

## Equivariant Diffusions on Principal Bundles

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Let  $\pi : P \rightarrow M$  be a smooth principal bundle with structure group  $G$ . This means that there is a  $C^\infty$  right multiplication  $P \times G \rightarrow P$ ,  $u \mapsto u \cdot g$  say, of the Lie group  $G$  such that  $\pi$  identifies the space of orbits of  $G$  with the manifold  $M$  and  $\pi$  is locally trivial in the sense that each point of  $M$  has an open neighbourhood  $U$  with a diffeomorphism

$$\begin{array}{ccc}
 \tau_U : \pi^{-1}(U) & \xrightarrow{\quad} & U \times G \\
 & \searrow & \swarrow \\
 & U & 
 \end{array}$$

over  $U$ , which is equivariant with respect to the right action of  $G$ , i.e. if  $\tau_u(b) = (\pi(b), k)$  then  $\tau_u(b \cdot g) = (\pi(b), kg)$ . Assume for simplicity that  $M$  is compact. Set  $n = \dim M$ . The fibres,  $\pi^{-1}(x)$ ,  $x \in M$  are diffeomorphic to  $G$  and their tangent spaces  $VT_uP (= \ker T_u\pi)$ ,  $u \in P$ , are the ‘vertical’ tangent spaces to  $P$ . A *connection* on  $P$ , (or on  $\pi$ ) assigns a complementary ‘horizontal’ subspace  $HT_uP$  to  $VT_uP$  in  $T_uP$  for each  $u$ , giving a smooth horizontal subbundle  $HTP$  of the tangent bundle  $TP$  to  $P$ . Given such a connection it is a classical result that for any  $C^1$  curve:  $\sigma : [0, T] \rightarrow M$  and  $u_0 \in \pi^{-1}(\sigma(0))$  there is a unique horizontal  $\tilde{\sigma} : [0, T] \rightarrow P$  which is a lift of  $\sigma$ , i.e.  $\pi(\tilde{\sigma}(t)) = \sigma(t)$  and has  $\tilde{\sigma}(0) = u_0$ .

In his startling ICM article [8] Itô showed how this construction could be extended to give horizontal lifts of the sample paths of diffusion processes. In fact he was particularly concerned with the case when  $M$  is given a Riemannian metric  $\langle \cdot, \cdot \rangle_x$ ,  $x \in M$ , the diffusion is Brownian motion on  $M$ , and  $P$  is the orthonormal frame bundle  $\pi : OM \rightarrow M$ . Recall that each  $u \in OM$  with  $u \in \pi^{-1}(x)$  can be considered as an isometry  $u : \mathbb{R}^n \rightarrow T_xM$ ,  $\langle \cdot, \cdot \rangle_x$  and a

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horizontal lift  $\tilde{\sigma}$  determines parallel translation of tangent vectors along  $\sigma$

$$\begin{aligned} //_t \equiv //(\sigma)_t : \quad & T_{\sigma(\cdot)}M \rightarrow T_{\sigma(t)}M \\ & v \mapsto \tilde{\sigma}(t)(\tilde{\sigma}(0))^{-1}v. \end{aligned}$$

The resulting parallel translation along Brownian paths extends also to parallel translation of forms and elements of  $\wedge^p TM$ . This enabled Itô to use his construction to obtain a semi-group acting on differential forms

$$P_t \phi = \mathbb{E}(//_t^{-1})_*(\phi) = \mathbb{E}\phi(//_t-).$$

As he pointed out this is not the semi-group generated by the Hodge-Kodaira Laplacian,  $\Delta$ . To obtain that generated by the Hodge-Kodaira Laplacian,  $\Delta$ , some modification had to be made since the latter contains zero order terms, the so called Weitzenbock curvature terms. The resulting probabilistic expression for the heat semi-groups on forms has played a major role in subsequent development.

In [5] we go in the opposite direction starting with a diffusion with smooth generator  $\mathcal{B}$  on  $P$ , which is  $G$ -invariant and so projects to a diffusion generator  $\mathcal{A}$  on  $M$ . We assume the symbol  $\sigma_{\mathcal{A}}$  has constant rank so determining a subbundle  $E$  of  $TM$ , (so  $E = TM$  if  $\mathcal{A}$  is elliptic). We show that this set-up induces a ‘semi-connection’ on  $P$  over  $E$  (a connection if  $E = TM$ ) with respect to which  $\mathcal{B}$  can be decomposed into a horizontal component  $\mathcal{A}^H$  and a vertical part  $\mathcal{B}^V$ . Moreover any vertical diffusion operator such as  $\mathcal{B}^V$  induces only zero order operators on sections of associated vector bundles.

There are two particularly interesting examples. The first when  $\pi : GLM \rightarrow M$  is the full linear frame bundle and we are given a stochastic flow  $\{\xi_t : 0 \leq t < \infty\}$  on  $M$ , generator  $\mathcal{A}$ , inducing the diffusion  $\{u_t : 0 \leq t < \infty\}$  on GLM by

$$u_t = T\xi_t(u_0).$$

Here we can determine the connection on GLM in terms of the LeJan-Watanabe connection of the flow [12], [1], as defined in [6], [7], in particular giving conditions when it is a Levi-Civita connection. The zero order operators arising from the vertical components, can be identified with generalized Weitzenbock curvature terms.

The second example slightly extends the above framework by letting  $\pi : P \rightarrow M$  be the evaluation map on the diffeomorphism group  $\text{Diff}M$  of  $M$  given by  $\pi(h) := h(x_0)$  for a fixed point  $x_0$  in  $M$ . The group  $G$  corresponds to the group of diffeomorphisms fixing  $x_0$ . Again we take a flow  $\{\xi_t(x) : x \in M, t \geq 0\}$  on  $M$ , but now the process on  $\text{Diff}M$  is just the right invariant process determined by  $\{\xi_t : 0 \leq t < \infty\}$ . In this case the horizontal lift to the diffeomorphism group of the diffusion  $\{\xi_t(x_0) : 0 \leq t < \infty\}$  on  $M$  is

obtained by ‘removal of redundant noise’, c.f. [7] while the vertical process is a flow of diffeomorphisms preserving  $x_0$ , driven by the redundant noise.

Here we report briefly on some of the main results to appear in [5] and give details of a more probabilistic version Theorem 2.5 below: a skew product decomposition which, although it has a statement not explicitly mentioning connections, relates to Itô’s pioneering work on the existence of horizontal lifts. The derivative flow example and a simplified version of the stochastic flow example are described in § 3.

The decomposition and lifting apply in much more generality than with the full structure of a principal bundle, for example to certain skew products and invariant processes on foliated manifolds. This will be reported on later. Earlier work on such decompositions includes [4] [13].

