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Gradient estimates and the smooth convergence of approximate travelling waves for reaction–diffusion equations

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Abstract. The space derivatives of Freidlin's travelling wave like solutions of generalized KPP equations are considered in this paper. We give estimates of the first two space derivatives on the wave front and show that the travelling wave is nearly flat on the trough and on the crest. Differentiation of heat semigroups, logarithmic transformation and semi-classical analysis based on stochastic analysis are the main tools used here.

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1. Introduction

A. Consider the following reaction–diffusion equation on R^n with parameter $\mu > 0$:

$$\begin{aligned}\frac{\partial u_t^\mu(x)}{\partial t} &= \frac{\mu^2}{2}\Delta u_t^\mu(x) + \frac{1}{\mu^2}c(t,x,u_t^\mu(x))u_t^\mu(x),\\ u^\mu(0,x) &= T_0(x)\exp\left\{-\frac{S_0(x)}{\mu^2}\right\}.\end{aligned} \tag{1.1}$$

Here Δ is the Laplace operator, c is a real valued measurable function, T_0 is a non-negative measurable function and S_0 is a C^2 function. We look at the behaviour of the solutions $\{u_t^\mu(x)\}$ for small values of the parameter μ. It was shown by Freidlin that under suitable conditions, with initial condition $u^\mu(0,x) = \chi_{x\leqslant 0}$, as $\mu \to 0$ the solution $u_t^\mu(x)$ to (1.1) converges to a 'travelling wave', i.e. there is a function $V(t,x)$ such that $\lim_{\mu\to 0} u_t^\mu(x) = 0$ if $V(t,x) > 0$ and $\lim_{\mu\to 0} u_t^\mu(x) = 1$ if $V(t,x) > 0$. The region $\{(t,x) : V(t,x) < 0\}$ is called the *trough* of the approximate travelling wave and $\{(t,x) : V(t,x) > 0\}$ is called the *crest*. See e.g. Freidlin [8, 9], via a stochastic approach and see Zhao and Elworthy [18], Elworthy *et al* [4] via classical mechanics and Freidlin's stochastic approach. For recent related work see Champneys *et al* [2], Karpelevich, Kelbert and Suhov [12], Evans and Sougandis [7] and Barles *et al* [1].

In this paper, we study the asymptotic behaviour of the space derivatives Du_t^μ of the solution of equation (1.1). We first use the probabilistic expression for the derivatives of the solutions in Elworthy and Li [5] to give gradient estimates of the wavefront and to conclude that $|Du_t^\mu|$ and $|D^2u_t^\mu|$ converge to zero exponentially fast on the trough. In sections 4 and 5 we assume that $c(t,x,u)$ is independent of t and x. Using the logarithmic transformation and semi-classical analysis we show that $\mu^2|\nabla \log u_t^\mu|$ is bounded on the wavefront and that $|\nabla u_t^\mu|$ is exponentially small on the crest of the travelling wave. For those results we need an assumption that $c'(u)$ is negative and some restriction on the initial functions. As an example we have global gradient estimates for the wavefront of the standard KPP equation and have shown that the approximate travelling wave for KPP equation is flat on the trough. In fact for good initial functions, including the Gaussian case, the travelling wave like solution of the KPP equation is also flat on the crest. In the companion paper Zhao [19], initial Dirac distributions are treated. See also Freidlin [10], Kolmogoroff, Petrovsky and Piscounoff [13], Sheu [15] and Evans and Sougandis [7] for related works.

B. Let $\bar{c}(x) = c(x, 0)$, assumed to be C^1. Let S_0 be a C^1 function. The function $V(t, x)$ defining the trough and the crest can be given by the classical mechanical system introduced in [2]:

$$\ddot{\Phi}_s(x) = -\nabla\bar{c}(\Phi_s(x)), \quad \Phi_0(x) = x, \quad \dot{\Phi}_0(x) = \nabla S_0(x) \quad s \geqslant 0. \tag{1.2}$$

The classical mechanical system is said to satisfy the *no caustic* condition if there exists $T > 0$ such that for $0 \leqslant s \leqslant T$, there is a solution Φ_s to (1.2) consisting of diffeomorphisms of R^n. Assume the no caustic condition. We define $V : [0, T] \times R^n \mapsto R$ by

$$V_t(x) = \int_0^t \bar{c}(\Phi_s(\Phi_t^{-1}(x)))\mathrm{d}s - S_0(\Phi_t^{-1}(x)) - \frac{1}{2}\int_0^t |\dot{\Phi}_s(\Phi_t^{-1}(x))|^2\mathrm{d}s, \quad 0 \leqslant t \leqslant T, \tag{1.3}$$

which solves the Hamilton–Jacobi equation

$$\frac{1}{2}|\nabla V_t(x)|^2 + \bar{c}(x) = \frac{\partial V_t(x)}{\partial t} \qquad 0 \leqslant t \leqslant T \tag{1.4}$$

with initial condition $-S_0$ (see [3]). We quote a result from [18]; it is a variation on Freildlin's results [8]. First are the conditions:

Condition (N)**.** *If* $V_t(x) = 0$ *then for* $s \in (0, t)$,

$$V_{t-s}(\Phi_{t-s}(\Phi_t^{-1}(x))) < 0. \tag{1.5}$$

The standard KPP conditions:

(I). Suppose $\bar{c}$ *is continuous, bounded above and* $c(x, u) \leqslant c(x, 0) = \bar{c}(x)$ *when* $u \geqslant 0$.

(II). $c(x, u) > 0$ *for* $0 < u < 1$ *and* $c(x, u) < 0$ *for* $u > 1$.

Theorem 1.1. *([18]) Assume condition (I) and the no caustic condition for* $0 \leqslant t \leqslant T$ *and that* c *and* S_0 *are* C^2 *with* S_0 *non-negative, and that* T_0 *is positive, continuous and bounded. Then for any compact subset* $\mathcal{K}$ *in* $\{(t, x) : V_t(x) < 0, 0 < t \leqslant T\}$*, there exist* $\delta(\mathcal{K}) > 0$ *and* $\mu_0(\mathcal{K}) > 0$ *such that for any* $0 < \mu < \mu_0$ *and* $(t, x) \in \mathcal{K}$,

$$u^\mu(t, x) < \exp\left\{-\frac{\delta}{\mu^2}\right\}. \tag{1.6}$$

If we also assume (II), (N)*, then*

$$\lim_{\mu\to 0} u_t^\mu(x) = 1, \tag{1.7}$$

uniformly on any compact subset of $\{(t, x) : V_t(x) > 0, 0 < t \leqslant T\}$.

Remarks. (i) The no caustic condition in theorem 1.1 can be replaced by a late caustic assumption, (see [4] for more details). It was also shown in [4] how a solution of a linear equation goes wrong at the caustic time and a solution of the corresponding nonlinear KPP equation is well defined for all time $t \geqslant 0$. Nevertheless the wavefront is infinitely fast due to the caustics. A similar phenomenon was also discussed in Harris and Williams [11].

(ii) From the proof given in [18] we see that the statment T_0 is bounded can be replaced by the statment 'u_0^μ is bounded uniformly in μ.'

(iii). For the case with drift $Z(x)$, we need semi-classical analysis for vector potentials, (see Truman and Zhao [17]). Freidlin [8] also discussed the wavefront with vector potentials. We consider the case $Z(x) \equiv 0$ except in section 2.

For simplicity, we sometimes write $v_t(x)$ for a function $v(t, x)$ of space and time.

2. Probabilistic expressions for the derivatives of solutions and some gradient estimates on the wavefront

Let c be a real valued function on $[0, \infty) \times R^n \times R^1$. Throughout this section let Z be a C^3 vector field such that $\langle x, Z(x)\rangle \leqslant k(1 + |x|^2)$ and $\langle DZ(x)(v), v\rangle \leqslant k[1 + \ln(1 + |x|)]|v|^2$ for some constant k. Consider the reaction–diffusion equation (1.1) with initial value u_0^μ.

