p$ *for* $1 < p < 2$ *and* c_p *a universal constant.*

Proof. Take f in BC^1. Noting that $e^{2h}\,dx$ is an invariant measure for the solution of (1), formula (24) gives

$$\begin{aligned}|\nabla(P_t f)(v)|_{L^2} &\leqslant \frac{1}{t}\sqrt{\int_M \left[Ef(F_t(x))\int_0^t \langle X(F_s(x))\,dB_s, TF_s(v)\rangle\right]^2 e^{2h}\,dx}\\ &\leqslant \frac{1}{t}\left(\sup_{x\in M} E\int_0^t |T_x F_s(v)|^2\,ds\right)^{1/2}\sqrt{\int_M Ef(F_t(x))^2\,e^{2h}\,dx}\\ &= \frac{1}{t}\left(\sup_{x\in M} E\int_0^t |T_x F_s(v)|^2\,ds\right)^{1/2}|f|_{L^2}.\end{aligned}$$

If $f \in L^2$, let f_n be a sequence in C_K^∞ converging to f in L^2, then $d(P_t f_n)$ converges in L^2 by the estimate with limit $d(P_t f)$. So formula (24) holds almost everywhere for L^2 functions.

On the other hand, if f also belongs to L^∞,

$$|P_t f|_{L^{\infty,1}} \leqslant \left(1+\sup_{x\in M}\frac{1}{t}\left(\int_0^t E\,|T_x F_s|^2\,ds\right)^{1/2}\right)|f|_{L^\infty}. \tag{34}$$

By the Reisz–Thorin interpolation theorem, we see for $f \in L^2 \cap L^p$, $2 \leqslant p \leqslant \infty$,

$$|(P_t f)|_{L^{p,1}} \leqslant \left(1+\frac{k}{t}\right)|f|_{L^p}. \tag{35}$$

Again we conclude that (24) holds for $f \in L^p$, $2 \leqslant p < \infty$. For $1 < p < 2$, let q be such that $1/p + 1/q = 1$. Then Hölder's inequality gives

$$\begin{aligned}|\nabla(P_t f)(v)|_{L^p} &\leqslant \frac{1}{t}\left(\int_M \left[Ef(F_t(x))\int_0^t \langle X(F_s(x))\,dB_s, TF_s(v)\rangle\right]^p e^{2h}\,dx\right)^{1/p}\\ &\leqslant \frac{1}{t}\left(\sup_{x\in M} E\left[\int_0^t \langle X\,dB_s, T_x F_s(v)\rangle\right]^q\right)^{1/q}\\ &\quad\times\left(\int_M E[f(F_t(x))]^p\,e^{2h}\,dx\right)^{1/p}\\ &= \frac{1}{t}\left(\sup_{x\in M} E\left[\int_0^t \langle X\,dB_s, T_x F_s(v)\rangle\right]^q\right)^{1/q}|f|_{L^p}.\end{aligned}$$

But

$$E\left[\int_0^t \langle X\,dB_s, T_xF_s(v)\rangle\right]^q \leqslant c_p E\left(\int_0^t |T_xF_s(v)|^2\,ds\right)^{q/2}$$

by the Burkholder–Davis–Gundy inequality. Here c_p is a constant. So again we have (33).

From (32) and Corollary 3.2 we see that (24) holds almost everywhere for $f \in L_\infty$ as therefore does (34).

EXAMPLE. Left invariant systems on Lie groups. Let G be a connected Lie group with identity element $\mathbf{1}$ and with L_g and R_g denoting left and right translation by G. Consider a left invariant s.d.e.

$$dx_t = X(x_t) \circ dB_t + A(x_t)\,dt \tag{36}$$

with solution $\{g_t : t \geqslant 0\}$ from $\mathbf{1}$. Then (36) has solution flow

$$F_t(u) = R_{g_t}u, \qquad t \geqslant 0, \quad u \in G.$$

Take a left-invariant Riemannian metric on G. Then by (24) for $f \in BC^1(G)$, $v_0 \in T_{\mathbf{1}}G$, if (36) is non-degenerate with $X(\mathbf{1})$: $R^m \to T_{\mathbf{1}}G$ an isometry,

$$\begin{aligned} dP_tf(v_0) &= \frac{1}{t}E\left\{f(g_t)\int_0^t \langle T_{\mathbf{1}}R_{g_s}(v_0), X(g_s)\,dB_s\rangle\right\} \\ &= \frac{1}{t}E\left\{f(g_t)\int_0^t \langle ad(g_s)^{-1}(v_0), d\tilde{B}_s\rangle_1\right\} \end{aligned}$$

where $\tilde{B}_s = X(\mathbf{1})B_s$. This gives

$$\nabla \log p_t(\mathbf{1}, y) = \frac{1}{t}E\left\{\int_0^t ad(g_s^{-1})^*\,d\tilde{B}_s \mid g_t = y\right\}.$$

4. For 1-Forms

Let M be a complete Riemannian manifold and $h: M \to R$ a smooth function with $L_{\nabla h}$ the Lie derivative in the direction of ∇h. Let $\Delta^h =: \Delta + 2L_{\nabla h}$ be the Bismut–Witten–Laplacian, and $\Delta^{h,q}$ its restriction to q-forms. It is then an essentially self-adjoint linear operator on $L^2(M, e^{2h(x)}\,dx)$ (see [13], extending [2] from the case $h=0$). We still use

Δ^h for its closure and use $D(\Delta^h)$ for its domain. By the spectral theorem, there is a smooth semigroup $e^{(1/2)t\Delta^h}$ solving the heat equation

$$\frac{\partial P_t}{\partial t} = \frac{1}{2}\Delta^h P_t.$$

A stochastic dynamical system (1) is called an *h-Brownian system* if it has generator $\frac{1}{2}\Delta^h$. Its solution is called an *h-Brownian motion.*

For clarity, we sometime use $P_t^{h,q}$ for the restriction of the semigroup $P_t^h := e^{(1/2)t\Delta^h}$ to q-forms. Denoting exterior differentiation by d with suitable domain, let δ^h be the adjoint of d in $L^2(M, e^{2h(x)}\,dx)$. Then $\Delta^h = -(d\delta^h + \delta^h d)$, and for $\phi \in D(\Delta^h)$,

$$d(P_t^{h,q}\phi) = P_t^{h,q+1}(d\phi). \tag{37}$$

Define

$$\int_0^t \phi \circ dx_s = \int_0^t \phi(X(x_s)\,dB_s) - \tfrac{1}{2}\int_0^t \delta^h\phi(x_s)\,ds, \tag{38}$$

for a 1-form ϕ. Theorem 2.1 has a generalization to closed differential forms. It is given in terms of the line integral $\int_0^t \phi \circ dx_s$ and a martingale; for it we need the following Itô's formula from [6]:

LEMMA 4.1 (Itô's Formula for one Forms). *Let T be a stopping time with $T < \xi$, then*

$$\begin{aligned}\phi(v_{t\wedge T}) = \phi(v_0) &+ \int_0^{t\wedge T} \nabla\phi(X(x_s)\,dB_s)(v_s) + \int_0^{t\wedge T} \phi(\nabla X(v_s)\,dB_s)\\ &+ \tfrac{1}{2}\int_0^{t\wedge T} \Delta^h\phi(x_s)(v_s)\,ds\\ &+ \tfrac{1}{2}\int_0^t \operatorname{trace}(\nabla\phi(X(x_s)(-))\,\nabla X(v_s)(-))\,ds. \quad\blacksquare\end{aligned}$$

THEOREM 4.2. *Consider an h-Brownian system. Assume there is no explosion, and*

$$\int_0^t E\,|T_x F_s|^2\,ds < \infty, \qquad \textit{for each } x \textit{ in } M.$$

Let ϕ be a closed 1-form in $D(\Delta^h) \cap L^\infty$, such that

$$\delta P_t \phi = e^{(1/2)t\Delta^{h,1}}\phi.$$

Then

$$P_t^{h,1}\phi(v_0)=\frac{1}{t}E\int_0^t \phi\circ dx_s \int_0^t \langle X(x_s)\,dB_s, TF_s(v_0)\rangle, \tag{39}$$

for all $v_0 \in T_{x_0}M$.

Proof. Following the proof for a compact manifold as in [6], let

$$Q_t(\phi)=-\tfrac{1}{2}\int_0^t P_0^h(\delta^h\phi)\,ds. \tag{40}$$

Differentiate Eq. (40) to obtain

$$\frac{\partial}{\partial t}Q_t\phi=-\tfrac{1}{2}P_t^h(\delta^h\phi).$$

We also have

$$\begin{aligned} d(Q_t\phi)&=-\tfrac{1}{2}\int_0^t d\delta^h(P_s^h\phi)\,ds\\ &=\tfrac{1}{2}\int_0^t \Delta^h(P_s^h\phi)\,ds\\ &=P_t^h\phi-\phi \end{aligned}$$

since $d\delta^h(P_s^h\phi)=P_s^h(d\delta^h\phi)$ is uniformly continuous in s and

$$d(P_s^h\phi)=P_s^h\,d\phi=0.$$

Consequently,

$$\Delta^h(Q_t(\phi))=-P_t^h(\delta^h\phi)+\delta^h\phi.$$

Apply Itô's formula to $(t,x)\mapsto Q_{T-t}\phi(x)$, which is sufficiently smooth because $P_s^h\phi$ is, to obtain

$$\begin{aligned} Q_{T-t}\phi(x_t)&=Q_T\phi(x_0)+\int_0^t d(Q_{T-s}\phi)(X(x_s)\,dB_s)\\ &\quad+\frac{1}{2}\int_0^t \Delta^h Q_{T-s}\phi(x_s)\,ds+\int_0^t\frac{\partial}{\partial s}Q_{T-s}\phi(x_s)\,ds\\ &=Q_T\phi(x_0)+\int_0^t P_{T-s}^h(\phi)(X(x_s)\,dB_s)-\int_0^t \phi\circ dx_s. \end{aligned}$$

Setting $t=T$, we obtain

$$\int_0^T \phi \circ dx_s = Q_T(\phi)(x_0) + \int_0^T P^h_{T-s}(\phi)(X(x_s)\,dB_s),$$

and thus

$$E\int_0^T \phi \circ dx_s \int_0^T \langle X(x_s)\,dB_s, TF_s(v_0)\rangle = E\int_0^T P^h_{T-s}\phi(TF_s(v_0))\,ds.$$

But

$$E\int_0^T P^h_{T-s}\phi(TF_s(v_0))\,ds = \int_0^T EP^h_{T-s}\phi(TF_s(v_0))\,ds, \tag{41}$$

by Fubini's theorem, since

$$\int_0^T E\,|P^h_{T-s}\phi(TF_s(v_0))|\,ds \leqslant |\phi|_\infty \int_0^T E\,|TF_T(v_0)|\,ds < \infty.$$

Next notice

$$EP^h_{T-s}\phi(TF_s(v_0)) = E\phi(TF_T(v_0)) = P^h_T\phi(v_0)$$

from the strong Markov property. We obtain

$$P^{h,1}_T\phi(v_0) = \frac{1}{T}E\left\{\int_0^T \phi \circ dx_s \int_0^T \langle X\,dB_s, TF_s(v_0)\rangle\right\}. \quad \blacksquare$$

Remark. If we assume $\sup_x E\,|T_xF_t|^2 < \infty$ for each t the result holds for all $\phi \in D(\Delta^h)$: first we have $\delta P_t\phi = e^{(1/2)t\Delta^h}\phi$ for $\phi \in L^2$ by continuity and also Eq. (41) holds from the following argument:

$$\int_0^T E\,|P^h_{T-s}\phi(TF_s(v))|\,ds \leqslant \int_0^T E\,|\phi(TF_T(v))|\,ds$$

$$\leqslant E\,|T_xF_T(v)|^2 \sup_x\left(\int_0^T E\,|\phi|^2_{F_T(x)}\,ds\right).$$

But $\int_M E\,|\phi|^2_{F_T(x)}\,e^{2h}\,dx = \int |\phi|^2\,e^{2h}\,dx < \infty$. So $E\,|\phi|^2_{F_T(x)} < \infty$ for each x by the continuity of $E\,|\phi|^2_{F_T(x)} = P_T(|\phi|^2)(x)$ in x.

COROLLARY 4.3. *Suppose $|\nabla X|$ is bounded and for all $v \in T_xM$, all $x \in M$,* $\text{Hess}(h)(v,v) - \frac{1}{2}\text{Ric}_x(v,v) \leqslant c\,|v|^2$ *for some constant c. Then* (39) *holds for all closed* 1*-forms in $D(\Delta^h)$.*

Proof. By Lemma A2, we have $\sup_x E\,|T_xF_t|^2 < \infty$ and $E\sup_{s\leqslant t}|T_xF_s| < \infty$. Thus Proposition A6 shows that $P_t^{h,1}\phi = \delta P_t(\phi)$. Theorem 4.2 now applies. ▌

Remark. Note that if $\phi = df$, formula (39) reduces to (24) using (37).