### §1. Construction

**A.** If  $\mathcal{A}$  is a second order differential operator on a manifold  $X$ , denote by  $\sigma^{\mathcal{A}} : T^*X \rightarrow TX$  its symbol determined by

$$df(\sigma^{\mathcal{A}}(dg)) = \frac{1}{2}\mathcal{A}(fg) - \frac{1}{2}\mathcal{A}(f)g - \frac{1}{2}f\mathcal{A}(g),$$

for  $C^2$  functions  $f, g$ . The operator is said to be *semi-elliptic* if  $df(\sigma^{\mathcal{A}}(df)) \geq 0$  for each  $f \in C^2(X)$ , and *elliptic* if the inequality holds strictly. Ellipticity is equivalent to  $\sigma^{\mathcal{A}}$  being onto. It is called a *diffusion operator* if it is semi-elliptic and annihilates constants, and is *smooth* if it sends smooth functions to smooth functions.

Consider a smooth map  $p : N \rightarrow M$  between smooth manifolds  $M$  and  $N$ . By a *lift* of a diffusion operator  $\mathcal{A}$  on  $M$  over  $p$  we mean a diffusion operator  $\mathcal{B}$  on  $N$  such that

$$(1) \quad \mathcal{B}(f \circ p) = (\mathcal{A}f) \circ p$$

for all  $C^2$  functions  $f$  on  $M$ . Suppose  $\mathcal{A}$  is a smooth diffusion operator on  $M$  and  $\mathcal{B}$  is a lift of  $\mathcal{A}$ .

**Lemma 1.1.** *Let  $\sigma^{\mathcal{B}}$  and  $\sigma^{\mathcal{A}}$  be respectively the symbols for  $\mathcal{B}$  and  $\mathcal{A}$ . The following diagram is commutative for all  $u \in p^{-1}(x)$ ,  $x \in M$ :*

$$\begin{array}{ccc} T_u^*N & \xrightarrow{\sigma_u^{\mathcal{B}}} & T_u N \\ (Tp)^* \uparrow & & \downarrow Tp \\ T_x^*M & \xrightarrow{\sigma_x^{\mathcal{A}}} & T_x M. \end{array}$$

**B. Semi-connections on principal bundles.** Let  $M$  be a smooth finite dimensional manifold and  $P(M, G)$  a principal fibre bundle over  $M$  with structure group  $G$  a Lie group. Denote by  $\pi : P \rightarrow M$  the projection and  $R_a$  the right translation by  $a$ .

**Definition 1.2.** Let  $E$  be a sub-bundle of  $TM$  and  $\pi : P \rightarrow M$  a principal  $G$ -bundle. An  $E$  semi-connection on  $\pi : P \rightarrow M$  is a smooth sub-bundle  $H^E TP$  of  $TP$  such that

- (i)  $T_u \pi$  maps the fibres  $H^E T_u P$  bijectively onto  $E_{\pi(u)}$  for all  $u \in P$ .
- (ii)  $H^E TP$  is  $G$ -invariant.

**Notes.**

(1) Such a semi-connection determines and is determined by, a smooth horizontal lift:

$$h_u : E_{\pi(u)} \rightarrow T_u P, \quad u \in P$$

such that

- (i)  $T_u \pi \circ h_u(v) = v$ , for all  $v \in E_x \subset T_x M$ ;
- (ii)  $h_{u \cdot a} = T_u R_a \circ h_u$ .

The horizontal subspace  $H^E T_u P$  at  $u$  is then the image at  $u$  of  $h_u$ , and the composition  $h_u \circ T_u P$  is a projection onto  $H^E T_u P$ .

(2) Let  $F = P \times V / \sim$  be an associated vector bundle to  $P$  with fibre  $V$ . An element of  $F$  is an equivalence class  $[(u, e)]$  such that  $(ug, g^{-1}e) \sim (u, e)$ . Set  $\tilde{u}(e) = [(u, e)]$ . An  $E$  semi-connection on  $P$  gives a covariant derivative on  $F$ . Let  $Z$  be a section of  $F$  and  $w \in E_x \subset T_x M$ , the covariant derivative  $\nabla_w Z \in F_x$  is defined, as usual for connections, by

$$\nabla_w Z = u(d\tilde{Z}(h_u(w))), \quad u \in \pi^{-1}(x) = F_x.$$

Here  $\tilde{Z} : P \rightarrow V$  is  $\tilde{Z}(u) = \tilde{u}^{-1} Z(\pi(u))$  considering  $\tilde{u}$  as an isomorphism  $\tilde{u} : V \rightarrow F_{\pi(u)}$ . This agrees with the ‘semi-connections on  $E$ ’ defined in Elworthy-LeJan-Li [7] when  $P$  is taken to be the linear frame bundle of  $TM$  and  $F = TM$ . As described there, any semi-connection can be completed to a genuine connection, but not canonically.

Consider on  $P$  a diffusion generator  $\mathcal{B}$ , which is *equivariant*, i.e.

$$\mathcal{B}f \circ R_a = \mathcal{B}(f \circ R_a), \quad \forall f, g \in C^2(P, R), \quad a \in G.$$

The operator  $\mathcal{B}$  induces an operator  $\mathcal{A}$  on the base manifold  $M$  by setting

$$(2) \quad \mathcal{A}f(x) = \mathcal{B}(f \circ \pi)(u), \quad u \in \pi^{-1}(x), \quad f \in C^2(M),$$

which is well defined since

$$\mathcal{B}(f \circ \pi)(u \cdot a) = \mathcal{B}((f \circ \pi))(u).$$

Let  $E_x := \text{Image}(\sigma_x^A) \subset T_x M$ , the image of  $\sigma_x^A$ . Assume the dimension of  $E_x = p$ , independent of  $x$ . Set  $E = \cup_x E_x$ . Then  $\pi : E \rightarrow M$  is a sub-bundle of  $TM$ .

**Theorem 1.3.** *Assume  $\sigma^A$  has constant rank. Then  $\sigma^B$  gives rise to a semi-connection on the principal bundle  $P$  whose horizontal map is given by*

$$(3) \quad h_u(v) = \sigma^B((T_u \pi)^* \alpha)$$

where  $\alpha \in T_{\pi(u)}^* M$  satisfies  $\sigma_x^A(\alpha) = v$ .