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= \frac{1}{2}\mu^2 \Delta u(t,x) + \langle Z(x), \nabla u(t,x)\rangle + \frac{1}{\mu^2} c\,(t, x, u(t,x))\, u(t,x)\\ u(0,x) &= u_0^\mu(x).\end{aligned} \tag{2.1}$$

Let u_0^μ be a bounded measurable function and u_t^μ a solution to (2.1). Here and in the following by *a solution* we mean a regular solution i.e. one which is C^2 in x and C^1 in t. As is known such a solution is given by the generalized solution (see e.g. Freidlin [8]):

$$u_t^\mu(x_0) = E u_0^\mu(x_t) \exp\left\{\frac{1}{\mu^2}\int_0^t c(t-s, x_s, u_{t-s}^\mu(x_s))\mathrm{d}s\right\}, \tag{2.2}$$

where $\{x_t \equiv F_t(x_0)\}$ is the solution from $x_0 \in R^n$ of the associated stochastic differential equation:

$$\mathrm{d}x_t = \mu \mathrm{d}B_t + Z(x_t)\mathrm{d}t, \tag{2.3}$$

for $\{B_t\}$ a R^n-valued Brownian motion on a given probability space $(\Omega, \mathcal{F}, P)$. Under the assumptions on Z this SDE does not explode.

There is also a functional integral expression for the derivatives of u_t^μ. Let v_t be the solution, from $v_0 \in R^n$, to

$$Dv_t = DZ(x_t)(v_t)\mathrm{d}t, \tag{2.4}$$

and let

$$G_t^\mu(x) = c(t, x, u^\mu(t,x))u^\mu(t,x) \tag{2.5}$$

so $\frac{1}{\mu^2}G_t^\mu(x)$ is the potential term in equation (2.1).

The following comes from the linear version suggested by Da Prato, 'note added in proof' in [5]. Let $\mu > 0$ be a fixed number.

Proposition 2.1. *Suppose $c(t, x, u)$ is bounded above, Hölder continuous in t and Lipschitz continuous in x,u on each time interval $[0, T] \times R^n \times R^1$. Let u_0^μ be a bounded measurable function. If u_t^μ is a $C^{2,1}$ solution of (2.1), then for $t > 0$*

$$\begin{aligned} Du_t^\mu(x_0)(v_0) &= \frac{1}{t}\frac{1}{\mu} Eu_0^\mu(x_t) \int_0^t \langle \mathrm{d}B_s, v_s\rangle \\ &\quad + \frac{1}{\mu^3}\int_0^t \frac{\mathrm{d}s}{t-s} EG_s^\mu(x_{t-s}) \int_0^{t-s} \langle \mathrm{d}B_r, v_r\rangle. \end{aligned} \tag{2.6}$$

Proof. Let $P_t u_0$ be the solution to the equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\mu^2 \Delta u(t,x) + \langle Z(x), \nabla u(t,x)\rangle$$

starting from u_0. Then $P_t u_0(x_0) = Eu_0(x_t)$ for all u_0 bounded measurable. Furthermore $P_t u_0$ is a regular solution and $DP_t u_0$ is given by (see theorem 2.1 in [5]):

$$D(P_t u_0)(x_0)(v_0) = \frac{1}{t}\frac{1}{\mu} Eu_0(x_t) \int_0^t \langle v_s, \mathrm{d}B_s\rangle. \tag{2.7}$$

On the other hand by the variation of constant formula u_t^μ satisfies:

$$u_t^\mu(x) = P_t u_0^\mu(x) + \frac{1}{\mu^2}\int_0^t P_{t-s}\left(c(s,\cdot,u_s^\mu(\cdot))u_s^\mu(\cdot)\right)(x)\mathrm{d}s. \tag{2.8}$$

Let $\{u_t^n\}$ be a sequence of functions approximating u_t^μ, given by:

$$u_t^n(x) = P_t u_0^\mu(x) + \frac{1}{\mu^2}\int_0^{t-1/n} P_{t-s}\left((c(s,\cdot,u_s^\mu(\cdot))u_s^\mu(\cdot)\right)(x)\mathrm{d}s. \tag{2.9}$$

Then each u_t^n is $C^{2,1}$ and converges to u_t^μ as n goes to infinity. Furthermore by (2.7)

$$\begin{aligned} Du_t^n(x_0)(v_0) &= DP_t u_0^\mu(x_0)(v_0) \\ &\quad + \frac{1}{\mu^3}\int_0^{t-1/n} \frac{\mathrm{d}s}{t-s} Ec(s, x_{t-s}, u_s^\mu(x_{t-s}))u_s^\mu(x_{t-s}) \int_0^{t-s} \langle \mathrm{d}B_r, v_r\rangle, \end{aligned}$$

Since $\sup_{x_0}\sup_{0\leqslant r\leqslant T} E|v_r|^2$ is bounded under the assumptions on Z [14], we see that the right-hand side of the above equation converges locally uniformly. It turns out that

$$\begin{aligned} &Du_t^\mu(v_0) = DP_t u_0^\mu(x_0)(v_0) \\ &+ \frac{1}{\mu^3}\int_0^t \frac{\mathrm{d}s}{t-s} Ec(s, x_{t-s}, u_s^\mu(x_{t-s}))u_s^\mu(x_{t-s}) \int_0^{t-s} \langle \mathrm{d}B_r, v_r\rangle. \end{aligned}$$

The required formula follows from (2.7) and the above equality. □

One of the remarkable properties of this formula is that it does not involve Du_0 or Dc. From here we can obtain a crude gradient estimate for the wavefront of the approximate travelling wave solution u_t^μ as μ goes to zero. Note that if the KPP condition (II) holds and if $\{u_0^\mu\}$ are non-negative and uniformly bounded in μ, then $\{u^\mu(t,x), \mu > 0\}$ are bounded uniformly in μ. And so are $\{G_t^\mu(x), \mu > 0\}$.

Corollary 2.2. *Suppose $\{G_t^\mu(x) : \mu > 0\}$ and $\{u_0^\mu : \mu > 0\}$ are uniformly bounded in μ on $[0,T] \times R^n$. Then under the conditions of proposition 2.1, for each $t \in (0,T]$ and μ small,*

$$\sup_{x\in R^n} \mu^3 |Du_t^\mu(x)| \leqslant k\left(\frac{\mu^2}{\sqrt{t}} + 1\right) \tag{2.10}$$

Here k is a constant independent of μ and t.

Proof. The estimate follows from (2.6) after taking account of the boundedness of $\sup_{x_0} \sup_{0\leqslant r\leqslant t} E|v_r|^2$. □

For further estimates of this type see theorem 4.3 below.

In the same spirit we have, let $v_0^1, v_0^2 \in R^n$, $v_t^1 = T_{x_0}F_t(v_0^1)$, $v_t^2 = T_{x_0}F_t(v_0^2)$ and $\mu > 0$ a fixed number:

Proposition 2.3. *Further to the conditions of proposition 2.1, suppose that c has first bounded derivatives in x and u and that $|DZ| + |D^2Z|$ are bounded. Let Dc_s^μ be the total derivative of $c(s,x,u_{t-s}^\mu(x))$ in x. Then for $t > 0$,*

$$\begin{aligned}
D^2u_t^\mu(x_0)(v_0^2, v_0^1) &= D^2(P_tu_0^\mu)(x_0)(v_0^2, v_0^1)\\
&\quad + \frac{1}{\mu^3}\int_0^t \frac{1}{t-s} EDc_s^\mu(x_{t-s})(v_{t-s}^2)u_s^\mu(x_{t-s})\int_0^{t-s}\langle \mathrm{d}B_r, v_r^1\rangle\\
&\quad + \frac{1}{\mu^4}\int_0^t \frac{\mathrm{d}s}{(t-s)s} Ec(s,x_{t-s},u_s^\mu(x_{t-s}))u_0^\mu(x_t)\int_{t-s}^t \langle \mathrm{d}B_r, v_r^2\rangle \int_0^{t-s}\langle \mathrm{d}B_r, v_r^1\rangle\\
&\quad + \frac{1}{\mu^6}\int_0^t \frac{\mathrm{d}s}{t-s} Ec(s,x_{t-s},u_s^\mu(x_{t-s}))\int_0^{t-s}\langle \mathrm{d}B_\beta, v_\beta^1\rangle\\
&\quad\quad \times \int_0^s \frac{dr}{s-r} G_r^\mu(x_{t-r})\int_{t-s}^{t-r}\langle \mathrm{d}B_\alpha, v_\alpha^2\rangle\\
&\quad + \frac{1}{\mu^3}\int_0^t \frac{\mathrm{d}s}{t-s} EG_s^\mu(x_{t-s})\int_0^{t-s}\langle \mathrm{d}B_r, D^2F_r(v_0^2, v_0^1)\rangle.
\end{aligned}$$