5. The Hessian Flow

A. Let $Z = A^X$ as in Section 3. Let $x_0 \in M$ with $\{x_t : 0 \leqslant t < \xi\}$, being the solution to (1) with initial value x_0 and explosion time ξ. Let W_t^Z be the solution flow to the covariant differential equation along $\{x_t\}$,

$$\frac{DW_t^Z(v_0)}{\partial t} = -\frac{1}{2}\,\mathrm{Ric}^{\#}(W_t^Z(v_0), -) + \nabla Z(W_t^Z(v_0)), \tag{42}$$

with $W_0^Z(v_0) = v_0$. It is called the Hessian flow. Here Ric denotes the Ricci curvature of the manifold, and $\#$ denotes the relevant raising or lowering of indices so that $\mathrm{Ric}^{\#}(v, -) \in T_xM$ if $v \in T_xM$. For $x \in M$ set

$$\rho(x) = \inf_{|v| \leqslant 1} \{\mathrm{Ric}_x(v, v) - 2\,\nabla Z(x)^{\#}(v, v)\}.$$

The following is a generalization of a result in [6].

Proposition 5.1 [13]. *Let $Z = \nabla h$ for h a smooth function on M. Suppose for some $T_0 > 0$,*

$$E \sup_{t \leqslant T_0} \chi_{t < \xi(x)} e^{-(1/2)\int_0^t \rho(F_s(x))\,ds} < \infty, \qquad 0 \leqslant t \leqslant T_0.$$

Then for a closed bounded C^2 1-form ϕ, we have for $0 < t \leqslant T_0$,

$$P_t^{h,1}\phi(v_0) = \frac{1}{t}\,E\int_0^t \phi \circ dx_s \int_0^t \langle X(x_s)\,dB_s, W_s^Z(v_0)\rangle. \tag{43}$$

The proof is as for (39) with TF_t, just noticing that under the conditions of the proposition, the s.d.e. does not explode and $P_t^{h,1}\phi = E\phi(W_t^h)$ for bounded 1-forms ϕ (see, e.g., [5 and 14]).

Remark. Taking $\phi = df$, we obtain, by (37),

$$dP_tf(v_0) = \frac{1}{t}\,Ef(x_t)\int_0^t \langle W_s^Z(v_0), X(x_s)\,dB_s\rangle \tag{44}$$

which leads to Bismut's formula [6] for $\nabla \log p_t(-, y)$ (proved there for $Z = 0$ and M compact). In fact, (44) can be proved directly, without assuming that Z is a gradient, by our basic method: Let $\phi_t = d(P_t f)$, then it solves $\partial\phi_t/\partial t = \frac{1}{2}\Delta^1\phi_t + L_{\nabla Z}\phi_t$ since $P_t f$ solves $\partial g/\partial t = \frac{1}{2}\Delta g + L_{\nabla Z} g$. Then Itô's formula (as in [6]) applied to $\phi_{t-s}(W_s^Z(v_0))$ shows that $\phi_t(v_0) = E\phi_0(W_t^Z(v_0))$ and our usual method can be used. Furthermore, if ρ is bounded from below so that $|W_t^Z|$ is bounded as in [6], then (44) holds for bounded measurable functions.

Note that it was shown, in [7], that for a gradient system on compact M, $E\{v_t \mid x_s : 0 \leqslant s \leqslant t\} = W_t^h(v_0)$. Recall that a *gradient system* is given by $X(\cdot)(e) = \nabla\langle f(\cdot), e\rangle$, $e \in R^m$, for $f\colon M \to R^m$, an isometric embedding. This relation between the derivative flow and the Hessian flow holds for non-compact manifolds if $E\int_0^t |\nabla X(x_s)|^2\, |v_s|^2\, ds < \infty$.

B. Let $V_\cdot(\cdot)\colon [0, \infty) \times R^n \to R$ be continuous, C^1 in x for each t and bounded above with derivative dV bounded on each $[0, T] \times R^n$. Consider the equation with potential V,

$$\frac{\partial u_t}{\partial t} = \frac{1}{2}\Delta u_t + L_Z u_t + V_t u_t.$$

Assume that the s.d.e. (1) does not explode. By the corresponding argument to that used for the case $V \equiv 0$, we get for $v_0 \in T_{x_0}M$,

$$du_t(v_0) = Eu_0(x_t)\, e^{\int_0^t V_{t-s}(x_s)\, ds} \int_0^t dV_{t-s}(W_s^Z(v_0))\, ds + E\, du_0(W_t^Z(v_0))\, e^{\int_0^t V_{t-s}(x_s)\, ds},$$

provided that $-\frac{1}{2}\mathrm{Ric}^{\#} + \nabla Z$ is bounded above as a linear operator and u_0 is BC^1. From this the proof analogous to that of (11) gives the following.

THEOREM 5.2. *Assume non-explosion and suppose* $-\frac{1}{2}\mathrm{Ric}^{\#} + \nabla Z$ *is bounded above and* dV *is bounded. Then for* u_0 *bounded measurable and* $t > 0$,

$$du_t(v_0) = \frac{1}{t} Eu_0(x_t)\, e^{\int_0^t V_{t-s}(x_s)\, ds} \int_0^t \langle W_s^Z, X(x_s)\, dB_s\rangle + \frac{1}{t} Eu_0(x_t) e^{\int_0^t V_{t-s}(x_s)\, ds} \int_0^t (t-r)\, dV_{t-r}(x_r)(W_r^Z)\, dr. \quad (45)$$

6. For Higher Order Forms and Gradient Brownian Systems

Recall that a gradient h-Brownian system is a gradient system with $A(x) =: \nabla h(x)$. For such systems $\sum_1^m \nabla X^i(X^i) = 0$. We assume there is no explosion as before.

If A is a linear map from a vector space E to E, then $(dA)^q A$ is the map from $E \times \cdots \times E$ to $E \times \cdots \times E$ defined by

$$(dA)^q A(v^1, ..., v^q) = \sum_{j=1}^{q} (v^1, ..., Av^j, ..., v^q).$$

Let $v_0 = (v_0^1, ..., v_0^q)$, for $v_0^i \in T_{x_0}M$. Denote by v_t the q vector induced by TF_t:

$$v_t = (TF_t(v_0^1), TF_t(v_0^2), ..., TF_t(v_0^q)).$$

LEMMA 6.1 [6]. *Let θ be a q form. Then, for a gradient h-Brownian system,*

$$\theta(v_t) = \theta(v_0) + \int_0^t \nabla\theta(X(x_s)\, dB_s)(v_s)$$
$$+ \int_0^t \theta((dA)^q (\nabla X(-)\, dB_s)(v_s)) + \int_0^t \tfrac{1}{2} \Delta^{h,q}(\theta)(v_s)\, ds.$$

Recall that if θ is a q form, then

$$(\delta P_t)\,\theta(v_0) = E\theta(v_t) \tag{46}$$

where defined. Define a $(q-1)$ form $\int_0^t \theta \circ dx_s$ by

$$\int_0^t \theta \circ dx_s(\alpha_0) =: \frac{1}{q}\int_0^t \theta(X(x_s)\, dB_s, TF_s(\alpha_0^1), ..., TF_s(\alpha_0^{q-1}))$$
$$-\frac{1}{2}\int_0^t \delta^h\theta(TF_s(\alpha_0^1), ..., TF_s(\alpha_0^{q-1}))\, ds \tag{47}$$

for $\alpha_0 = (\alpha_0^1, ..., \alpha_0^{q-1})$ a $(q-1)$-vector. Then we have the following extension of Theorem 4.2.