*Proof.* To prove  $h_u$  is well defined we only need to show  $\psi(\sigma^B(T_u \pi^*(\alpha))) = 0$  for every 1-form  $\psi$  on  $P$  and for every  $\alpha$  in  $\ker \sigma_x^A$ . Now  $\sigma^A \alpha = 0$  implies by Lemma 1.1 that

$$0 = \alpha \sigma^A(\alpha) = (T\pi)^*(\alpha) \sigma^B((T\pi)^*(\alpha)).$$

Thus  $T\pi^*(\alpha) \sigma^B(T\pi^*(\alpha)) = 0$ . On the other hand we may consider  $\sigma^B$  as a bilinear form on  $T^*P$  and then for all  $\beta \in T_u^* P$ ,

$$\begin{aligned} & \sigma^B(\beta + t(T\pi)^*(\alpha), \beta + t(T\pi)^*(\alpha)) \\ &= \sigma^B(\beta, \beta) + 2t\sigma^B(\beta, (T\pi)^*(\alpha)) + t^2\sigma^B((T\pi)^*\alpha, (T\pi)^*\alpha) \\ &= \sigma^B(\beta, \beta) + 2t\sigma^B(\beta, (T\pi)^*(\alpha)). \end{aligned}$$

Suppose  $\sigma^B(\beta, (T\pi)^*(\alpha)) \neq 0$ . We can then choose  $t$  such that

$$\sigma^B(\beta + t(T\pi)^*(\alpha), \beta + t(T\pi)^*(\alpha)) < 0,$$

which contradicts the semi-ellipticity of  $\mathcal{B}$ .

We must verify (i)  $T_u \pi \circ h_u(v) = v$ ,  $v \in E_x \subset T_x M$  and (ii)  $h_{u \cdot a} = T_u R_a \circ h_u$ . The first is immediate by Lemma 1.1 and for the second use the fact that  $T\pi \circ TR_a = T\pi$  for all  $a \in G$  and the equivariance of  $\sigma^B$ .  $\blacksquare$

## §2. Horizontal lifts of diffusion operators and decompositions of equivariant operators

**A.** Denote by  $C^\infty \Omega^p$  the space of smooth differential  $p$ -forms on a manifold  $M$ . To each diffusion operator  $\mathcal{A}$  we shall associate a unique operator  $\delta^A$ . The horizontal lift of  $\mathcal{A}$  can be defined to be the unique operator such that the associated operator  $\bar{\delta}$  vanishes on vertical 1-forms and such that  $\bar{\delta}$  and  $\delta^A$  are intertwined by the lift map  $\pi^*$  acting on 1-forms.

**Proposition 2.1.** *For each smooth diffusion operator  $\mathcal{A}$  there is a unique smooth differential operator  $\delta^A : C^\infty(\Omega^1) \rightarrow C^\infty \Omega^0$  such that*

$$(1) \quad \delta^A(f\phi) = df\sigma^A(\phi)_x + f \cdot \delta^A(\phi)$$

$$(2) \quad \delta^{\mathcal{A}}(df) = \mathcal{A}(f).$$

For example if  $\mathcal{A}$  has Hörmander representation

$$\mathcal{A} = \frac{1}{2} \sum_{j=1}^m \mathcal{L}_{X^j} \mathcal{L}_{X^j} + \mathcal{L}_A$$

for some  $C^1$  vector fields  $X^i$ ,  $A$  then

$$\delta^{\mathcal{A}} = \frac{1}{2} \sum_{j=1}^m \mathcal{L}_{X^j} \iota_{X^j} + \iota_A$$

where  $\iota_A$  denotes the interior product of the vector field  $A$  acting on differential forms.

**Definition 2.2.** Let  $S$  be a  $C^\infty$  sub-bundle of  $TN$  for some smooth manifold  $N$ . A diffusion operator  $\mathcal{B}$  on  $N$  is said to be along  $S$  if  $\delta^{\mathcal{B}}\phi = 0$  for all 1-forms  $\phi$  which vanish on  $S$ ; it is said to be strongly cohesive if  $\sigma^{\mathcal{B}}$  has constant rank and  $\mathcal{B}$  is along the image of  $\sigma^{\mathcal{B}}$ .

To be along  $S$  implies that any Hörmander form representation of  $\mathcal{B}$  uses only vector fields which are sections of  $S$ .

**Definition 2.3.** When a diffusion operator  $\mathcal{B}$  on  $P$  is along the vertical foliation  $VTP$  of the  $\pi : P \rightarrow M$  we say  $\mathcal{B}$  is vertical, and when the bundle has a semi-connection and  $\mathcal{B}$  is along the horizontal distribution we say  $\mathcal{B}$  is horizontal.

If  $\pi : P \rightarrow M$  has an  $E$  semi-connection and  $\mathcal{A}$  is a smooth diffusion operator along  $E$  it is easy to see that  $\mathcal{A}$  has a unique horizontal lift  $\mathcal{A}^H$ , i.e. a smooth diffusion operator  $\mathcal{A}^H$  on  $P$  which is horizontal and is a lift of  $\mathcal{A}$  in the sense of (1). By uniqueness it is equivariant.

**B.** The action of  $G$  on  $P$  induces a homomorphism of the Lie algebra  $\mathfrak{g}$  of  $G$  with the algebra of right invariant vector fields on  $P$ : if  $\alpha \in \mathfrak{g}$ ,

$$A^\alpha(u) = \left. \frac{d}{dt} \right|_{t=0} u \exp(t\alpha),$$

and  $A^\alpha$  is called the fundamental vector field corresponding to  $\alpha$ . Take a basis  $A_1, \dots, A_k$  of  $\mathfrak{g}$  and denote the corresponding fundamental vector fields by  $\{A_i^*\}$ .

We can now give one of the main results from [5]:

**Theorem 2.4.** *Let  $\mathcal{B}$  be an equivariant operator on  $P$  with  $\mathcal{A}$  the induced operator on the base manifold. Assume  $\mathcal{A}$  is strongly cohesive. Then there is a unique semi-connection on  $P$  over  $E$  for which  $\mathcal{B}$  has a decomposition*

$$\mathcal{B} = \mathcal{A}^H + \mathcal{B}^V,$$

where  $\mathcal{A}^H$  is horizontal and  $\mathcal{B}^V$  is vertical. Furthermore  $\mathcal{B}^V$  has the expression  $\sum \alpha^{ij} \mathcal{L}_{A_i^*} \mathcal{L}_{A_j^*} + \sum \beta^k \mathcal{L}_{A_k^*}$ , where  $\alpha^{ij}$  and  $\beta^k$  are smooth functions on  $P$ , given by  $\alpha^{k\ell} = \tilde{\omega}^k(\sigma^{\mathcal{B}}(\tilde{\omega}^\ell))$ , and  $\beta^\ell = \delta^{\mathcal{B}}(\tilde{\omega}^\ell)$  for  $\tilde{\omega}$  any connection 1-form on  $P$  which vanishes on the horizontal subspaces of this semi-connection.