Proof. This follows from differentiating (2.6). □

Note that by a formula in [5],

$$\begin{aligned}
D^2(P_tu_0^\mu)(v_0^2, v_0^1) &= \frac{4}{t^2}\frac{1}{\mu^2}E\left(u_0^\mu(x_t)\int_{\frac{t}{2}}^t \langle v_s^1, \mathrm{d}B_s\rangle \int_0^{\frac{t}{2}}\langle v_s^2, \mathrm{d}B_s\rangle\right)\\
&\quad + \frac{2}{t}\frac{1}{\mu}E\left(u_0^\mu(x_t)\int_0^{\frac{t}{2}}\langle D^2F_s(v_0^2, v_0^1), \mathrm{d}B_s\rangle\right).
\end{aligned} \tag{2.11}$$

From (2.11) and proposition 2.3 we conclude that

Corollary 2.4. *Assume KPP condition (II) and the conditions of proposition 2.3. Suppose that* $\{u_0^\mu : \mu > 0\}$ *are bounded uniformly in* μ *on* $[0, T] \times R^n$*. Then for each* $t \in (0, T]$ *there is a constant* k *such that for all* μ

$$\sup_{x \in R^n} \mu^6 |D^2 u_t^\mu(x)| \leqslant k \left(\frac{\mu^4}{t} + 1 \right). \tag{2.12}$$

This follows from a similar argument to that used in the proof of corollary 2.2.

There are further estimates on the wavefront in section 4. The estimates in this section are not very delicate especially on the trough and on the crest. In sections 3 and 4 we shall mainly study the gradients on the trough and in section 5 we shall devote to the study on the crest and show that $|\nabla u_t^\mu|$ is exponentially small as μ goes to zero.

3. Gradient estimates for u_t^μ at the trough of the travelling waves

Assume the potential term in equation (2.1) does not depend on t explicitly so $c(t, x, u) = c(x, u)$. Take the initial function u_0^μ to be bounded and of the following form

$$u_0^\mu(x) = T_0(x) \exp\left\{ -\frac{1}{\mu^2} S_0(x) \right\} \tag{3.1}$$

for $T_0 : R^n \to R$ a non-negative measurable function and $S_0 : R^n \to R$ a C^2 function. So we are considering:

$$\begin{aligned} \frac{\partial u^\mu(t, x)}{\partial t} &= \frac{\mu^2}{2} \Delta u^\mu(t, x) + \frac{1}{\mu^2} c(x, u^\mu(t, x)) u^\mu(t, x) \\ u^\mu(0, x) &= T_0(x) \exp\left\{ -\frac{1}{\mu^2} S_0(x) \right\}. \end{aligned} \tag{3.2}$$

Under certain conditions $u_t^\mu(x)$ converges to zero exponentially fast (cf theorem 1.1) in the region $\{(t, x) : V(t, x) < 0\}$. Here we show that $|\nabla u_t^\mu|$ also converges to zero at the trough with virtually no extra conditions and essentially the same proof. Recall that $V(t, x)$ is defined by (1.3).

Let $\bar{c}(x) = c(x, 0)$. Assume KPP condition (I), that is $\bar{c}$ is C^1 and bounded above, and $c(x, u) \leqslant \bar{c}(x)$.

Theorem 3.1. *Let* $c(x, u)$ *be a* C^2 *function bounded above, Lipschitz continuous in both variables and satisfying the KPP condition (I) and the no caustic condition. Suppose* $\bar{c} \geqslant 0$ *and* S_0 *is* C^3*. Assume that, for some constant* k_0*,*
(i) $|T_0(x)| \leqslant k_0(1 + |x|)$ *and* $|c(x, u_t^\mu(x))| \leqslant k_0$ *for* $\mu \leqslant 1$ *on* $[0, T] \times R^n$*.*
(ii) $|\nabla V(t, x)| \leqslant k_0(1 + |x|)$ *and* $|\Delta V(t, x)| \leqslant k_0$ *on* $[0, T] \times R^n$*.*

Then if $u_t^\mu(x)$ *is a regular solution, there is a constant* k *such that for small enough* $\mu > 0$*, and* $(t, x) \in (0, T] \times R^n$*,*

$$|Du_t^\mu(x)| \leqslant k \left(1 + \frac{1}{\sqrt{t}\mu^4} \right) \exp\left\{ \frac{1}{\mu^2} V(t, x) \right\}. \tag{3.3}$$

In particular $|Du_t^\mu(x)|$ *converges to zero uniformly and exponentially as* $\mu \to 0$, *on compact subsets of* $\{(t,x) : V(t,x) < 0\}$.

Proof. First we rewrite formulae (2.6) in the following form:

$$\begin{aligned}Du_t^\mu(x_0)(v_0) &= \frac{1}{t}\frac{1}{\mu} E u_0^\mu(x_t) \int_0^t \langle \mathrm{d}B_s, v_s\rangle \\ &\quad + \frac{1}{\mu^3}\int_0^t \frac{\mathrm{d}s}{t-s} E c(x_{t-s}, u_s^\mu(x_{t-s})) \times u_0^\mu(x_t) \\ &\quad \times \exp\left\{\frac{1}{\mu^2}\int_{t-s}^t c(x_r, u_{t-r}^\mu(x_r))\mathrm{d}r\right\}\int_0^{t-s}\langle \mathrm{d}B_r, v_r\rangle.\end{aligned}$$

This formula can be simplified by the following transform:

$$\mathrm{d}y_s = \mu \mathrm{d}B_s + \nabla V_{t-s}(y_s)\mathrm{d}s, \tag{3.4}$$

which has no explosion from the linear growth of $|\nabla V(t,x)|$. Let

$$\mathcal{M}_t = \exp\left\{-\frac{1}{\mu}\int_0^t \langle \nabla V_{t-s}(y_s), \mathrm{d}B_s\rangle - \frac{1}{2\mu^2}\int_0^t |\nabla V_{t-s}(y_s)|^2\,\mathrm{d}s\right\}. \tag{3.5}$$

Using (1.4), it was shown in [18]:

$$\mathcal{M}_t = \mathrm{e}^{\left(\frac{S_0(y_t)}{\mu^2}\right)}\mathrm{e}^{\left(\frac{V(t,x)}{\mu^2}\right)}\mathrm{e}^{\left(-\frac{1}{2}\int_0^t \Delta V_{t-s}(y_s)\mathrm{d}s\right)}\mathrm{e}^{\left(-\frac{1}{\mu^2}\int_0^t \bar{c}(y_s)\mathrm{d}s\right)}. \tag{3.6}$$

Note in this case

$$v_t = v_0.$$

The Maruyama–Girsanov–Cameron–Martin formula gives:

$$\begin{aligned}Du_t^\mu(x_0)(v_0) &= \frac{1}{t}\frac{1}{\mu^2} E u_0^\mu(y_t)\cdot \mathcal{M}_t \cdot \left(\langle \mu B_t, v_0\rangle + \int_0^t \langle \nabla V_{t-s}(y_s), v_0\rangle \mathrm{d}s\right) \\ &\quad + \frac{1}{\mu^4}\int_0^t \frac{\mathrm{d}s}{t-s}\cdot E\mathcal{M}_t c(y_{t-s}, u_s^\mu(y_{t-s}))u_0(y_t) \\ &\quad \times \exp\left\{\frac{1}{\mu^2}\int_{t-s}^t c(y_r, u_{t-r}^\mu(y_r))\mathrm{d}r\right\}(\langle \mu B_{t-s}, v_0\rangle + \int_0^{t-s}\langle \nabla V_{t-r}(y_r), v_0\rangle \mathrm{d}r).\end{aligned}$$

Let

$$N_t = T_0(y_t)\exp\left\{-\frac{1}{2}\int_0^t \Delta V_{t-s}(y_s)\mathrm{d}s\right\}.$$

