

On extensions of Myers' theorem

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Abstract

Let M be a compact Riemannian manifold and h a smooth function on M . Let $\rho^h(x) = \inf_{|v|=1} (Ric_x(v, v) - 2Hess(h)_x(v, v))$. Here Ric_x denotes the Ricci curvature at x and $Hess(h)$ is the Hessian of h . Then M has finite fundamental group if $\Delta^h - \rho^h < 0$. Here $\Delta^h =: \Delta + 2L_{\nabla h}$ is the Bismut-Witten Laplacian. This leads to a quick proof of recent results on extension of Myers' theorem to manifolds with mostly positive curvature. There is also a similar result for noncompact manifolds.

An early result of Myers says a complete Riemannian manifold with Ricci curvature bounded below by a positive number is compact and has finite fundamental group. See e.g. [9]. Since then efforts have been made to get the same type of result but to allow a little bit of negativity of the curvature (see Bérard and Besson[2]). Wu [12] showed that Myers' theorem holds if the manifold is allowed to have negative curvature on a set of small diameter, while Elworthy and Rosenberg [8] considered manifolds with some negative curvature on a set of small volume, followed by recent work of Rosenberg and Yang [10]. We use a method of Bakry [1] to obtain a result given in terms of the potential kernel related to $\rho(x) = \inf_{|v|=1} Ric_x(v, v)$, which gives a quick probabilistic proof of recent results on extensions of Myers' theorem. Here Ric_x denotes the Ricci curvature at x .

Let M be a complete Riemannian manifold, and h a smooth real-valued function on it. Assume $Ric - 2Hess(h)$ is bounded from below, where $Hess(h)$

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is the hessian of h . Denote by Δ^h the Bismut-Witten Laplacian (with probabilistic sign convention) defined by: $\Delta^h = \Delta + 2L_{\nabla h}$ on C_K^∞ the space of smooth differential forms with compact support. Here $L_{\nabla h}$ is the Lie derivative in direction of ∇h . Then the closure of Δ^h is a negative-definite self-adjoint differential operator on L^2 functions (or L^2 differential forms) with respect to $e^{2h}dx$ for dx the standard Lebesgue measure on M . We shall use the same notation for Δ^h and its closure. By the spectral theorem there is a heat semigroup P_t^h satisfying the following heat equation:

$$\frac{\partial u(x, t)}{\partial t} = \frac{1}{2} \Delta^h u(x, t).$$

We shall denote by $P_t^h \phi$ the solution with initial value ϕ . For clarity, we also use $P_t^{h,1}$ for the corresponding heat semigroup for one forms. Then for a function f in C_K^∞ ,

$$dP_t^h f = P_t^{h,1}(df). \quad (1)$$

On M there is a h-Brownian motion $\{F_t(x) : t \geq 0\}$, i.e. a path continuous strong Markov process with generator $\frac{1}{2}\Delta^h$ for each starting point x . For a fixed point $x_0 \in M$, we shall write $x_t = F_t(x_0)$. Then $P_t^h f(x) = Ef(F_t(x))$ for all bounded L^2 functions.

Let $\{W_t^h(-), t \geq 0\}$ be the solution flow to the following covariant equation along h-Brownian paths $\{x_t\}$:

$$\begin{cases} \frac{D}{dt} W_t^h(v_0) &= -\frac{1}{2} Ric_{x_t} (W_t^h(v_0), -)^\# + Hess(h)_{x_t} (W_t^h(v_0), -)^\#, \\ W_0^h(v_0) &= v_0, \quad v_0 \in T_{x_0}M. \end{cases} \quad (2)$$

Here $\#$ stands for the adjoint. The solution flow W_t^h is called the Hessian flow.

Let ϕ be a bounded 1-form, then for $x_0 \in M$, and $v_0 \in T_{x_0}M$

$$E\phi(W_t^h(v_0)) = P_t^{h,1}\phi(v_0). \quad (3)$$

if Ricci-2 Hess(h) is bounded from below. See e.g. [4] and [5].