THEOREM 6.2. *Let M be a complete Riemannian manifold. Consider a gradient h-Brownian system on it. Suppose it has no explosion and for each $t > 0$ and $x \in M$,*

$$\int_0^t E\,|T_x F_s|^{2q}\, ds < \infty.$$

Let θ be a closed bounded C^2 q form in $D(\Delta^{h,q})$ with

$$\delta P_t \theta = P_t^{h,q}\theta.$$

Then

$$(P_t^{h,q}\theta)_{x_0} = \frac{1}{t} E \int_0^t \langle X(x_s)\, dB_s, T_{x_0}F_s(\cdot)\rangle \wedge \int_0^t \theta \circ dx_s. \tag{48}$$

Proof. Let $Q_t\theta$ be the $(q-1)$ form given by

$$Q_t(\theta)(\alpha_0) = -\tfrac{1}{2}\int_0^t (\delta^h P_s^{h,q}\theta)(\alpha_0)\, ds, \tag{49}$$

for $\alpha_0 \in \bigwedge^{q-1} T_x M$.

Note that $P_t^{h,q}(\theta)$ is smooth on $[0, T] \times M$ by parabolic regularity, so

$$\frac{\partial}{\partial t} Q_t(\theta) = -\tfrac{1}{2}\,\delta^h(P_t^h \theta),$$

$$d(Q_t(\theta)) = -\tfrac{1}{2}\int_0^t d\delta^h(P_s^{h,q}\theta)\, ds,$$

$$\delta^h Q_t(\theta) = -\tfrac{1}{2}\int_0^t \delta^h\delta^h(P_s^{h,q}\theta)\, ds = 0.$$

In particular,

$$d(Q_t(\theta)) = \tfrac{1}{2}\int_0^t \Delta^{h,q}(P_s^{h,q}\theta)\, ds = P_t^{h,q}\theta - \theta \tag{50}$$

since $\Delta^{h,q}\theta = -d\delta^h\theta$. Therefore,

$$\Delta^{h,q-1}(Q_t(\theta)) = -P_t^{h,q-1}(\delta^h\theta) + \delta^h\theta.$$

Next we apply Itô's formula (the previous lemma) to $(t, \alpha) \mapsto Q_{T-t}(\theta)(\alpha)$, writing $\alpha_t = (TF_t(\alpha_0^1), ..., TF_t(\alpha_0^{q-1}))$:

$$\begin{aligned} Q_{T-t}\theta(\alpha_t) &= Q_T\theta(\alpha_0) + \int_0^t \nabla Q_{T-s}\theta(X(x_s)\, dB_s)(\alpha_s) \\ &+ \int_0^t Q_{T-s}\theta((dA)^{q-1}(\nabla X(-)\, dB_s)(\alpha_s)) \\ &+ \frac{1}{2}\int_0^t \Delta^h Q_{T-s}\theta(\alpha_s)\, ds + \int_0^t \frac{\partial}{\partial s}(Q_{T-s})\,\theta(\alpha_s)\, ds. \end{aligned}$$

From the calculations above we obtain

$$Q_{T-t}\theta(\alpha_t) = Q_T\theta(\alpha_0) + \int_0^t \nabla Q_{T-s}\theta(X(x_s)\,dB_s)(\alpha_s)$$
$$+ \int_0^t Q_{T-s}\theta((dA)^{q-1}(\nabla X(-)\,dB_s)(\alpha_s))$$
$$+ \tfrac{1}{2}\int_0^t \delta^h\theta(\alpha_s)\,ds,$$

By definition and the equality above,

$$\int_0^T \theta \circ dx_s(\alpha_0) = Q_T\theta(\alpha_0) + \frac{1}{q}\int_0^T \theta(X(x_s)\,dB_s, \alpha_s)$$
$$+ \int_0^T \nabla Q_{T-s}\theta(X(x_s)\,dB_s)(\alpha_s)$$
$$+ \int_0^T Q_{T-s}\theta((dA)^{q-1}(\nabla X(-)\,dB_s)(\alpha_s)). \qquad (51)$$

We calculate the expectation of each term of $\int_0^t \theta \circ dx_s$ in (51) after wedging with $\int_0^T \langle X(x_s)\,dB_s, TF_s(-)\rangle\,ds$. The first term clearly vanishes. The last term vanishes as well for a gradient h-Brownian system since $\sum_i \nabla X^i(X^i(-)) = 0$.

Take $v_0 = (v_0^1, ..., v_0^q)$. Write $v_s^i = TF_s(v_0^i)$, and denote by $w_s(\cdot)$ the linear map

$$w_s(\cdot) = (\overbrace{TF_s(\cdot), ..., TF_s(\cdot)}^{q-1}).$$

Then

$$\frac{1}{q} E\int_0^T \theta(X(x_s)\,dB_s, w_s(\cdot)) \wedge \int_0^T \langle X(x_s)\,dB_s, TF_s(\cdot)\rangle(v_0)$$
$$= \frac{1}{q}\sum_{i=1}^q (-1)^{q-i} E\int_0^T \theta(v_s^i, v_s^1, ..., \widehat{v_s^i}, ..., v_s^q)\,ds$$
$$= \frac{1}{q}\sum_{i=1}^q (-1)^{q-i}(-1)^{i-1} E\int_0^T \theta(v_s^1, ..., v_s^q)\,ds$$
$$= (-1)^{q-1} E\int_0^T \theta(v_s^1, ..., v_s^q)\,ds$$
$$= (-1)^{q-1}\int_0^T P_s^h\theta(v)\,ds.$$

The last step uses the assumption $\int_0^T E\,|T_xF_s|^{2q}\,ds < \infty$. Similar calculations show

$$E\left\{\int_0^T \nabla Q_{T-s}\theta(X(x_s)\,dB_s)(w_s(\cdot)) \wedge \int_0^T \langle X(x_s)\,dB_s, TF_s(\cdot)\rangle\right\}(v_0)$$

$$= \sum_{i=1}^{q} (-1)^{q-i} E\int_0^T \nabla(Q_{T-s}\theta)(v_s^i)(v_s^1, ..., \widehat{v_s^i}, ..., v_s^q)\,ds$$

$$= (-1)^{q-1} E\int_0^T (d(Q_{T-s}\theta))(v_s^1, ..., v_s^q)\,ds$$

$$= (-1)^{q-1} \int_0^T P_s^h(P_{T-s}^h(\theta) - \theta)(v)\,ds$$

$$= (-1)^{q-1}\left[T(P_T^h(\theta)(v) - \int_0^T P_s^h\theta(v)\,ds\right].$$

Comparing these with (51), we have

$$P_T^{h,q}\theta = \frac{1}{T} E\int_0^T \langle X(x_s)\,dB_s, TF_s(\cdot)\rangle \wedge \int_0^T \theta \circ dx_s. \quad \blacksquare$$

Note. With an additional condition, $\sup_{x\in M} E\,|T_xF_s|^{2q} < \infty$, the formula in the above proposition holds for forms which are not necessarily bounded. See the remark at the end of Section 4.