Then EN_t^2 is finite and

$$\begin{aligned}&Du_t^\mu(x_0)(v_0)\exp\left\{-\frac{V(t,x)}{\mu^2}\right\} \\ &= \frac{1}{t}\frac{1}{\mu^2}E\left\{N_t \exp\left\{-\frac{1}{\mu^2}\int_0^t \bar{c}(y_r)\mathrm{d}r\right\}\cdot\left(\langle \mu B_t, v_0\rangle + \int_0^t \langle \nabla V_{t-s}(y_s), v_0\rangle \mathrm{d}s\right)\right\} \\ &+ \frac{1}{t}\frac{1}{\mu^4}E\left\{N_t\int_0^t \frac{\mathrm{d}s}{t-s} c(y_{t-s}, u_s^\mu(y_{t-s}))\cdot\exp\left\{\frac{1}{\mu^2}\int_{t-s}^t [c(y_r, u_{t-r}(y_r)) - \bar{c}(y_r)]\mathrm{d}r\right\}\right. \\ &\left.\times\exp\left\{-\frac{1}{\mu^2}\int_0^{t-s}\bar{c}(y_r)\mathrm{d}r\right\}\cdot\left(\langle \mu B_{t-s}, v_0\rangle + \int_0^{t-s}\langle \nabla V_{t-r}(y_r), v_0\rangle \mathrm{d}r\right)\right\}.\end{aligned} \tag{3.7}$$

Therefore,

$$
\begin{aligned}
&|Du_t^\mu(x_0)(v_0)| \exp\left\{-\frac{V(t,x)}{\mu^2}\right\} \\
&\qquad \leqslant \frac{1}{t}\frac{1}{\mu^2} E N_t\left(\langle \mu B_t, v_0\rangle + \int_0^t \langle \nabla V_{t-s}(y_s), v_0\rangle \mathrm{d}s\right) \\
&\qquad + \frac{k_0}{t}\frac{1}{\mu^4} E\left\{N_t \int_0^t \frac{\mathrm{d}s}{t-s}\left(\langle \mu B_{t-s}, v_0\rangle + \int_0^{t-s} \langle \nabla V_{t-r}(y_r), v_0\rangle \mathrm{d}r\right)\right\}.
\end{aligned}
$$

However

$$E\int_0^t |\nabla V_{t-s}(y_s)|^2 \mathrm{d}s < \infty.$$

So there is a constant k such that for small enough μ, and all $(t,x) \in (0,T] \times R^n$,

$$|Du_t^\mu(x_0)| \exp\left\{-\frac{1}{\mu^2} V(t,x)\right\} \leqslant k\left(1 + \frac{1}{\sqrt{t}\mu^4}\right).$$

□

Remarks. (i) Note that $\{c(x, u_t^\mu(x)) : \mu > 0\}$ are uniformly bounded if either of the following holds: (a) $c(x,u) : R^n \times R^1 \to R$ is a bounded function (b) c is bounded above and satisfies the KPP condition (II). Also T_0 is non-negative and S_0 is bounded from below, u_0^μ is bounded uniformly in μ. This is because under those conditions $\{u_t^\mu : \mu > 0\}$ are uniformly bounded and so are $\{c(x, u_t^\mu(x)) : \mu > 0\}$.

(ii) The condition on the boundedness of $\Delta V(t,x)$ is unnecessary. It can be removed using an argument in Elworthy and Zhao [6].

(iii) Applying the above transform to the formulae for $D^2 u_t^\mu$ we see that under suitable conditions

$$\lim_{\mu\to 0} |D^2 u_t^\mu(x)| = 0$$

on the trough.

4. Further estimates on the wavefront and on the trough

A. Let c be a real-valued C^1 function bounded above, $S_0 : R^n \to R$ be a C^2 function bounded from below and T_0 a strictly positive C^1 function on R^n. Assume $u^\mu(0,x) = T_0(x)\exp\left\{-\frac{1}{\mu^2}S_0(x)\right\}$ is bounded. Consider

$$
\begin{aligned}
\frac{\partial u^\mu(t,x)}{\partial t} &= \frac{\mu^2}{2}\Delta u^\mu(t,x) + 1\mu^2 c(u^\mu(t,x))u^\mu(t,x) \\
u^\mu(0,x) &= T_0(x)\exp\left\{-\frac{1}{\mu^2}S_0(x)\right\}.
\end{aligned}
\tag{4.1}
$$

Let $u^\mu(t,x)$ be a regular solution to (4.1). Set $v^\mu(t,x) = Du^\mu(t,x)$. We observe that $v^\mu(t,x)$ satisfies:

$$\begin{aligned}\frac{\partial}{\partial t}v_t^\mu(x) &= \frac{1}{2}\mu^2\Delta v_t^\mu(x) + \frac{1}{\mu^2}[c(u_t^\mu(x)) + c'(u_t^\mu(x))u_t^\mu(x)]v_t^\mu(x),\\ v_0^\mu(x) &= Du_0(x).\end{aligned} \tag{4.2}$$

Now $u_t^\mu(x)$ is bounded and so therefore is $c'(u_t^\mu(x))u_t^\mu(x)$. Assuming that $|Du_0^\mu|$ is bounded, then by the Feynman–Kac formula we can write down the solution to (4.3) explicitly:

$$\begin{aligned}&v_t^\mu(x_0) = Du_t^\mu(x)\\ =&\frac{1}{\mu^2}EDu_0^\mu(x_t)\exp\left\{\frac{1}{\mu^2}\int_0^t[c(u_{t-s}^\mu(x_s)) + c'(u_{t-s}^\mu(x_s))u_{t-s}^\mu(x_s)]\mathrm{d}s\right\}.\end{aligned} \tag{4.3}$$

Here $x_s = x + \mu B_s$.

B. In the following we shall apply a logarithmic transformation to (4.1) in order to give some uniform estimate on $|Du_t^\mu(x)|$. Set

$$J^\mu(t,x) = -\mu^2\log u^\mu(t,x).$$

Then J^μ is $C^{2,1}$ and satisfies the nonlinear Hamilton–Jacobi–Bellman equation:

$$\frac{\partial}{\partial t}J^\mu(t,x) + \frac{1}{2}|\nabla J^\mu(t,x)|^2 + c(u^\mu(t,x)) = \frac{1}{2}\mu^2\Delta J^\mu(t,x), \tag{4.4}$$

with $J^\mu(0,x) = -\mu^2\log T_0(x) + S_0(x)$, For $s \leqslant t$ consider

$$\mathrm{d}z_s^\mu = \mu\mathrm{d}B_s - \nabla J^\mu(t-s, z_s^\mu)\mathrm{d}s \quad z_0^\mu = x. \tag{4.5}$$

We shall first give estimates of $|\nabla u_t^\mu|$ for each μ to conclude that (4.5) does not explode up to time t, and then use (4.5) to simplify (4.3).

Let $t > 0$ be fixed. We solve (1.2) to get $\Phi_s(x) = x + s\nabla S_0(x)$. Assume that each $\Phi_s : R^n \to R^n$ is a diffeomorphism for $0 \leqslant s \leqslant T$, for some $T > 0$ (i.e. the no caustic condition holds). Then by (1.3)

$$V_t(x) = c(0)t - S_0(\Phi_t^{-1}(x)) - \frac{t}{2}|\nabla S_0(\Phi_t^{-1}(x))|^2.$$

Set, for $0 \leqslant t \leqslant T$,

$$\phi_t(x) = |\det\nabla\Phi_t^{-1}(x)|$$

and

$$\psi_t(x) = T_0(\Phi_t^{-1}(x))\sqrt{\phi_t(x)}.$$

Note that $\phi_t(x) > 0$ from the assumption and

$$\nabla\Phi_t^{-1}(x) = \left[Id + tD(\nabla S_0(\Phi_t^{-1}(x)))\right]^{-1}$$

Let BC^1 be the space of bounded C^1 functions with bounded first derivatives, and

$$T_d^\mu(x) = \mu^2 DT_0(x) - T_0(x) DS_0(x) \tag{4.6}$$

so that

$$Du_0^\mu(x) = \frac{1}{\mu^2} T_d^\mu(x) \exp\left\{-\frac{S_0(x)}{\mu^2}\right\}.$$

Lemma 4.1. *Assume that* $c \in C^1$ *and bounded above and* $u_0 = T_0 \exp\left\{-\frac{S_0}{\mu^2}\right\}$ *is* BC^1 *with* $T_0 > k_1 > 0$. *Let* $\mu \in (0, 1]$. *Then*

(i) If $||T_d^\mu||_\infty \leqslant k_2$, *then on each* $[0, t] \times R^n$,

$$|\mu^2 D \log u_s^\mu(x)| \leqslant \frac{k_2}{k_1} \mathrm{e}^{\frac{ks}{\mu^2}}.$$