We shall only prove the first part of Theorem 2.4 here. The semi-connection is the one given by Theorem 1.3, and we define  $\mathcal{A}^H$  to be the horizontal lift of  $\mathcal{A}$ . The proof that  $\mathcal{B}^V := \mathcal{B} - \mathcal{A}^H$  is vertical is simplified by using the fact that a diffusion operator  $\mathcal{D}$  on  $P$  is vertical if and only if for all  $C^2$  functions  $f_1$  on  $P$  and  $f_2$  on  $M$

$$(4) \quad \mathcal{D}(f_1(f_2 \circ \pi)) = (f_2 \circ \pi)\mathcal{D}(f_1).$$

Set  $\tilde{f}_2 = f_2 \circ \pi$ . Note

$$(\mathcal{B} - \mathcal{A}^H)(f_1 \tilde{f}_2) = \tilde{f}_2(\mathcal{B} - \mathcal{A}^H)f_1 + f_1(\mathcal{B} - \mathcal{A}^H)\tilde{f}_2 + 2(df_1)\sigma^{\mathcal{B}-\mathcal{A}^H}(d\tilde{f}_2).$$

Therefore to show  $(\mathcal{B} - \mathcal{A}^H)$  is vertical we only need to prove

$$f_1(\mathcal{B} - \mathcal{A}^H)\tilde{f}_2 + 2(df_1)\sigma^{\mathcal{B}-\mathcal{A}^H}(d\tilde{f}_2) = 0.$$

Recall Lemma 1.1 and use the natural extension of  $\sigma^{\mathcal{A}}$  to  $\sigma^{\mathcal{A}} : E^* \rightarrow E$  and the fact that by (3)  $h \circ \sigma_x^{\mathcal{A}} = \sigma^{\mathcal{B}}(T_u \pi)^*$  to see

$$\begin{aligned} \sigma^{\mathcal{A}^H}(d\tilde{f}_2) &= (h \circ \sigma^{\mathcal{A}} h^*)(df_2 \circ T\pi) = h \circ \sigma^{\mathcal{A}} df_2 \\ &= \sigma^{\mathcal{B}}(df_2 \circ T\pi) = \sigma^{\mathcal{B}}(d\tilde{f}_2), \end{aligned}$$

and so  $\sigma^{(\mathcal{B}-\mathcal{A}^H)}(d\tilde{f}_2) = 0$ . Also by equation (1)

$$(\mathcal{B} - \mathcal{A}^H)\tilde{f}_2 = \mathcal{A}f_2 - \mathcal{A}^H\tilde{f}_2 = 0.$$

This shows that  $\mathcal{B} - \mathcal{A}^H$  is vertical. ■

Define  $\alpha : P \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  and  $\beta : P \rightarrow \mathfrak{g}$  by

$$\alpha(u) = \sum \alpha^{ij}(u) A_i \otimes A_j$$

$$\beta(u) = \sum \beta^k(u) A_k.$$

It is easy to see that  $\mathcal{B}^V$  depends only on  $\alpha, \beta$  and the expression is independent of the choice of basis of  $\mathfrak{g}$ . From the invariance of  $\mathcal{B}$  we obtain

$$\begin{aligned}\alpha(ug) &= (ad(g) \otimes ad(g))\alpha(u), \\ \beta(ug) &= ad(g)\beta(u)\end{aligned}$$

for all  $u \in P$  and  $g \in G$ .

**C.** Theorem 2.4 has a more directly probabilistic version. For this let  $\pi : P \rightarrow M$  be as before and for  $0 \leq l < r < \infty$  let  $C(l, r; P)$  be the space of continuous paths  $y : [l, r] \rightarrow P$  with its usual Borel  $\sigma$ -algebra. For such write  $l_y = l$  and  $r_y = r$ . Let  $C(*, *; P)$  be the union of such spaces. It has the standard additive structure under concatenation: if  $y$  and  $y'$  are two paths with  $r_y = l_{y'}$  and  $y(r_y) = y'(l_{y'})$  let  $y + y'$  be the corresponding element in  $C(l_y, r_{y'}; P)$ . The *basic*  $\sigma$ -algebra of  $C(*, *, P)$  is defined to be the pull back by  $\pi$  of the usual Borel  $\sigma$ -algebra on  $C(*, *, M)$ .

Consider the laws  $\{\mathbb{P}_a^{l,r} : 0 \leq l < r, a \in P\}$  of the process running from  $a$  between times  $l$  and  $r$ , associated to a smooth diffusion operator  $\mathcal{B}$  on  $P$ . Assume for simplicity that the diffusion has no explosion. Thus  $\{\mathbb{P}_a^{l,r}, a \in P\}$  is a kernel from  $P$  to  $C(l, r; P)$ . The right action  $R_g$  by  $g$  in  $G$  extends to give a right action, also written  $R_g$ , of  $G$  on  $C(*, *, P)$ . Equivariance of  $\mathcal{B}$  is equivalent to

$$\mathbb{P}_{ag}^{l,r} = (R_g)_* \mathbb{P}_a^{l,r}$$

for all  $0 \leq l \leq r$  and  $a \in P$ . If so  $\pi_*(\mathbb{P}_a^{l,r})$  depends only on  $\pi(a), l, r$  and gives the law of the induced diffusion  $\mathcal{A}$  on  $M$ . We say that such a diffusion  $\mathcal{B}$  is *basic* if for all  $a \in P$  and  $0 \leq l < r < \infty$  the basic  $\sigma$ -algebra on  $C(l, r; P)$  contains all Borel sets up to  $\mathbb{P}_a^{l,r}$  negligible sets, i.e. for all  $a \in P$  and Borel subsets  $B$  of  $C(l, r; P)$  there exists a Borel subset  $A$  of  $C(l, r, M)$  s.t.  $\mathbb{P}_a(\pi^{-1}(A)\Delta B) = 0$ .

For paths in  $G$  it is more convenient to consider the space  $C_{id}(l, r; G)$  of continuous  $\sigma : [l, r] \rightarrow G$  with  $\sigma(l) = id$  for 'id' the identity element. The corresponding space  $C_{id}(*, *, G)$  has a multiplication

$$C_{id}(s, t; G) \times C_{id}(t, u; G) \longrightarrow C_{id}(s, u; G)$$

$$(g, g') \mapsto g \times g'$$

where  $(g \times g')(r) = g(r)$  for  $r \in [s, t]$  and  $(g \times g')(r) = g(t)g'(r)$  for  $r \in [t, u]$ .

Given probability measures  $\mathbb{Q}, \mathbb{Q}'$  on  $C_{id}(s, t; G)$  and  $C_{id}(t, u; G)$  respectively this determines a convolution  $\mathbb{Q} * \mathbb{Q}'$  of  $\mathbb{Q}$  with  $\mathbb{Q}'$  which is a probability measure on  $C_{id}(s, u; G)$ .