Recall that $\rho(x)$ is the distance function between x and a fixed point in M, and $\partial h/\partial\rho := dh(\nabla\rho)$.

COROLLARY 6.3. *Consider a gradient h-Brownian system. Formula* (48) *holds for a closed C^2q-form in $D(\Delta^h)$, if one of the following conditions holds.*

1. *The related second fundamental form is bounded and* $\frac{1}{2}\mathrm{Ric} - \mathrm{Hess}(h)$ *is bounded from below.*

2. *The second fundamental form is bounded by* $c[1+\ln(1+\rho(x))]^{1/2}$, *and also*

$$\frac{\partial h}{\partial\rho} \leqslant c[1+\rho(x)],$$

$$\mathrm{Hess}(h)(x)(v,v) \leqslant c[1+\ln(1+\rho(x))]\,|v|^2.$$

Proof. This follows since ([6]) for $v_1, v_2 \in T_xM$ and $e\in R^m$,

$$\langle \nabla X(v_1)e, v_2\rangle_x = \langle \alpha(v_1,v_2), e\rangle_{R^m}.$$

Lemmas A2 and A3 give $E \sup_{s \leq t} |T_x F_s|^{2q} < \infty$ for all q. The second part of Proposition A6 now gives $\delta P_t \theta = P_t^{h,q}\theta$. So the conditions of the theorem are satisfied, with the remark above used to avoid assuming θ is bounded. ∎

We now have the extension of our basic differentiation result to the case of q-forms.

COROLLARY 6.4. *Consider a gradient h-Brownian system on a complete Riemannian manifold. Suppose there is no explosion and* $\int_0^t E\,|T_x F_s|^{2q}\,ds < \infty$. *Let* ϕ *be a* $q-1$ *form such that* $d\phi$ *is a bounded* C^2 *form in* $D(\Delta^{h,q})$ *with* $P_t^{h,q}(d\phi) = \delta P_t(d\phi)$. *Then*

$$d(P_t^{h,q-1}(\phi)) = \frac{1}{t} E\left(\int_0^t \langle X(x_s)\,dB_s, TF_s(\cdot)\rangle \wedge \phi(\overbrace{TF_t(\cdot), ..., TF_t(\cdot)}^{q-1})\right).$$

Proof. By (47), if $\theta = d\phi$,

$$\int_0^t \theta \circ dx_s(-) = \frac{1}{q}\int_0^t d\phi(X(x_s)\,dB_s, \overbrace{TF_s(\cdot), ..., TF_s(\cdot)}^{q-1})(-)$$

$$+ \frac{1}{2}\int_0^t \Delta^h \phi(\overbrace{TF_s(\cdot), ..., TF_s(\cdot)}^{q-1})(-)\,ds. \tag{53}$$

On the other hand, if $\alpha_0 = (\alpha_0^1, ..., \alpha_0^{q-1})$ for $\alpha_0^i \in T_{x_0}M$, then by Itô's formula

$$\phi(TF_t(\alpha_0^1), ..., TF_t(\alpha_0^{q-1}))$$

$$= \phi(\alpha_0) + \int_0^t \nabla\phi(X(x_s)\,dB_s)(TF_s(\alpha_0^1), ..., TF_s(\alpha_0^{q-1}))$$

$$+ \tfrac{1}{2}\int_0^t \Delta^h \phi(TF_s(\alpha_0^1), ..., TF_s(\alpha_0^{q-1}))\,ds. \tag{54}$$

However,

$$E\int_0^t \langle X(x_s)\,dB_s, TF_s(\cdot)\rangle \wedge \int_0^t d\phi(X(x_s)\,dB_s, TF_s(\cdot), ..., TF_s(\cdot))$$

$$= qE\int_0^t \langle X(x_s)\,dB_s, TF_s(\cdot)\rangle \wedge \int_0^t \nabla\phi(X(x_s)\,dB_s)(TF_s(\cdot), ..., TF_s(\cdot)).$$

Compare Eqs. (53) and (54) to obtain

$$E\int_0^t \langle X\,dB_s, TF_s(-)\rangle \wedge \int_0^t d\phi \circ dx_s$$

$$= E\int_0^t \langle X\,dB_s, TF_s(-) \wedge \int_0^t \nabla\phi(X\,dB_s)(\overbrace{TF_s(-), ..., TF_s(-)}^{q-1})$$

$$+ \tfrac{1}{2}\int_0^t \langle X\,dB_s, TF_s(-)\rangle \wedge \int_0^t \Delta^h\phi(\overbrace{TF_t(-), ..., TF_t(-)}^{q-1})\,ds$$

$$= E\left(\int_0^t \langle X\,dB_s, TF_s(-)\rangle \wedge \phi(\overbrace{TF_t(-), ..., TF_t(-)}^{q-1})\right).$$

This gives the required result by the formula for $P_t^{h,q}(d\phi)$ in the previous theorem. ∎

Remarks. (i) This can be proved directly as for the case $q=0$ in Theorem 2.1.

(ii) Equation (52) can be given the following interpretation: Our stochastic differential equation determines a 1-form valued process $\Psi_t = \Psi_t^{X,A}$, $t \geqslant 0$, given by

$$\Psi_{t,x_0}(v_0) = \int_0^t \langle X(x_s)\,dB_s, T_{x_0}F_s(v_0)\rangle,$$

i.e.,

$$\Psi_{t,x_0} = \int_0^t (T_{x_0}F_s)^* (\langle X(x_s)\,dB_s, -\rangle_{x_s}).$$

(So for each x_0, $\{\Psi_{t,x_0} : t \geqslant 0\}$ determines a local martingale on $T_{x_0}^* M$ with tensor quadratic variation given by $\int_0^t T.F_s^* T.F_s\,ds$. Note that the Malliavin covariance matrix is given by $\int_0^t (T.F_s^* T.F_s)^{-1}\,ds$.) In fact, Ψ_t is exact: $\Psi_t = d\psi_t$ where $\psi_t : M \times \Omega \to R$ is given by

$$\psi_t(x) = \int_0^t \langle f(F_s(x)), dB_s\rangle_{R^m}$$

for $f: M \to R^m$, the given embedding.