Here k *is a constant such that* $c'(u_s(x))u_s(x) \leqslant k$. *If furthermore* $c'(u) \leqslant 0$, *then*

$$|\mu^2 D \log u_s^\mu(x)| \leqslant \frac{k_2}{k_1}.$$

(ii) Assume c *is* C^2 *and satisfies the no caustic condition, and* S_0 *is* C^3. *Suppose that* $\Delta V(s, x)$ *is uniformly bounded and* $|\nabla V(s, x)| \leqslant k_0(1+|x|)$ *on* $[0, t] \times R^n$ *for some constant* k_0. *Then on* $[0, t]$

$$|\mu^2 D \log u_s^\mu(x)| \leqslant \left(\frac{1}{k_1}\right)^{\frac{1}{q}} \mathrm{e}^{\frac{k_3 s}{\mu^2}} (1 + |x|),$$

if the function $(\frac{|T_d^\mu|^q}{T_0{}^{q-1}})^{\frac{1}{q}}$ *has linear growth for some* $q > 1$. *Here* k_3 *is a constant.*

(iii) Assume c *is* C^2 *and satisfies the no caustic condition, and* S_0 *is* C^3. *Suppose there is a number* $k_0 > 0$ *such that* $|\Delta V(s, x)| + |\Delta \psi_s/\psi_s| \leqslant k_0$ *and* $|\nabla V(s, x)| + |\nabla \log \psi_s(x)| + \frac{|T_d^\mu|}{T_0}(x) \leqslant k_0(1 + |x|)$ *on* $[0, t] \times R^n$ *for each* $\mu > 0$. *Then there is a constant* k *such that*

$$|\mu^2 D \log u_s^\mu(x)| \leqslant k \mathrm{e}^{\frac{kT}{\mu^2}} (1 + |x|).$$

In particular (4.5) has no explosion under any of the conditions.

Proof. (i) By Feynman–Kac formulae (2.2) for u_t,

$$\begin{aligned} u_s^\mu(x_0) &= ET_0(x_s) \exp\left\{-\frac{S_0(x_s)}{\mu^2}\right\} \exp\left\{\int_0^s \frac{1}{\mu^2} c(u_{s-r}(x_r))\mathrm{d}r\right\} \\ &\geqslant k_1 E \exp\left\{-\frac{S_0(x_s)}{\mu^2}\right\} \exp\left\{\int_0^s \frac{1}{\mu^2} c(u_{s-r}(x_r))\mathrm{d}r\right\}. \end{aligned}$$

And by (4.3) we have for $0 \leqslant s \leqslant t$,

$$|Du_s^\mu(x)| \leqslant \frac{1}{\mu^2} \cdot k_2 \mathrm{e}^{\frac{ks}{\mu^2}} E \exp\left\{-\frac{S_0(x_s)}{\mu^2}\right\} \exp\left\{\int_0^s \frac{1}{\mu^2} c(u_{s-r}(x_r))\mathrm{d}r\right\}.$$

Therefore,

$$|D\log u^{\mu}(s,x)| \leqslant \frac{1}{\mu^2}\frac{k_2}{k_1}\mathrm{e}^{\frac{ks}{\mu^2}}.$$

So $\mu^2|D\log u^{\mu}(s,x)|$ is bounded on $[0,t]\times R^n$ and (4.5) has no explosion.

(ii) Write $f = u_0(x_t)\exp\left\{\frac{1}{\mu^2}\int_0^t c(u^{\mu}_{t-s}(x_s))\mathrm{d}s\right\}$ and

$$g = \frac{T^{\mu}_d}{T_0}(x_t)\exp\left\{\frac{1}{\mu^2}\int_0^t c'(u^{\mu}_{t-s}(x_s))u^{\mu}_{t-s}(x_s)\mathrm{d}s\right\},$$

so that $Du^{\mu}_t(x) = \frac{1}{\mu^2}Efg$. Then for any conjugate numbers $p, q > 1$,

$$\begin{aligned}
Du^{\mu}_t(x) &= \frac{1}{\mu^2}Ef^{\frac{1}{p}}f^{\frac{1}{q}}g \leqslant \frac{1}{\mu^2}(Ef)^{\frac{1}{p}}(Efg^q)^{\frac{1}{q}}\\
&= \frac{1}{\mu^2}u(t,x)^{\frac{1}{p}}\left[E\frac{(T^{\mu}_d)^q}{T_0^{q-1}}(x_t)\exp\left\{-\frac{S_0(x_t)}{\mu^2}\right.\right.\\
&\quad\left.\left.+\frac{1}{\mu^2}\int_0^t[c(u^{\mu}_{t-s}(x_s)) + qc'(u^{\mu}_{t-s}(x_s))u^{\mu}_{t-s}(x_s)]\mathrm{d}s\right\}\right]^{\frac{1}{q}}.
\end{aligned}\tag{4.7}$$

Let $\{y_t\}$ be defined by (3.3). Applying the Girsanov transform respectively to (4.7) and to the Feyman–Kac formula (2.2) for u^{μ}_t we see that

$$\begin{aligned}
\mu^2\frac{Du^{\mu}_t(x)}{u(t,x)^{\frac{1}{p}}} &= \exp^{\frac{1}{q\mu^2}V(t,x)}\left[E\exp\left\{-\frac{1}{2}\int_0^t \Delta V_{t-s}(y_s)\mathrm{d}s\right\}\right.\\
&\quad\times\frac{(T^{\mu}_d)^q}{T_0^{q-1}}(y_t)\exp\left\{\frac{1}{\mu^2}\int_0^t[c(u^{\mu}_{t-s}(y_s)) - c(0)\right.\\
&\quad\left.\left.+ qc'(u^{\mu}_{t-s}(y_s))u^{\mu}_{t-s}(y_s)]\mathrm{d}s\right\}\right]^{\frac{1}{q}}
\end{aligned}\tag{4.8}$$

and

$$\begin{aligned}
u^{\mu}_t(x) = &\exp\left\{\frac{1}{\mu^2}V(t,x)\right\}E\left[T_0(y_t)\exp\left\{-\frac{1}{2}\int_0^t \Delta V_{t-s}(y_s)\mathrm{d}s\right\}\right.\\
&\left.\exp\left\{\frac{1}{\mu^2}\int_0^t[c(u^{\mu}_{t-s}(y_s)) - c(0)]\mathrm{d}s\right\}\right].
\end{aligned}\tag{4.9}$$

In particular there is a lower bound for $u^{\mu}_t(x)$:

$$u^{\mu}_t(x) \geqslant k_1\mathrm{e}^{-\frac{kt}{\mu^2}}\mathrm{e}^{-\frac{kt}{2}}\exp\left\{\frac{1}{\mu^2}V(t,x)\right\}.\tag{4.10}$$

Here k is a constant such that $|\Delta V_t(x)| \leqslant k$ and $c(u^{\mu}_{t-s}(x)) - c(0) > -k$. And by (4.8)

$$\mu^2\frac{|Du^{\mu}_t(x_0)|}{u^{\mu}_t(x)} \leqslant \mu^2\frac{|Du^{\mu}_t(x_0)|}{u^{\mu}_t(x)^{\frac{1}{p}}}\cdot\frac{1}{u^{\mu}_t(x)^{\frac{1}{q}}} \leqslant \left(\frac{1}{k_1}\right)^{\frac{1}{q}}\exp\left\{\frac{k_3 t}{\mu^2}\right\}\left(E\left|\frac{(T^{\mu}_d)^q}{T_0^{q-1}}(y_t)\right|\right)^{\frac{1}{q}}$$

for some constant $k_3 > 0$. Since $|\nabla V(t,x)|$ has linear growth we see that

$$\mu^2 \frac{|Du_t^\mu(x)|}{u_t^\mu(x)} \leqslant \left(\frac{1}{k_1}\right)^{\frac{1}{q}} \exp\left\{\frac{k_3 t}{\mu^2}\right\}(1+|x|),$$

and (4.5) has no explosion.