**Theorem 2.5.** *Given the laws  $\{\mathbb{P}_a^{l,r} : a \in P, 0 \leq l < r < \infty\}$  of an equivariant diffusion  $\mathcal{B}$  as above with  $\mathcal{A}$  strongly cohesive there exist probability kernels  $\{\mathbb{P}_a^{H,l,r} : a \in P\}$  from  $P$  to  $C(l,r;P)$ ,  $0 \leq l < r < \infty$  and  $\mathbb{Q}_y^{l,r}$ , defined  $\mathbb{P}^{l,r}$  a.s. from  $C(l,r,P)$  to  $C_{id}(l,r;G)$  such that*

- (i)  $\{\mathbb{P}_a^{H,l,r} : a \in P\}$  is equivariant, basic and determining a strongly cohesive generator.
- (ii)  $y \mapsto \mathbb{Q}_y^{l,r}$  satisfies

$$\mathbb{Q}_{y+y'}^{l_y, r_{y'}} = \mathbb{Q}_y^{l_y, r_y} * \mathbb{Q}_{y'}^{l_{y'}, r_{y'}}$$

for  $\mathbb{P}^{l_y, r_y} \otimes \mathbb{P}^{l_{y'}, r_{y'}}$  almost all  $y, y'$  with  $r_y = l_{y'}$ .

- (iii) For  $U$  a Borel subset of  $C(l,r,P)$ ,

$$\mathbb{P}_a^{l,r}(U) = \int \int \chi_U(y \cdot g) \mathbb{Q}_y^{l,r}(dg) \mathbb{P}_a^{H,l,r}(dy).$$

The kernels  $\mathbb{P}_a^{H,l,r}$  are uniquely determined as are the  $\{\mathbb{Q}_y^{l,r} : y \in \mathbb{R}\}$ ,  $\mathbb{P}_a^{H,l,r}$  a.s. in  $y$  for all  $a$  in  $P$ . Furthermore  $\mathbb{Q}_y^{l,r}$  depends on  $y$  only through its projection  $\pi(y)$  and its initial point  $y_l$ .

*Proof.* Fix  $a$  in  $P$  and let  $\{b_t : l \leq r \leq t\}$  be a process with law  $\mathbb{P}_a^{l,r}$ . By Theorem 2.4 we can assume that  $b$ . is given by an s.d.e. of the form

$$(5) \quad db_t = \tilde{X}(b_t) \circ dB_t + \tilde{X}^0(b_t)dt + A(b_t) \circ d\beta_t + V(b_t)dt$$

where  $\tilde{X} : P \times \mathbb{R}^p \rightarrow TP$  is the horizontal lift of some  $X : M \times \mathbb{R}^p \rightarrow E$ ,  $\tilde{X}^0$  is the horizontal lift of a vector field  $X^0$  on  $M$ , while  $A : P \times \mathbb{R}^1 \rightarrow TP$  and the vector field  $V$  are vertical and determine  $\mathcal{B}^V$ . Here  $B$ . and  $\beta$ . are independent Brownian motions on  $\mathbb{R}^p$  and  $\mathbb{R}^q$  respectively, some  $q$ , and we are using the semi-connection on  $P$  induced by  $\mathcal{B}$  as in Theorem 1.3.

Let  $\{\tilde{x}_t : l \leq t \leq r\}$  satisfy

$$(6) \quad \begin{aligned} d\tilde{x}_t &= \tilde{X}(\tilde{x}_t) \circ dB_t + \tilde{X}^0(\tilde{x}_t)dt \\ \tilde{x}_l &= a \end{aligned}$$

so  $\tilde{x}$ . is the horizontal lift of  $\{\pi(b_t) : l \leq t \leq r\}$ . Then there is a unique continuous process  $\{g_t : l \leq t \leq r\}$  in  $G$  with  $g_l = id$  such that

$$\tilde{x}_t g_t = b_t.$$

We have to analyse  $\{g_t : l \leq t \leq r\}$ . Using local trivialisations of  $\pi : P \rightarrow M$  we see it is a semi-martingale. As in [9], Proposition 3.1 on page 69,

$$db_t = TR_{g_t}(\circ d\tilde{x}_t) + A^{g_t^{-1} \circ dg_t}(b_t)$$

giving

$$\tilde{\omega}(\circ db_t) = \tilde{\omega}\left(A^{g_t^{-1} \circ dg_t}(b_t)\right) = g_t^{-1} \circ dg_t$$

for any smooth connection form  $\tilde{\omega} : P \rightarrow \mathfrak{g}$  on  $P$  which vanishes on  $H^E TP$ . Thus

$$(7) \quad \begin{aligned} dg_t &= TL_{g_t} \tilde{\omega}(A(\tilde{x}_t g_t) \circ d\beta_t + V(\tilde{x}_t g_t) dt) \\ g_l &= id, \quad l \leq t \leq r. \end{aligned}$$

For  $y \in C(l, r : P)$  let  $\{g_t^y : l \leq t \leq r\}$  be the solution of

$$(8) \quad \begin{aligned} dg_t^y &= TL_{g_t^y} \tilde{\omega}(A(y_t g_t^y) \circ d\beta_t + V(y_t g_t^y) dt) \\ g_l^y &= id \end{aligned}$$

(where the Stratonovich equation is interpreted with ' $dy_t d\beta_t = 0$ '). Since  $\beta$ . and  $B$ . and hence  $\beta$ . and  $\tilde{x}$ . are independent we see  $g = g^{\tilde{x}}$  almost surely. For a discussion of some technicalities concerning skew products, see [16].

For  $y$ . in  $C(*, *; P)$  let  $\{h(y)_t : l_y \leq t \leq r_y\}$  be the horizontal lift of  $\pi(y)$ ., starting at  $y_{l_y}$ . This exists for almost all  $y$  as can be seen either by the extension of Itô's result to general principal bundles, e.g. using (6), or by the existence of measurable sections using the fact that  $\mathcal{A}^H$  is basic. Define  $\mathbb{P}_a^{H,l,r}$  to be the law of  $\tilde{x}$ . above and  $Q_y^{l,r}$  to be that of  $g^{h(y)}$ . Clearly conditions (i) is satisfied.

To check (ii) take  $y$  and  $y'$  with  $r_y = l_{y'}$ . Then

$$h(y + y') = h(y) + h(y') \left(g_{r_y}^{h(y)}\right)^{-1},$$

writing  $y = h(y)g^{h(y)}$  and  $y' = h(y')g^{h(y')}$ . For  $r_y \leq t \leq r_{y'}$  this shows

$$(y + y')_t = h(y')_t \left(g_{r_y}^{h(y)}\right)^{-1} g_t^{h(y+y')}.$$

But  $(y + y')_t = y'_t = h(y')_t g_t^{h(y')}$  and so we have  $g_t^{h(y+y')} = g_{r_y}^{h(y)} g_t^{h(y')}$  for  $t \geq r_y$ , giving  $g^{h(y+y')} = g^{h(y)} \times g^{h(y')}$  almost surely. This proves (ii).