Equation (52) states

$$dP_t^{h,q-1}\phi = \frac{1}{t} E\{\Psi_t \wedge (F_t)^* \phi\}$$

$$= \frac{1}{t} E\{d\psi_t \wedge (F_t)^* \phi\}.$$

(iii) Note that (50) gives an explicit cohomology between $P_t^{h,q}\theta$ and θ.

APPENDIX: Differentiation under the Expectation

Consider the stochastic differential equation:

$$dx_t = X(x_t) \circ dB_t + A(x_t)\, dt \tag{55}$$

on a complete n-dimensional Riemannian manifold. We need the following result on the existence of a partial flow taken from [8], following Kunita.

THEOREM A1. *Suppose X and A are in C^r, for $r \geqslant 2$. Then there is a partially defined flow $(F_t(\cdot), \xi(\cdot))$ such that for each $x \in M$, $(F_t(x), \xi(x))$ is a maximal solution to (55) with lifetime $\xi(x)$, and if*

$$M_t(\omega) = \{x \in M,\ t < \xi(x, \omega)\},$$

then there is a set Ω_0 of full measure such that for all $\omega \in \Omega_0$:

1. *$M_t(\omega)$ is open in M for each $t > 0$, i.e., $\xi(\cdot, \omega)$ is lower semicontinuous.*

2. *$F_t(\cdot, \omega)$: $M_t(\omega) \to M$ is in C^{r-1} and is a diffeomorphism onto an open subset of M. Moreover, the map $: t \mapsto F_t(\cdot, \omega)$ is continuous into $C^{r-1}(M_t(\omega))$, with the topology of uniform convergence on compacta of the first $r-1$ derivatives.*

3. *Let K be a compact set and $\xi^K = \inf_{x \in K} \xi(x)$. Then*

$$\lim_{t \nearrow \xi^K(\omega)} \sup_{x \in K} d(x_0, F_t(x)) = \infty \tag{56}$$

almost surely on the set $\{\xi^K < \infty\}$. (Here x_0 is a fixed point of M and d is any complete metric on M.)

From now on, we use (F_t, ξ) for the partial flow defined in theorem A1 unless otherwise stated.

Recall that a stochastic differential equation is called *strongly p-complete* it its solution can be chosen to be jointly continuous in time and space for all time when restricted to a smooth singular p-simplex. A singular p-simplex is a map σ from a standard p-simplex to M. We also use the term "singular p-simplex" for its image. If a s.d.e. is strongly p-complete, $\xi^K = \infty$ almost surely for each smooth singular p-simplex K [12].

Let $x \in M$ and $v \in T_x M$. Define H_p as follows:

$$H_p(x)(v, v) = 2\langle \nabla A(x)(v), v\rangle + \sum_1^m \langle \nabla^2 X^i(X^i, v), v\rangle + \sum_1^m |\nabla X^i(v)|^2 + \sum_1^m \langle \nabla X^i(\nabla X^i(v)), v\rangle + (p-2)\sum_1^m \frac{1}{|v|^2}\langle \nabla X^i(v), v\rangle^2.$$

There are simplifications of H_p:

For s.d.e. (3) on R^n,

$$H_p(x)(v, v) = 2\langle DZ(x)(v), v\rangle + \sum_1^m |DX^i(v)|^2 + (p-2)\sum_1^m \frac{\langle DX^i(v), v\rangle^2}{|v|^2}.$$

For (1) with generator $\frac{1}{2}\Delta + L_Z$,

$$H_p(x)(v, v) = -\mathrm{Ric}_x(v, v) + 2\langle \nabla Z(x)(v), c\rangle + \sum_1^m |\nabla X^i(v)|^2 + (p-2)\sum_1^m \frac{1}{|v|^2}\langle \nabla X^i(v), v\rangle^2.$$

There are the following lemmas from [12].

LEMMA A2. *Assume the stochastic differential equation* (1) *is complete. Then*

(i) *It is strongly* 1*-complete if* $H_1(v, v) \leqslant c\,|v|^2$ *for some constant* c. *Furthermore, if also* $|\nabla X|$ *is bounded, then it is strongly complete and* $\sup_x E(\sup_{s\leqslant t} |T_x F_s|^p)$ *is finite for all* $p > 0$ *and* $t > 0$.

(ii) *Suppose* $H_p(v, v) \leqslant c\,|v|^2$, *then* $\sup_{x\in M} E\,|T_x F_t|^p \leqslant k e^{cp(t/2)}$ *for* $t > 0$. *Here* k *is a constant independent of* p.

For a more refined result, let c and c_1 be two constants, let $\rho(x)$ be the distance between x and a fixed point p of M, and assume $\mathscr{A} = \frac{1}{2}\Delta + Z$.

LEMMA A3 [12]. *Assume that the Ricci curvature at each point x of M is bounded from below by $-c(1+\rho^2(x))$. Suppose $dr(Z(x)) \leqslant c[1+\rho(x)]$, then there is no explosion. If furthermore $|\nabla X(x)|^2 \leqslant c[1+\ln(1+\rho(x))]$, and*

$$\mathrm{Ric}_x(v, v) - 2\langle \nabla Z(x)(v), v\rangle \geqslant c_1[1+\ln(1+\rho(x))]\,|v|^2,$$

then the system is strongly complete and

$$\sup_{x\in K} E(\sup_{s\leqslant t} |T_x F_s|^p) < k_1 e^{k_2 t}$$

for all compact sets K. Here k_1 and k_2 are constants independent of t.

We first use strong 1-completeness to differentiate under expectations in the sense of distribution. For this furnish M with a complete Riemannian metric and let dx denote the corresponding volume measure of M. Let A be a smooth vector field on M. For $f\in L^1_{\mathrm{loc}}(M, R)$, the space of locally integrable functions on M, we say that $g\in L^1_{\mathrm{loc}}(M, R)$ is the *weak Lie derivative of f* in the direction A and write

$$g = Ł_A f, \qquad \text{weakly},$$

if for all $\phi: M\to R$ in C_K^∞, the space of smooth functions with compact support, we have

$$\int_M \phi(x)\, g(x)\, dx = -\int_M f(x)[\langle \nabla\phi(x), A(x)\rangle_x + \phi(x)\,\mathrm{div}\, A(x)]\, dx.$$

A locally integrable 1-form ψ on M is the *weak derivative* of f

$$df = \psi \qquad \text{weakly}$$

if $\psi(A(\cdot)) = Ł_A f$ weakly for all C_K^∞ vector fields A on M.