(iii) This is proved by essentially the same method as used in the proof (ii). The only difference is that we add an extra drift in the Girsanov transform to (4.3) and (2.2):

$$\mathrm{d}y_s^\mu = \mu \mathrm{d}B_s + \nabla V(t-s, y_s)\mathrm{d}s + \mu^2 \nabla \log \psi_{t-s}(y_s)\mathrm{d}s.$$

Let $\{\bar{y}_t\}$ be the solution to the above SDE starting from x. Then for any continuous $F : C([0,t] \to R^n) \to R$ (for the proof see [16]):

$$EF(x.)T_0(x_t)\mathrm{e}^{-\frac{S_0(x_t)}{\mu^2}} = \mathrm{e}^{\frac{V_t(x)}{\mu^2}} \psi_t(x) EF(\bar{y}.)\mathrm{e}^{-\frac{c(0)t}{\mu^2}} \exp\left\{\frac{\mu^2}{2}\int_0^t \frac{\Delta\psi_{t-s}(\bar{y}_s)}{\psi_{t-s}(\bar{y}_s)}\mathrm{d}s\right\}. \tag{4.11}$$

So

$$\frac{Du_t^\mu(x)}{u_t^\mu(x)} = \frac{1}{\mu^2} \times \frac{E\frac{T_d^\mu(\bar{y}_t^\mu)}{T_0(\bar{y}_t^\mu)} \exp\left\{\frac{1}{\mu^2}\int_0^t [c(u_{t-s}^\mu(\bar{y}_s)) + c'(u_{t-s}^\mu(\bar{y}_s))u_{t-s}^\mu(\bar{y}_s) + \frac{\mu^4}{2}\frac{\Delta\psi_{t-s}(\bar{y}_s)]}{\psi_{t-s}(\bar{y}_s)}\mathrm{d}s\right\}}{E\exp\left\{\frac{1}{\mu^2}\int_0^t [c(u_{t-s}^\mu(\bar{y}_s)) + \frac{\mu^4}{2}\frac{\Delta\psi_{t-s}(\bar{y}_s)}{\psi_{t-s}(\bar{y}_s)}]\mathrm{d}s\right\}}.$$

The required estimate follows since $\{c(u_{t-s}^\mu(x))\}$ and $\{c'(u_{t-s}^\mu(x))\}$ are bounded on $[0,t]\times R^n$ for each μ. □

Observations

Let c be a C^2 function satisfying the KPP condition (I). Then by the Girsanov transform in the proof of lemma 4.1 we obtain results on the flatness of the approximate travelling waves on the trough. More precisely,

(1) Assume condition (ii) of lemma 4.1 (this does not include that $\frac{|T_d^\mu|^q}{T_0^{q-1}}$ has linear growth). Then by (4.9)

$$|u_t^\mu(x)| \leqslant k \exp\left\{\frac{V(t,x)}{\mu^2}\right\} E|T_0(y_t)|,$$

And if furthermore $|Du_0^\mu|$ is bounded and $c'(u) \leqslant 0$, then by the same Girsanov transform to (4.3) we obtain:

$$|Du_t^\mu(x)| \leqslant k\frac{1}{\mu^2}\exp\left\{\frac{V(t,x)}{\mu^2}\right\} E|T_d^\mu(y_t)|.$$

(2) Assume condition (iii) of lemma 4.1. Then by (4.11), (2.2), for $\mu \leqslant 1$,

$$|u_t^\mu(x)| \leqslant k \exp\left\{\frac{V(t,x)}{\mu^2}\right\} \psi_t(x)$$

and if $|Du_0^\mu|$ is bounded and $c'(u) \leqslant 0$ then by (4.11) and (4.3),

$$|Du_t^\mu(x)| \leqslant \mathrm{e}^{\frac{\mu^2}{2}kt} \exp\left\{\frac{V(t,x)}{\mu^2}\right\} \psi_t(x) E\left|\frac{T_d^\mu(\bar{y}_t)}{T_0(\bar{y}_t)}\right|$$
$$\leqslant \mathrm{e}^{\frac{\mu^2}{2}kt}(1+|x|)\psi_t(x)\exp\left\{\frac{V(t,x)}{\mu^2}\right\}.$$

Theorem 4.2. *Assume $c \in C^2$ and bounded above and u_0 is BC^1 with T_0 bounded below by a positive constant. Then*

$$D\log u_t^\mu(x) = \frac{1}{\mu^2} E\frac{T_d^\mu(z_t^\mu)}{T_0(z_t^\mu)} \exp\left\{\frac{1}{\mu^2}\int_0^t c'(u_{t-s}^\mu(z_s^\mu))u_{t-s}^\mu(z_s^\mu)\mathrm{d}s\right\}, \tag{4.12}$$

if (4.5) does not explode. In particular (4.12) holds if any one of the conditions in lemma 4.1 holds.

Proof. This is an application of the Maruyama–Girsanov transform to (4.3) and (4.5). Let

$$\mathcal{M}_t^\mu = \exp\left\{-\mu\int_0^t \langle \mathrm{d}B_s, D\log u^\mu(t-s, z_s^\mu)\rangle\right.$$
$$\left. -\frac{\mu^2}{2}\int_0^t |D\log u^\mu(t-s,z_s^\mu)|^2 \mathrm{d}s\right\}.$$

It can be simplified as in Elworthy and Truman [3]: apply Itô's formula to $(s,x) \to \log u^\mu(t-s,x)$ and use (4.4) to see

$$\log u^\mu(t-s,z_s^\mu) = \log u^\mu(t,x) + \mu\int_0^s \langle \mathrm{d}B_r, \nabla \log u^\mu(t-r,z_r^\mu)\rangle$$
$$+\frac{\mu^2}{2}\int_0^s |D\log u^\mu(t-r,z_r^\mu)|^2\mathrm{d}r - \frac{1}{\mu^2}\int_0^s c(u^\mu(t-r,z_r^\mu))\mathrm{d}r. \tag{4.13}$$

It follows that

$$\mathcal{M}_t^\mu = \frac{u^\mu(t,x)}{u_0^\mu(z_t^\mu)} \cdot \exp\left\{-\frac{1}{\mu^2}\int_0^t c(u_{t-r}(z_r^\mu))\mathrm{d}r\right\},$$

and so by the Maruyama–Girsanov–Cameron–Martin formula, for each t,

$$Du^\mu(t,x) = \frac{1}{\mu^2}u^\mu(t,x)E\frac{T_d^\mu(z_t^\mu)}{T_0(z_t^\mu)}\exp\left\{\frac{1}{\mu^2}\int_0^t c'(u_{t-s}^\mu(z_s^\mu))u_{t-s}^\mu(z_s^\mu)\mathrm{d}s\right\}.$$

□

Since u is non-negative this gives us the first estimate on $|\nabla \log u_t^\mu(x)|$:

Theorem 4.3. *Assume (4.5) has no explosion and $c'(u) \leqslant 0$. Let K be a compact subset of $R^1 \times R^n$. If for $(t,x) \in K$, $E\frac{|T_d^\mu(z_t^\mu)|}{T_0(z_t^\mu)} \leqslant k$ for some constant k , then on K*

$$\mu^2\frac{|\nabla u^\mu(t,x)|}{u^\mu(t,x)} \leqslant E\frac{|T_d^\mu(z_t^\mu)|}{T_0(z_t^\mu)} \leqslant k. \tag{4.14}$$

Remark. Comparing corollary 2.2 and theorem 4.3, we notice that the estimate in theorem 4.3 is much more accurate. But it uses hypotheses on $\{z_t^\mu\}$, the stochastic flow by the logarithmic transformation as well as the assumption that $c'(u) \leqslant 0$. The condition on $\{z_t^\mu\}$ will turn out to be conditions on the initial functions. We hope in the future we can combine these two approaches to relax the conditions on c' and on Du_0. Note also that if $c'(u) \leqslant 0$ then by (4.12),

$$|Du_t^\mu(x)| \leqslant \frac{1}{\mu^2} u_t^\mu(x) \left| E \frac{T_d^\mu(z_t^\mu)}{T_0(z_t^\mu)} \right| .$$

So it is essential to get estimates on $\left| E \frac{T_d^\mu(z_t^\mu)}{T_0(z_t^\mu)} \right|$.

In the following lemma we generalize a result in Sheu [15] where linear parabolic equations were considered.

Lemma 4.4. *Let c be* C^2 *function satisfying the KPP condition (II) and the no-caustic condition. Suppose (4.5) has no explosion,* $\{u_0^\mu\}$ *are bounded uniformly in* μ *with* T_0 *strictly positive and* S_0 C^3. *If there exists* $k_0 > 0$ *such that* $|\Delta V(s,x)| \leqslant k_0$ *and*

$$|\nabla V(s,x)| + \left| \frac{T_d(x)}{T_0(x)} \right| \leqslant k_0(1+|x|)$$

on $[0,T] \times R^n$, *then*

$$\sup_{(t,x)\in K} E \left| \frac{T_d^\mu(z_t^\mu)}{T_0(z_t^\mu)} \right|^2$$

are bounded uniformly in μ *for each compact set* $K \subset [0,T] \times R^n$.