For uniqueness suppose we have another set of probability measures denoted  $\tilde{\mathbb{Q}}_y^{l,r}$  and  $\tilde{P}_a^{H,l,r}$  which satisfy (i), (ii), (iii). Since  $\{\tilde{\mathbb{P}}_a^{H,l,r}\}$  is equivariant and induces  $\mathcal{A}$  on  $M$  we can apply the preceding argument to it in place of  $\{\mathbb{P}_a^{l,r}\}$ . However since it is basic the term involving  $\beta$  in the stochastic differential equation (6) must vanish. Since it is also strongly cohesive the vertical part  $V$  must vanish also and we have  $\tilde{\mathbb{P}}_a^{H,l,r} = \mathbb{P}_a^{H,l,r}$ . On the other hand in the decomposition  $b_t = \tilde{x}_t g_t^{\tilde{x}_t}$  the law of  $g^{\tilde{x}}$  is determined by those of  $b$ . and  $\tilde{x}$ . but  $\mathbb{Q}_y^{l,r}$  is the conditional law of  $g^{\tilde{x}}$ . given  $\tilde{x} = y$ . and so is uniquely determined as described. ■

In fact  $\mathbb{Q}_y^{l,r}$  is associated to the time dependent generator which at  $g \in G$  and  $t \in [l, r]$  is  $\sum \alpha^{ij}(h(y)_t g) \mathcal{L}_{A_i} \mathcal{L}_{A_j} + \sum \beta^k(h(y)_t g) \mathcal{L}_{A_k}$  for  $\alpha^{ij}$  and  $\beta^k$  as defined in Theorem 2.4 while  $\mathbb{P}^{H,l,r}$  is associated to  $\mathcal{A}^H$ .

### §3. Stochastic flows and derivative flows

**A. Derivative flows.** Let  $\mathcal{A}$  on  $M$  be given in Hörmander form

$$\mathcal{A} = \frac{1}{2} \sum_{j=1}^m \mathcal{L}_{X^j} \mathcal{L}_{X^j} + \mathcal{L}_A$$

for some vector fields  $X^1, \dots, X^m, A$ . As before let  $E_x = \text{span}\{X^1(x), \dots, X^m(x)\}$  and assume  $\dim E_x$  is constant,  $p$ , say, giving a sub-bundle  $E \subset TM$ . The  $X^1(x), \dots, X^m(x)$  determine a vector bundle map of the trivial bundle  $\underline{\mathbb{R}}^m$

$$X : \underline{\mathbb{R}}^m \longrightarrow TM$$

with  $\sigma^{\mathcal{A}} = X(x)X(x)^*$ . We can, and will, consider  $X$  as a map  $X : \underline{\mathbb{R}}^m \rightarrow E$ .

As such it determines (a) a Riemannian metric  $\{\langle \cdot, \cdot \rangle_x : x \in M\}$  on  $E$  (the same as that determined by  $\sigma^{\mathcal{A}}$ ) and (b) a metric connection  $\check{\nabla}$  on  $E$  uniquely defined by the requirement that for each  $x$  in  $M$ ,

$$\check{\nabla}_v X(e) = 0$$

for all  $v \in T_x M$  whenever  $e$  is orthogonal to the kernel of  $T_x M$ . Then for any differentiable section  $U$  of  $E$ ,

$$(9) \quad \check{\nabla}_v U = Y(x)d(Y(U(\cdot)))(v), \quad v \in T_x M,$$

where  $Y$  is the  $\mathbb{R}^m$  valued 1-form on  $M$  given by

$$\langle Y_x(v), e \rangle_{\mathbb{R}^m} = \langle X(x)(e), v \rangle_x, \quad e \in \mathbb{R}^m, v \in E_x, x \in M$$

e.g. [7] where it is referred to as the LeJan-Watanabe connection in this context. By a theorem of Narasimhan and Ramanan [14] all metric connections on  $E$  arise this way, see [15], [7].

For  $\{B_t : 0 \leq t < \infty\}$  a Brownian motion on  $\mathbb{R}^m$ , the stochastic differential equation

$$(10) \quad dx_t = X(x_t) \circ dB_t + A(x_t)dt$$

determines a Markov process with differential generator  $\mathcal{A}$ . Over each solution  $\{x_t : 0 \leq t < \rho\}$ , where  $\rho$  is the explosion time, there is a ‘derivative’ process  $\{v_t : 0 \leq t < \rho\}$  in  $TM$  which we can write as  $\{T\xi_t(v_0) : 0 \leq t < \rho\}$

with  $T\xi_t : T_{x_0}M \rightarrow T_{x_t}M$  linear. This would be the derivative of the flow  $\{\xi_t : 0 \leq t < \rho\}$  of the stochastic differential equation when the stochastic differential equation is strongly complete. In general it is given by a stochastic differential equation on the tangent bundle  $TM$ , or equivalently by a covariant equation along  $\{x_t : 0 \leq t < \rho\}$ :

$$Dv_t = \nabla X(v_t) \circ dB_t + \nabla A(v_t)dt$$

with respect to any torsion free connection. Take  $P$  to be the linear frame bundle  $GL(M)$  of  $M$ , treating  $u \in GL(M)$  as an isomorphism  $u : \mathbb{R}^n \rightarrow T_{\pi(u)}M$ . For  $u_0 \in GLM$  we obtain a process  $\{u_t : 0 \leq t < \rho\}$  on  $GLM$  by

$$u_t = T\xi_t \circ u_0.$$

Let  $\mathcal{B}$  be its differential generator. Clearly it is equivariant and a lift of  $\mathcal{A}$ .

A proof of the following in the context of stochastic flows, is given later. For  $w \in E_x$ , set

$$(11) \quad Z^w(y) = X(y)Y(x)(w).$$

**Theorem 3.1.** *The semi-connection  $\nabla$  induced by  $\mathcal{B}$  is the adjoint connection of the LeJan-Watanabe connection  $\check{\nabla}$  determined by  $X$ , as defined by (9), [7]. Consequently  $\nabla_w V = L_{Z^w} V$  for any vector field  $V$  and  $w \in E$  also  $\nabla_{V(x)} Z^w$  vanishes if  $w \in E_x$ .*

In the case of the derivative flow the  $\alpha, \beta$  of Theorem 2.4 have an explicit expression: for  $u \in GLM$ ,

$$(12) \quad \begin{cases} \alpha(u) = \frac{1}{2} \sum \left( u^{-1}(-) \check{\nabla}_{u(-)} X^p \right) \otimes \left( u^{-1}(-) \check{\nabla}_{u(-)} X^p \right) \\ \beta(u) = -\frac{1}{2} \sum u^{-1} \check{\nabla}_{\check{\nabla}_{u(-)} X^p} X^p - \frac{1}{2} u^{-1} \text{Ric} \# u(-). \end{cases}$$

Here  $\check{R}$  is the curvature tensor for  $\check{\nabla}$  and  $\check{Ric}^\# : TM \rightarrow E$  the Ricci curvature defined by  $\check{Ric}^\#(v) = \sum_{j=1}^p \check{R}(v, e^j) e^j$ ,  $v \in T_x M$ .