Let A be a C_K^∞ vector field on M and for each x in M let $K(x)$ be the integral curve of A through x.

LEMMA A4. *Suppose the s.d.e.* (55) *is complete. Then for $t\geqslant 0$,*

(i) *with probability one $M_t(\omega) = \{x: t < \xi(x, \omega)\}$ has full measure in M. In particular, $f\circ F_t(-, \omega)$ determines an element of $L^1_{\mathrm{loc}}(M, R)$ with probability one for each bounded measurable $f: M\to R$.*

If also (55) *is strongly* 1*-complete and f is BC^1 then with probability* 1:

(ii) *$t < \xi^K(\omega)$ for any compact subset K of $K(x)$ for almost all x in M; and*

(iii) *the Lie derivative* $\text{Ł}_A(f \circ F_t(-, \omega))$ *exists almost everywhere on* M *in the classical sense, is equal to the Lie derivative in the weak sense almost everywhere, and*

$$\text{Ł}_A(f \circ F_t(-, \omega)) = df \circ TF_t(-, \omega)(A(-)) \qquad \textit{weakly}.$$

Proof. Completeness of (55) implies that $\xi(x, \omega) = \infty$ with probability 1 for each x in M so that $\{(x, \omega) \in M \times \Omega : t < \xi(x, \omega)\}$ has full $\lambda \otimes P$ measure. Fubini's theorem gives (i). The same argument applied to $\{(x, \omega) \in M \times \Omega : t < \xi^{K(x)}(\omega)\}$ yields (ii).

From (ii) we know that if $f \in BC^1$, then $f \circ F_t(-, \omega)$ is C^1 on almost all $\{K(x) : x \in M\}$ with probability one. In particular, it is absolutely continuous along the trajectories of A through almost all points of M with probability one. It follows e.g. by Schwartz [18, Chap. 2, Sect. 5] that $\text{Ł}_A(f \circ F_t(-, \omega))$ exists almost everywhere. However, at each point x of $M_t(\omega)$ this classical derivative is just $df \circ T_x F_t(-, \omega)(A(x))$, which is in L^1_{loc}. By [18] it is therefore equal to the weak Lie derivative almost everywhere, with probability 1. ∎

THEOREM A5. *Suppose the stochastic differential equation* (55) *is strongly* 1-*complete and* $E\,|T_x F_t| \in L^1_{\text{loc}}$ *in* x. *Then for* f *in* BC^1, $P_t f$ *has weak derivative given by*

$$d(P_t f) = \delta P_t(df) \qquad \textit{weakly}. \tag{57}$$

In particular, this holds if $H_1(v, v) \leqslant c\,|v|^2$.

Proof. Let A be a C_K^∞ vector field on M. Then by Lemma A4 and Fubini's theorem,

$$\begin{aligned}
\int_M P_t f(x) \operatorname{div} A(x)\, dx &= \int_M Ef(F_t(x)) \operatorname{div} A(x)\, dx \\
&= E \int_M f \circ F_t(x, \omega) \operatorname{div} A(x)\, dx \\
&= -E \int_M \text{Ł}_A(f \circ F_t(-, \omega))(x)\, dx \\
&= -E \int_M df \circ T_x F_t(-, \omega)(A(x))\, dx \\
&= -\int_M \delta P_t(df)(A(x))\, dx
\end{aligned}$$

as required. The last part comes from Lemma A2. ∎

Remark. Under the conditions of the theorem it follows as in [18] that the derivatives $Ł_A(P_t f)$ exist in the classical sense a.e. for each smooth vector field A and are given by $\delta P_t(df)(A(\cdot))$.

If also the stochastic differential equation (55) is non-degenerate (so that its generator is elliptic) and $x \to E|T_x F_t|$ is continuous on each compact set, then, by parabolic regularity and a direct proof in [12], Eq. (57) holds in the classical sense at all points of x.

In the elliptic case there are the following criteria.

PROPOSITION A6 ([5, 13]). *For a complete h-Brownian system on a complete Riemannian manifold*:

(i) *suppose* $E\sup_{s \leqslant t}|T_x F_s| < \infty$ *for all* $x \in M$ *and* $t > 0$, *then for every bounded* C^2, *closed 1-form* ϕ_0, $\delta P_t(\phi_0)$ *is the unique solution to the heat equation* $(\partial\phi_t/\partial t) = \frac{1}{2}\Delta^{h,1}\phi_t$ *with initial condition* ϕ_0. *If* $\phi = df$, *this gives* $dP_t f(x) = \delta P_t(df)(x)$ *for all x and for all bounded* C^3 *functions with bounded first derivatives.*

(ii) *If the system considered is a gradient system, then*

$$\delta P_t \psi = e^{(1/2)t\Delta^h}\psi, \tag{58}$$

for all bounded C^2 *q-forms* ψ, *provided that* $E(\sup_{s \leqslant t}|TF_s|^q)$ *is finite for each* $t > 0$.

In particular, these hold if $|\nabla X|$ *is bounded and* H_1 *is bounded above.*

However, the following often has advantage when $P_t f$ is known to be BC^1.

PROPOSITION A7. *Assume* $\mathscr{A} = \frac{1}{2}\Delta + L_Z$. *Let M be a complete Riemannian manifold with Ricci curvature bounded from below by* $-c(1+\rho^2(x))$. *Suppose* $d\rho(Z(x)) \leqslant c[1+\rho(x)]$ *and* $H_{1+\delta}(x)(v,v) \leqslant c\ln[1+\rho(x)]\,|v|^2$ *for all x and v. Here c and* $\delta > 0$ *are constants. Then*

$$dP_t f = \delta P_t(df)$$

for all f in C_K^∞ *provided* $d(P_t f)$ *is bounded uniformly in each* $[0, T]$.