Proof. Recall that $J^\mu(r,x) = -\mu^2 \log u_r^\mu(x)$. Let

$$\tau_N = \inf_{0\leqslant s\leqslant t} \left\{ |\nabla J^\mu(t-s, z_s^\mu)| \geqslant N \right\}.$$

Then by (4.13),

$$\begin{aligned} &\tfrac{1}{2}\int_0^{s\wedge\tau_N} |DJ^\mu(t-r, z_r^\mu)|^2 \mathrm{d}r = -J^\mu(t-s\wedge\tau_N, z_{s\wedge\tau_N}^\mu) + J^\mu(t,x) \\ &- \mu \int_0^{s\wedge\tau_N} \langle \mathrm{d}B_r, \nabla J^\mu(t-r, z_r^\mu)\rangle + \int_0^{s\wedge\tau_N} c(u^\mu(t-r, z_r^\mu))\mathrm{d}r. \end{aligned}$$

Let $s \to t$ and take expectations of both sides to get:

$$\begin{aligned} &\frac{1}{2} E \int_0^{t\wedge\tau_N} |DJ^\mu(t-r, z_r^\mu)|^2 \mathrm{d}r \\ &= -EJ_{t-t\wedge\tau_N}^\mu(z_{t\wedge\tau_N}^\mu) + J^\mu(t,x) - E\int_0^{t\wedge\tau_N} c(u^\mu(t-r, z_r^\mu))\mathrm{d}r \end{aligned}$$

But $\{u_t^\mu(x)\}$ are uniformly bounded by KPP (II) and the assumption on $\{u_0^\mu\}$, and by (4.10)

$$\mu^2 \log u_s^\mu(x) \geqslant \mu^2 \log k_1 - \frac{1}{2}\mu^2 ks - ks + V(s,x)$$

for two constants k and k_1 independent of μ, s and x. Thus

$$-EJ^\mu_{t-t\wedge\tau_N}(z^\mu_{t\wedge\tau_N}) + J^\mu(t,x) - E\int_0^{t\wedge\tau_N} c(u^\mu(t-r, z_r^\mu))\mathrm{d}r \leqslant k$$

for some constant k on K. By Fatou's lemma

$$\sup_{(t,x)\in K} \frac{1}{2}E\int_0^t |DJ^\mu(t-r, z_r^\mu)|^2 \mathrm{d}r \leqslant k. \tag{4.15}$$

Note that this bound k is independent of μ. Now

$$\langle z_t^\mu, z_t^\mu\rangle = \langle x, x\rangle + 2\int_0^t \langle z_s^\mu, \mu \mathrm{d}B_s\rangle + 2\int_0^t \langle z_s^\mu, \mu^2\nabla J^\mu(t-s, z_s^\mu)\rangle \mathrm{d}s + n\mu^2 t$$

and also

$$\begin{aligned}
&E\left|\int_0^t \langle z_s^\mu, J^\mu(t-s, z_s^\mu)\rangle \mathrm{d}s\right| \\
&\leqslant \left(\int_0^t E\langle z_s^\mu, z_s^\mu\rangle\right)^{\frac{1}{2}} \left(\int_0^t E|J^\mu(t-s, z_s^\mu)|^2 \mathrm{d}s\right)^{\frac{1}{2}} \\
&\leqslant k\left(\int_0^t E\langle z_s^\mu, z_s^\mu\rangle\right)^{\frac{1}{2}} \leqslant \frac{1}{2}\left(k^2 + \int_0^t E\langle z_s^\mu, z_s^\mu\rangle \mathrm{d}s\right).
\end{aligned}$$

It follows from Gronwall's inequality that $\sup_{(t,x)\in K} E\langle z_t^\mu, z_t^\mu\rangle < \infty$ so that

$$\sup_{(t,x)\in K} E\left|\frac{T_d^\mu(z_t^\mu)}{T_0(z_t^\mu)}\right|^2$$

is bounded, from the linear growth assumption. □

5. Gradient estimtes on the crest

In the following we investigate the behaviour of $Du_t^\mu(x)$ on the crest. First we recall that $u_t^\mu(x)$ converges to 1 uniformlly on compact sets of the crest for KPP equation with suitable initial value. Let $\mu \leqslant 1$.

Theorem 5.1. *Let c be a C^2 function bounded above such that $c'(u) \leqslant 0$ and $c'(1) < 0$. Let $\{u_0^\mu\}$ be BC^1 with T_0 bounded below by a positive constant.*

(i) Assume (4.5) does not explode, and for some $p > 1$ $E|\frac{T_d^\mu(z_t^\mu)}{T_0(z_t^\mu)}|^p$ is bounded for (t,x) in compact subsets of $[0,T]\times R^n$ unifomrly in μ for small μ.

(ii) Suppose $\lim_{\mu\to 0} u_t^\mu(x) = 1$ uniformly on compact subsets of $C = \{(t,x) : V(t,x) > 0\}$.

Then for any $K \subset C$ *compact, there exists* $\mu_0(K) > 0$, $\delta(K) > 0$ *such that for* $0 < \mu < \mu_0$ *and* $(t, x) \in K$,

$$|\nabla u^\mu(t,x)| \leqslant \exp\left\{-\frac{\delta}{\mu^2}\right\}, \tag{5.1}$$

Condition (i) and (ii) are satisfied under the following conditions: (a) $\{u_0^\mu\}$ *are bounded uniformly in* μ *with* S_0 *non-negative, (b) condition (iii) of lemma 4.1, (c) the KPP conditions (I) (II) and condition (N).*

Proof. First, by the continuity of c', there exists $\gamma \in (0, \frac{1}{2}]$ such that if $|u-1| < \gamma$,

$$c'(u) < \frac{c'(1)}{2}. \tag{5.2}$$

Let K_1 be a compact set in C containing K with $d(\partial K, \partial K_1)$ positive. Here d is the distance function in R^{n+1}. More precisely if $s, t \in R^1$ and $x, y \in R^n$, then $d((s,x),(t,y)) = |s-t| + |x-y|$. By theorem 1.1, $\lim_{\mu\to 0} u^\mu(t,x) = 1$ uniformly on compact subsets of C. So for any $\epsilon \in (0, \gamma \wedge d(\partial K, \partial K_1))$ there is a number $\mu_0 > 0$ such that

$$|u^\mu(t,x) - 1| < \epsilon < \gamma \tag{5.3}$$

whenever $(t,x) \in K_1$ and $\mu < \mu_0$. From (5.2), for such (t,x) and μ

$$c'(u^\mu(t,x)) < \frac{c'(1)}{2}. \tag{5.4}$$

Now by (4.12) in theorem 4.2 and (5.3),

$$|\nabla u^\mu(t,x)| \leqslant \frac{1+\gamma}{\mu^2} E\left|\frac{T_d^\mu}{T_0}(z_t^\mu)\right| \exp\left\{\frac{1}{\mu^2}\int_0^t c'(u_{t-s}^\mu(z_s^\mu))u_{t-s}^\mu(z_s^\mu)\mathrm{d}s\right\}. \tag{5.5}$$

Next we show that $c'(u_{t-s}^\mu(z_s^\mu))$ is strictly negative on an interval with large probability. By theorem 4.3, $|\mu^2 D \log u_t^\mu(x)|$ is bounded. Let k_0 be a number such that on K_1,

$$|\mu^2 D \log u_t^\mu(x)| + E\left|\frac{T_d^\mu(z_t^\mu)}{T_0(z_t^\mu)}\right|^p \leqslant k_0.$$

Let $h > 0$ be a constant smaller than $\frac{\epsilon}{2}$ and such that $k_0 h < \frac{\epsilon}{2}$. Define

$$\Omega_0 = \{\omega : (t-s, z_s^\mu(\omega)) \in K_1, \text{ for } s \in [0,h], (t,x) \in K\}.$$

If $\mu < \mu_0$ then by (5.3) and (5.4),

$$\begin{aligned} &1+\gamma > u_{t-s}^\mu(z_s^\mu) > 1-\gamma \geqslant \frac{1}{2}, \\ &c'(u^\mu(t-s, z_s^\mu)) < \frac{c'(1)}{2} \qquad \text{on } \Omega_0. \end{aligned} \tag{5.6}$$