Formula (3) gives the following estimates on the heat semigroup:

$$|P_t^{h,1}\phi| \leq |\phi|_\infty E|W_t^h|. \quad (4)$$

Let $\rho^h(x) = \inf_{|v|=1} (Ric_x(v, v) - 2Hess(h)(x)(v, v))$ and write ρ for ρ^h if $h = 0$. Then covariant equation (2) gives:

$$E|W_t^h| \leq Ee^{-\frac{1}{2} \int_0^t \rho^h(x_s) ds} \quad (5)$$

as in [4].

Let $P_t^{\rho^h}$ be the L^2 semigroup generated by the Schrödinger operator $\frac{1}{2}\Delta^h - \frac{1}{2}\rho^h$. Then

$$P_t^{\rho^h} f(x_0) = E \left[f(x_t) e^{-\frac{1}{2} \int_0^t \rho^h(x_s) ds} \right]$$

by the Feynman-Kac formula. So equation (5) is equivalent to

$$E|W_t^h| \leq P_t^{\rho^h} 1.$$

Let Uf be the corresponding potential kernel defined by:

$$Uf(x_0) = \int_0^\infty E \left[f(x_t) e^{-\frac{1}{2} \int_0^t \rho^h(x_s) ds} \right] dt.$$

Following Bakry's paper [1], we have the following theorem:

Theorem 1 *Let M be a complete Riemannian manifold with $Ric - 2Hess(h)$ bounded from below. Suppose*

$$\sup_{x \in K} U1(x) < \infty \quad (6)$$

for each compact set K . Then M has finite h -volume (i.e. $\int_M e^{2h(x)} dx < \infty$), and finite fundamental group.

Proof: We follow [1]. Let $f \in C_K^\infty$, then $Hf = \lim_{t \rightarrow \infty} P_t f$ is an L^2 harmonic function. Assume $h\text{-vol}(M) = \infty$, then $Hf = 0$. We shall prove this is impossible. Let $f, g \in C_K^\infty$, then:

$$\begin{aligned} & \int_M (P_t^h f - f) g e^{2h} dx \\ &= \int_M \int_0^t \left(\frac{\partial}{\partial s} P_s^h f \right) g e^{2h} dx \\ &= \int_0^t \int_M \langle \nabla P_s^h f, \nabla g \rangle e^{2h} dx ds \end{aligned}$$

$$\begin{aligned}
&\leq |\nabla f|_\infty \int_M |\nabla g| \left(\int_0^t E|W_s^h| ds \right) e^{2h} dx, \\
&\leq |\nabla f|_\infty \left(\sup_{x \in \text{sup}(g)} \int_0^\infty E|W_s^h| ds \right) |\nabla g|_{L^1} \\
&\leq c |\nabla f|_\infty |\nabla g|_{L^1}.
\end{aligned}$$

Here $c = \sup_{x \in \text{sup}(g)} [U1(x)] = \sup_{x \in \text{sup}(g)} \left(\int_0^\infty E \left(e^{-\int_0^t \rho^h(F_s(x)) ds} \right) dt \right)$, and $\text{sup}(g)$ denotes the support of g .

Next take $f = h_n$, for h_n an increasing sequence of smooth functions approximating 1 with $0 \leq h_n \leq 1$ and $|\nabla h_n| \leq \frac{1}{n}$, see e.g. [1].

Then

$$\int_M (P_t^h h_n - h_n) g e^{2h} dx \leq c \frac{1}{n} |\nabla g|_{L^1}.$$

First let t go to infinity, then let $n \rightarrow \infty$ to obtain:

$$-\int_M g e^{2h} dx \leq 0.$$

This gives a contradiction with a suitable choice of g . So we conclude $\text{h-vol}(M) < \infty$.

Let $p: \tilde{M} \rightarrow M$ be the universal covering space for M with induced Riemannian metric on \tilde{M} . For $p(\tilde{x}) = x$, let $\{\tilde{F}_t(\tilde{x}), t \geq 0\}$ be the horizontal lift of $\{F_t(x)\}$ to \tilde{M} . Denote by \tilde{Ric} the Ricci curvature on \tilde{M} , \tilde{h} the lift of h to \tilde{M} , and $\tilde{\rho}^h$ the corresponding lower bound for $\tilde{Ric} - 2\text{Hess}(\tilde{h})$. Then the induced $\{\tilde{F}_t(\tilde{x}), t \geq 0\}$ is a h-Brownian motion on \tilde{M} . See e.g. [4]. Note also $\tilde{\rho}^h$ satisfies