Proof. Let ϕ_0 be a bounded C^2 1-form. We show that a solution $P_t\phi$ to $\partial\phi_t/\partial t = \Delta^1\phi_t + L_Z\phi_t$ starting from ϕ_0 and bounded on $[0, T] \times M$ is given by $E\phi_0(v_t)$ and then note that $d(P_t f) = P_t(df)$ for smooth functions to finish the proof. Let $\tau_n(x_0)$ be the first exit time of $F_t(x_0)$ from the ball

$B(n)$ radius n, centred at p. Since $P_t\phi$ is smooth, we apply Itô's formula to obtain

$$P_{T-t}\phi(v_t) = P_T\phi(v_0) + \int_0^t \nabla P_{T-s}\phi(X\,dB_s) + \int_0^t P_{T-s}\phi(\nabla X(v_s)\,dB_s).$$

Replace t by $t \wedge \tau^n$ in the above inequality to obtain

$$P_{T-t\wedge\tau^n}\phi(v_{t\wedge\tau^n}) = P_T\phi(v_0) + \int_0^{t\wedge\tau^n} \nabla P_{T-s}\phi(X\,dB_s)$$
$$+ \int_0^{t\wedge\tau^n} P_{T-s}\phi(\nabla X(v_s)\,dB_s).$$

This gives

$$E\phi(v_T)\,\chi_{T\leqslant\tau^n} + EP_{T-\tau^n}\phi(v_{\tau^n})\,\chi_{\tau^n<T} = P_T\phi(v_0). \tag{59}$$

But under the condition $H_{1+\delta}(x) \leqslant c\ln[+\rho(x)]$,

$$E\,|v_{\tau^n}|^{1+\delta}\,\chi_{\tau^n<T} \leqslant e^{C_nT/2}. \tag{60}$$

Here $C_n = \sup_{x\in B(n)} \sup_{|v|\leqslant 1} H_{1+\delta}(x)(v,v) \leqslant c\ln(1+n)$. See [12] for the details. On the other hand [12], there is a constant $k_0>0$ such that for each $\beta>0$,

$$P(\tau^n(x)<T) \leqslant \frac{1}{n^\beta}\,[1+\rho(x)]^\beta\, e^{k_0[1+\beta^2]T}. \tag{61}$$

Take numbers $\delta'>0$, and $p>1$, $q>1$ such that $1/p+1/q=1$ and $p(1+\delta')=1+\delta$. Then

$$\sup_n E\,|P_{T-\tau^n}\phi(v_{\tau^n})\,X_{\tau^n<T}|^{1+\delta'}$$
$$\leqslant k\sup_n\,[E\,|v_{\tau^n}\chi_{\tau^n<T}|^{p(1+\delta')}]^{1/p}\,[P(\tau^n<T)]^{1/q}.$$

Here k is a constant. We have used the assumption that $P_t\phi$ is uniformly bounded on $[0, T]$.

By choosing β sufficiently big, we see, from (60) and (61), that the right-hand side of the inequality is finite. Thus $|P_{T-\tau^n}\phi(v_{\tau^n})\,\chi_{\tau^n<T}|$ is uniformly integrable. Passing to the limit $n\to\infty$ in (59), we have shown

$$E\phi(v_T) = P_T\,\phi(v_0). \quad \blacksquare$$

There are also parallel results for higher order forms.

PROPOSITION A8. *Suppose the s.d.e.* (55) *is strongly* 1*-complete and* T_xF_t *is also strongly* 1*-complete. Let* $f \in BC^2$, *then*

$$\nabla d(P_t f)(u, v) = E\nabla(df)(T_xF_t(u), T_xF_t(v)) + E\, df(\nabla(TF_t)(u, v)) \tag{62}$$

for all $u, v \in T_xM$, *if for each* $t > 0$ *and compact set* K *there is a constant* $\delta > 0$ *such that*

$$\sup_{x \in K} E\, |T_xF_t|^{2+\delta} < \infty \tag{63}$$

and

$$\sup_{x \in K} E\, |\nabla T_xF_t|^{1+\delta} < \infty. \tag{64}$$

In particular, (62) *holds if the first three derivatives of* X *and the first two derivatives of* A *are bounded.*

Proof. First, $dP_t f = \delta P_t(df)$ from a result in [12]. Let $u, v \in T_xM$. Take a smooth map $\sigma_1\colon [0, s_0] \to M$ such that $\dot\sigma_1(0) = u$. Let $v(s) \in T_{\sigma_1(s)}M$ be the parallel translate of v along σ_1. Suppose its image is contained in a compact set K. Then $df_{F_t(\sigma_1(s))}(T_{\sigma_1(s)}F_t(v(s)))$ is a.s. differentiable in s for each $t > 0$. So for almost all ω,

$$\begin{aligned} I_s &= \frac{df_{F_t(\sigma_1(s))}(T_{\sigma_1(s)}F_t(v(s))) - df(T_xF_t(v))}{s} \\ &= \frac{1}{s}\int_0^s \frac{\partial}{\partial r}\,[df(T_{(\sigma_1(r))}F_t(v(r)))]\, dr \\ &= \frac{1}{s}\int_0^s \nabla\, df(TF_t(\dot\sigma_1(r)), TF_t(v(r)))\, dr + \frac{1}{s}\int_0^s df(\nabla TF_t(\dot\sigma_1(r), v(r)))\, dr. \end{aligned}$$

But the integrand of the right-hand side is continuous in r in L_1, so $E \lim_{s \to 0} I_s = \lim_{s \to 0} EI_s$. Thus

$$\begin{aligned} \nabla d(P_t f)(u, v) &= \nabla(\delta P_t(df))(u, v) \\ &= E\nabla(df)(TF_t(u), TF_t(v)) + E\, df(\nabla(TF_t)(u, v)). \end{aligned}$$

For the last part observe that if the s.d.e. is strongly 2-complete then TF_t is strongly 1-complete, and apply Lemma A2. ∎

For elliptic systems, we may use the previous weak derivatives argument. Just note that for two C_K^∞ vector fields A_1 and A_2,

$$L_{A_2}Ł_{A_1}(P_t f)(x) = \nabla^2 P_t f(x)(A_2(x), A_1(x)) + \langle \nabla P_t f(x), \nabla A_1(A_2(x)) \rangle_x$$

and

$$Ł_{A_2}\, df \circ T_x F_t(A_1(x)) = \nabla\, df(T_x F_t(A_2), T_x F_t(A_1) + df \circ \nabla T_x F_t(A_2, A_1) + df \circ T_x F_t(\nabla A_1(A_2(x))).$$

In this case the number δ in the assumption can be taken to be zero, but the required equality (62) holds only almost surely. However, this is usually enough for our purposes.

Note added in proof. Professor G. Da Prato has pointed out to us that for the case of a generalized $\mathscr{A}_t$ with a zero order term V_t, as considered in Section 2.2, use of the formula

$$u_t(x) = P_0^t u_0(x) + \int_0^t P_s^t(V_s(\cdot)\, u_s(\cdot))(x)\, ds$$

for the solution to (10) in terms of the evolution $\{P_s^t : 0 \leqslant s \leqslant t\}$ for the generator with the zero order term removed, together with our formula (11) for $d(P_s^t(V_s u_s)$, gives an alternative formula to (11) which does not involve derivatives of V_t. It therefore demonstrates the smoothing property of (10) even when V is not differentiable, and gives estimates in terms of L^∞ norms of $\{V_s : 0 \leqslant s \leqslant t\}$. There is a corresponding alternative to (45).

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