Define

$$\tau(\omega) = \inf\left\{s : (t-s, z_s^\mu(\omega)) \notin K_1, (t,x) \in K\right\}.$$

Then $\Omega_0 = \{\omega : \tau(\omega) \geqslant h\}$. Now

$$z_\tau^\mu = x + \mu B_\tau - \int_0^\tau \nabla J(t-s, z_s^\mu)\mathrm{d}s$$

and

$$\begin{aligned}|z_\tau^\mu - x| &\leqslant \mu|B_\tau| + \int_0^\tau |\nabla J(t-s, z_s^\mu)|\mathrm{d}s\\ &\leqslant \mu|B_\tau| + k_0 h\\ &\leqslant \mu|B_\tau| + \frac{\epsilon}{4}\end{aligned}$$

However $(t-\tau, z_\tau) \in \partial K_1$ and $(t,x) \in K$. So on $\{\tau \leqslant h\}$,

$$|z_\tau - x| \geqslant d(\partial K, \partial K_1) - \tau \geqslant d(\partial K, \partial K_1) - h \geqslant \frac{\epsilon}{2}.$$

Consequently

$$\mu|B_\tau| \geqslant \frac{\epsilon}{4}.$$

It turns out that

$$\begin{aligned}P(\Omega - \Omega_0) &= P\{\omega : \tau(\omega) \leqslant h\}\\ &\leqslant P\left\{\omega : \mu \sup_{0\leqslant s\leqslant h} |B_s| \geqslant \frac{\epsilon}{4}\right\}\\ &\leqslant \exp\left\{-\frac{\epsilon^2}{32h\mu^2}\right\}.\end{aligned}$$

Note that $c' \leqslant 0$ and $u_t(x) \geqslant 0$. Let q be the conjugate number to p and suppose $z_0^\mu = x$. Then by (5.5) and (5.6),

$$\begin{aligned}
|Du^\mu(t,x)| &\leqslant \frac{1+\gamma}{\mu^2} E\left|\frac{T_d^\mu(z_t^\mu)}{T_0(z_t^\mu)}\right| \exp\left\{\frac{1}{\mu^2}\int_0^t c'(u_{t-s}^\mu(z_s^\mu))u_{t-s}^\mu(z_s^\mu)\mathrm{d}s\right\}\\
&= \frac{1+\gamma}{\mu^2} E[\chi_{\Omega_0} + \chi_{\Omega-\Omega_0}]\left|\frac{T_d^\mu(z_t^\mu)}{T_0(z_t^\mu)}\right| \exp\left\{\frac{1}{\mu^2}\int_0^t c'(u_{t-s}^\mu(z_s^\mu))u_{t-s}^\mu(z_s^\mu)\mathrm{d}s\right\}\\
&\leqslant \frac{1+\gamma}{\mu^2} E\left|\frac{T_d^\mu(z_t^\mu)}{T_0(z_t^\mu)}\right| \chi_{\Omega_0} \exp\left\{\frac{1}{\mu^2}\int_0^h c'(u_{t-s}^\mu(z_s^\mu))u_{t-s}^\mu(z_s^\mu)\mathrm{d}s\right\}\\
&\quad + \frac{1+\gamma}{\mu^2} E\left|\frac{T_d^\mu(z_t^\mu)}{T_0(z_t^\mu)}\right| \chi_{\Omega-\Omega_0}\\
&\leqslant \frac{1+\gamma}{\mu^2} E\left|\frac{T_d^\mu(z_t^\mu)}{T_0(z_t^\mu)}\right| \chi_{\Omega_0} \exp\left\{\frac{c'(1)h}{4\mu^2}\right\} + \frac{1+\gamma}{\mu^2}\left(E\left|\frac{T_d^\mu(z_t^\mu)}{T_0(z_t^\mu)}\right|^p\right)^{\frac{1}{p}} [P(\Omega-\Omega_0)]^{\frac{1}{q}}\\
&\leqslant \frac{1+\gamma}{\mu^2} E\left|\frac{T_d^\mu(z_t^\mu)}{T_0(z_t^\mu)}\right| \exp\left\{\frac{c'(1)h}{4\mu^2}\right\} + \frac{1+\gamma}{\mu^2}\exp\left\{-\frac{\epsilon^2}{32\mu^2 hq}\right\}\left(E\left|\frac{T_d^\mu(z_t^\mu)}{T_0(z_t^\mu)}\right|^p\right)^{\frac{1}{p}}\\
&= \frac{1+\gamma}{\mu^2}\left(E\left|\frac{T_d^\mu(z_t^\mu)}{T_0(z_t^\mu)}\right|^p\right)^{\frac{1}{p}}\left(\exp\left\{\frac{c'(1)h}{4\mu^2}\right\} + \exp\left\{-\frac{\epsilon^2}{32\mu^2 hq}\right\}\right)
\end{aligned}$$

Now ϵ and h are fixed numbers for the compact set K and $c'(1) < 0$. The result follows for small μ.

For the last part of the theorem we first note that (4.5) does not explode from lemma 4.1 and (4.12) holds. Applying theorem 4.3 and lemma 4.4 we obtain the required integrability condition on $\left|\frac{T_d^\mu(z_t^\mu)}{T_0(z_t^\mu)}\right|$. The rest follows from theorem 1.1. □

Example. Consider the KPP equation

$$\frac{\partial}{\partial t}u^\mu(t,x) = \frac{1}{2}\mu^2\Delta u^\mu(t,x) + \frac{\hat{c}}{\mu^2}(1-u^\mu(t,x))u^\mu(t,x) \tag{5.7}$$

on R^1. Here $\hat{c}$ is a positive constant and μ takes value in $(0,1]$. It is known that for each positive C^2 initial condition there is a unique $C^{2,1}$ solution to the KPP equation.

Let $u_0^\mu(x) = T_0(x)\exp\left\{-\frac{S_0(x)}{\mu^2}\right\}$ be bounded uniformly in μ and $T_0 \geqslant 0$. Then $\{\mu^3|Du_t(x)| : \mu > 0\}$ and $\{|\mu^6 D^2 u_t(x)| : \mu > 0\}$ are bounded on $[\delta, T]\times R^n$ for each $\delta > 0$, by corollary 2.2 and corollary 2.4.

Now suppose $S_0(x) = x^2$ and so $u_0^\mu(x) = T_0(x)\exp\left\{-\frac{x^2}{\mu^2}\right\}$. Then the semi-classical flow is given by $\Phi_s(x) = (1+2s)x$ and $V(t,x) = \hat{c}t - \frac{x^2}{2t+1}$. So $\Phi_t^{-1}(x) = \frac{x}{1+2t}$ and the no-caustic condition is satisfied for all t. It is also clear that ΔV is bounded and $|\nabla V(t,x)|$ has linear growth. So by theorem 3.1 and the remarks followed we have: on $(t,x) : \frac{x^2}{2t+1} > \hat{c}t$,

$$\lim_{\mu\to 0}|\nabla u_t^\mu(x)| = 0.$$

Let $T_0(x) = 1$. Then $T_d^\mu(x) = -2x$ and condition (ii) of lemma 4.1 holds and so does (4.11), the formula for $D\log u_t^\mu(x)$.

For general T_0,

$$\psi_t(x) = T_0\left(\frac{x}{1+2t}\right)\sqrt{\frac{1}{1+2t}}$$

and so

$$\frac{\Delta\psi_t(x)}{\psi_t(x)} = (1+2t)^{-2}\frac{D^2T_0(\frac{x}{1+2t})}{T_0(\frac{x}{1+2t})}$$

wherever it exists.

Let $T_0(x) = 1 + x^2$. Then $T_d^\mu(x)/T_0(x) = \mu^2\frac{2x}{1+x^2} - 2x$, $\nabla\log\psi_s(x) = \frac{2x}{(1+2s)^2+x^2}$ has linear growth, and $\Delta\psi_s(x)/\psi_s(x) = (1+2s)\frac{2}{(1+2s)^2+x^2}$ is bounded. Thus by lemma 4.1, the stochastic differential equation (4.5) by the logarithmic transformation has no explosion. From theorem 5.1 we conclude that $|\nabla u_t^\mu(x)|$ converges to 0 at the crest.

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