Equivariant operators on  $GLM$  determine operators on associated bundles, such as  $\wedge^q TM$ . If the original operator was vertical this turns out to be a zero order operator (as is shown in [5] for general principal bundles) and in the case of  $\wedge^q TM$  these operators are the generalized Weitzenböck curvature operators described in [7]. In particular for differential 1-forms the operator is  $\phi \mapsto \phi(\text{Ric}^\# -)$ . To see this, as an illustrative example, given a 1-form  $\phi$

define  $\tilde{\phi} : GLM \rightarrow L(\mathbb{R}^n; \mathbb{R})$  by  $\tilde{\phi}(u) = \phi_{\pi u}$  so  $\tilde{\phi}(ug) = \phi_{\pi u}(ug-)$ . Then

$$\begin{aligned} L_{A_j^*}(\tilde{\phi})(u) &= \frac{d}{dt} \tilde{\phi}(u \cdot e^{A_j t})|_{t=0} \\ &= \frac{d}{dt} \phi_{\pi u}(u \cdot e^{A_j t})|_{t=0} \\ &= \phi_{\pi u}(u A_j -) = \tilde{\phi}(u)(A_j -). \end{aligned}$$

Iterating we have

$$\begin{aligned} \mathcal{B}^V(\tilde{\phi})(u) &= \sum_{i,j} \alpha^{i,j}(u) \phi_{\pi u}(u A_j A_i -) + \sum_k \beta^k(u) \phi_{\pi u}(u A_k -) \\ &= -\frac{1}{2} \tilde{\phi}(u)(u^{-1} Ric^\#(u-)) \end{aligned}$$

as required, by using the map  $gl(n) \otimes gl(n) \rightarrow gl(n)$ ,  $S \otimes T \mapsto S \circ T$ , and equation (12).

**B. Stochastic flows.** In fact Theorem 3.1 can be understood in the more general context of stochastic flows as diffusions on the diffeomorphism groups. For this assume that  $M$  is compact and for  $r \in \{1, 2, \dots\}$  and  $s > r + \dim(M)/2$  let  $\mathcal{D}^s = \mathcal{D}^s M$  be the  $C^\infty$  manifold of diffeomorphisms of  $M$  of Sobolev class  $H^s$ , (for example see Ebin-Marsden [2] or Elworthy [3].) Alternatively we could take the space  $\mathcal{D}^\infty$  of  $C^\infty$  diffeomorphisms with differentiable structure as in [11]. Fix a base point  $x_0$  in  $M$  and let  $\pi : \mathcal{D}^s \rightarrow M$  be evaluation at  $x_0$ . This makes  $\mathcal{D}^s$  into a principal bundle over  $M$  with group the manifold  $\mathcal{D}_{x_0}^s$  of  $H^s$ -diffeomorphisms  $\theta$  with  $\theta(x_0) = x_0$ , acting on the right by composition (although the action of  $\mathcal{D}^{s+r}$  is only  $C^r$ , for  $r = 0, 1, 2, \dots$ ).

Let  $\{\xi_t^s : 0 \leq s \leq t < \infty\}$  be the flow of (10) starting at time  $s$ . Write  $\xi_t$  for  $\xi_t^0$ . The more general case allowing for infinite dimensional noise is given in [5]. We define probability measures  $\{\mathbb{P}_\theta^{s,t} : \theta \in \mathcal{D}^s\}$  on  $C([s,t]; M)$  by letting  $\mathbb{P}_\theta^{s,t}$  be the law of  $\{\xi_r^s \circ \theta : s \leq r \leq t\}$  (These correspond to the diffusion process on  $\mathcal{D}^s$  associated to the right-invariant stochastic differential equation on  $\mathcal{D}^s$  satisfied by  $\{\xi_t : 0 \leq t < \infty\}$  as in [3].) These are equivariant and project by  $\pi$  to the laws given by the stochastic differential equation on  $M$ . Assuming that these give a strongly cohesive diffusion on  $M$  we are essentially in the situation of Theorem 2.5.

Let  $K(x) : \mathbb{R}^m \rightarrow \mathbb{R}^m$  be the orthogonal projection onto the kernel of  $X(x)$ , each  $x \in M$ . set  $K^\perp(x) = id - K(x)$ . Consider the  $\mathcal{D}^\infty$ -valued process  $\{\theta_t : 0 \leq t < \infty\}$  given by (or as the flow of)

$$(13) \quad d\theta_t(x) = X(\theta_t(x))K^\perp(\theta_t(x_0)) \circ dB_t + X(\theta_t(x))Y(\theta_t(x_0))A(\theta_t(x_0))$$

for given  $\theta_0$  in  $\mathcal{D}^\infty$  and, define a  $\mathcal{D}_{x_0}^\infty$ -valued process  $\{g_t : 0 \leq t < \infty\}$  by

$$(14) \quad \begin{aligned} dg_t &= T\theta_t^{-1} \{X(\theta_t g_t)K(\theta_t x_0) \circ dB_t \\ &\quad + A(\theta_t g_t)dt - X(\theta_t g_t)Y(\theta_t x_0)A(\theta_t x_0)dt\} \\ g_0 &= id. \end{aligned}$$

Set  $x_t^\theta = \xi_t(\theta_0(x_0))$ . Note that  $\pi(\theta_t) = \theta_t(x_0) = x_t^\theta$  since

$$X(\theta_t(x_0))K^\perp(\theta_t(x_0)) = X(\theta_t(x_0))$$

and

$$X(\theta_t(x_0))Y(\theta_t(x_0))A(\theta_t(x_0)) = A(\theta_t(x_0)).$$

Thus  $\{\theta_t : 0 \leq t < \infty\}$  is a lift of  $\{x_t^\theta, 0 \leq t < \infty\}$ . It can be considered to be driven by the ‘relevant noise’, (from the point of view of  $\xi_t(\theta_0(x_0))$ ), i.e. by the Brownian motion  $\tilde{B}$ . given by

$$\tilde{B}_t = \int_0^t \tilde{\jmath}(x_s^\theta)^{-1} K^\perp(x_s^\theta) dB_s$$

where  $\{\tilde{\jmath}(x_s^\theta), 0 \leq s < \infty\}$  is parallel translation along  $x^\theta$  with respect to the connection on the trivial bundle  $M \times \mathbb{R}^m \rightarrow M$  determined by  $K$  and  $K^\perp$ , so that

$$\tilde{\jmath}(x_s^\theta) : \mathbb{R}^m \rightarrow \mathbb{R}^m$$

is orthogonal and maps the kernel of  $X(\theta_s(x_0))$  onto the kernel of  $X(x_s^\theta)$  for  $0 \leq s < \infty$ , see [7](chapter 3).