$$\sup_{\tilde{x} \in \tilde{K}} \int_0^\infty E \left(e^{-\frac{1}{2} \int_0^t \tilde{\rho}^h(\tilde{F}_s(\tilde{x})) ds} \right) dt = \sup_{x \in p(\tilde{K})} \int_0^\infty E \left(e^{-\frac{1}{2} \int_0^t \rho^h(F_s(x)) ds} \right) dt$$

for any $\tilde{K} \subset \tilde{M}$ compact. The same calculation as above will show that \tilde{M} has finite h-volume, therefore p is a finite covering and so M has finite fundamental group. ■

Let $r(x)$ be the Riemannian distance between x and a fixed point of M and take h to be identically zero:

Corollary 2 *Let M be a complete Riemannian manifold with*

$$\text{Ric}_x > -\frac{n}{n-1} \frac{1}{r^2(x)}, \quad \text{when } r > r_0 \tag{7}$$

for some $r_0 > 0$. Then the manifold is compact if $\sup_{x \in K} U_1(x) < \infty$ for each compact set K .

Proof: This is a consequence of the result of [3]: A complete Riemannian manifold with (7) has infinite volume. See also [11]. ■

For another extension of Myers' compactness theorem, see [3] where a diameter estimate is also obtained.

In the following we shall assume M is compact and get the following corollary:

Corollary 3 *Let M be a compact Riemannian manifold and h a smooth function on it. Then M has finite fundamental group if $\Delta^h - \rho^h < 0$.*

Proof: Let λ_0 be the minimal eigenvalue of $\Delta^h - \rho^h$. Then

$$\overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \sup_M \log E \left(e^{-\frac{1}{2} \int_0^t \rho^h(x_s) ds} \right) \leq \lambda_0 < 0.$$

See e.g. [7]. Thus there is a number T_0 such that if $t \geq T_0$,

$$\sup_M E \left(e^{-\frac{1}{2} \int_0^t \rho^h(x_s) ds} \right) \leq e^{\lambda_0 t}.$$

Therefore

$$\begin{aligned} & \sup_M \int_0^\infty E \left(e^{-\frac{1}{2} \int_0^t \rho^h(x_s) ds} \right) dt \\ & \leq \int_0^{T_0} E \left(e^{-\frac{1}{2} \inf_{x \in M} [\rho^h(x)] t} \right) dt + \int_{T_0}^\infty E \left(e^{-\frac{1}{2} \int_0^t \rho^h(x_s) ds} \right) dt \\ & < \infty. \end{aligned}$$

The result follows from theorem 1. ■

Let $d_h = e^h de^{-h}$. It has adjoint $\delta_h = e^{-h} \delta e^h$ on $L^2(M, dx)$. Let \square_h be the Witten Laplacian defined by:

$$\square_h = -(d_h + \delta_h)^2.$$

By conjugacy of \square_h on $L^2(M, dx)$ with Δ^h on $L^2(M, e^{2h} dx)$, the condition " $\Delta^h - \rho^h < 0$ " becomes:

$$\square_h - \rho^h < 0$$

on $L^2(M, dx)$. On the other hand,

$$\square_h = \Delta - \|dh\|^2 - \Delta h.$$

See e.g. [6]. This gives: a compact manifold has finite fundamental group if

$$\Delta - \|dh\|^2 - \Delta h - \rho^h < 0$$

on $L^2(M, dx)$.

Corollary 3 leads to the following theorem from [10]: Let $\mathcal{N} = \mathcal{N}(K, D, V, n)$ be the collection of n -dimensional Riemannian manifolds with Ricci curvature bounded below by K , diameter bounded above by D , and volume bounded below by V .

Corollary 4 (Rosenberg& Yang) *Choose $R_0 > 0$. There exists $a = a(\mathcal{N}, R_0)$ such that a manifold $M \in \mathcal{N}$ with $\text{vol}\{x : \rho(x) < R_0\} < a$ has finite fundamental group. Here "vol" denotes the volume of the relevant set.*

Proof: Let $h = 0$ in corollary 3. Then under the assumptions in the corollary, $\Delta - \rho < 0$ according to [8]. The conclusion follows from corollary 3. ■

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