Correspondingly there is the ‘redundant noise’, the Brownian motion  $\{\beta_t : 0 \leq t < \infty\}$  given by

$$\beta_t = \int_0^t \tilde{\jmath}(x_s^\theta)^{-1} K(x_s^\theta) dB_s.$$

Then, as shown in [7](chapter 3),

- (i)  $\tilde{B}$ . has the same filtration as  $\{x_s^\theta : 0 \leq s < \infty\}$
- (ii)  $\beta$ . and  $\tilde{B}$ . are independent
- (iii)  $dB_t = \tilde{\jmath}_t d\beta_t + \tilde{\jmath}_t d\tilde{B}_t$ .

We wish to see how  $g$ . is driven by  $\beta$ .. For this observe

$$\int_0^t K(x_s^\theta) \circ dB_s = \int_0^t K(x_s^\theta) dB_s + \int_0^t \Lambda(x_s^\theta) ds$$

for  $\Lambda : M \rightarrow \mathbb{R}$  given by the Stratonovich correction term. By (iii)

$$\int_0^t K(x_s^\theta) dB_s = \int_0^t \tilde{\jmath}_s d\beta_s = \int_0^t \tilde{\jmath}_s \circ d\beta_s$$

since  $\tilde{\mathbb{J}}$  is independent of  $\beta$  by (i) and (ii). Thus equation (14) for  $g$  can be written as

$$dg_t = T\theta_t^{-1} \left\{ X(\theta_t g_t) \tilde{\mathbb{J}}(\theta_t(x_0))_t \circ d\beta_t + X(\theta_t g_t) \Lambda(\theta_t(x_0)) dt + A(\theta_t g_t) dt - X(\theta_t g_t) Y(\theta_t x_0) A(\theta_t x_0) dt \right\}$$

and if we define

$$\begin{aligned} dg_t^y &= Ty_t^{-1} \left\{ X(y_t g_t) \tilde{\mathbb{J}}(y_t(x_0))_t \circ d\beta_t + X(y_t g_t) \Lambda(y_t(x_0)) dt + A(y_t g_t) dt - X(y_t g_t) Y(y_t x_0) A(y_t x_0) dt \right\} \\ g_0 &= id \end{aligned}$$

for any continuous  $y : [0, \infty) \rightarrow \mathcal{D}^\infty$ , we see, by the independence of  $\beta$  and  $\theta$  that  $g = g^\theta$ .

By Itô's formula on  $\mathcal{D}^s$ , for  $x \in M$ ,

$$d(\theta_t g_t(x)) = (\circ d\theta_t)(g_t(x)) + T\theta_t(\circ dg_t^\theta(x)).$$

Now

$$\begin{aligned} T\theta_t(\circ dg_t^\theta(x)) &= \{ X(\theta_t g_t(x)) K(\theta_t x_0) \circ dB_t + A(\theta_t g_t(x)) dt - X(\theta_t g_t(x)) Y(\theta_t x_0) A(\theta_t x_0) dt \} \end{aligned}$$

and so by (13) we see that  $\theta_t g_t = \xi_t \circ \theta_0$ , a.s.

Taking  $\theta_0 = id$  we have

**Proposition 3.2.** *The flow  $\xi$  has the decomposition*

$$\xi_t = \theta_t g_t^\theta, \quad 0 \leq t < \infty$$

for  $\theta$  and  $g^\theta \equiv g$  given by (13) and (14) above. For almost all  $\sigma : [0, \infty) \rightarrow M$  with  $\sigma(0) = x_0$  and bounded measurable  $F : C(0, \infty; \mathcal{D}^\infty) \rightarrow \mathbb{R}$

$$\mathbb{E} \{ F(\xi_\cdot) | \xi_\cdot(x_0) = \sigma \} = \mathbb{E} \{ F(\tilde{\sigma} g^\theta) \}$$

where  $\tilde{\sigma} : [0, \infty) \rightarrow \mathcal{D}^\infty$  is the horizontal lift of  $\sigma$  with  $\tilde{\sigma}(0) = id$ .

To define the 'horizontal lift' above we can use the fact, from (i) above, that  $\theta$  has the same filtration as  $\xi_\cdot(x_0)$  and so furnishes a lifting map.

In terms of the semi-connection induced on  $\pi : \mathcal{D}^s \rightarrow M$  over  $E$ , from above, by uniqueness or directly, we see the horizontal lift

$$\begin{aligned} h_\theta &: E_{\theta(x_0)} \longrightarrow T_\theta \mathcal{D}^s \\ h_\theta(v) &: M \longrightarrow TM \end{aligned}$$

is given by  $h_\theta(v) = X(\theta(x))Y(\theta(x_0))v$  and the horizontal lift  $\tilde{\sigma}$  from  $\tilde{\sigma}_0$  of a  $C^1$  curve  $\sigma$  on  $M$  with  $\tilde{\sigma}_0(x_0) = \sigma_0$  and  $\dot{\sigma}(t) \in E_{\sigma(t)}$ , all  $t$ , is given by

$$\frac{d}{dt}\tilde{\sigma}_t = X(\tilde{\sigma}_t)Y(\sigma_t)\dot{\sigma}_t$$

for  $\tilde{\sigma}_0 = id$ . The lift is the solution flow of the differential equation

$$\dot{y}_t = Z^{\dot{\sigma}}(y_t)$$

on  $M$ .

For each frame  $u : \mathbb{R}^n \rightarrow T_{x_0}M$  there is a homomorphism of principal bundles

$$(15) \quad \begin{array}{l} \mathcal{D}^s \rightarrow GL(M) \\ \theta \mapsto T_{x_0}\theta \circ u. \end{array}$$

This sends  $\{\xi_t : t \geq 0\}$  to the derivative process  $T_x\xi_t \circ u$ . (If the latter satisfies the strongly cohesive condition we could apply our analysis to this submersion  $\mathcal{D}^s \rightarrow GLM$  and get another decomposition of  $\xi$ .)

Results in Kobayashi-Nomizu [9] (Proposition 6.1 on page 79) apply to the homomorphism  $\mathcal{D}^s \rightarrow GL(M)$  of (15). This gives a relationship between the curvature and holonomy groups of the semi-connection  $\hat{\nabla}$  on  $GLM$  determined by the derivative flow and those of the connection induced by the diffusion on  $\mathcal{D}^s \xrightarrow{\pi} M$ . It also shows that the horizontal lift  $\{\tilde{x}_t : t \geq 0\}$  through  $u$  of  $\{x_t : t \geq 0\}$  to  $GL(M)$  is just  $T_{x_0}\theta_t \circ u$  for  $\{\theta_t : t \geq 0\}$  the flow given by (13) with  $\theta_0 = id$ , i.e. the solution flow of the stochastic differential equation

$$dy_t = Z^{\circ dx_t}(y_t).$$

From this and Lemma 1.3.4 of [7] we see that  $\hat{\nabla}$  is the adjoint of the LeJan-Watanabe connection determined by the flow, so proving Theorem 3.1 above. However the present construction applies with GLM replaced by any natural bundle over  $M$  (e.g. jet bundles, see Kolar-Michor-Slovak [10]), to give semi-connections on these bundles.

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