Infinite Dimensional Analysis, Quantum Probability
and Related Topics
Vol. 6, No. 3 (2003) 1–27

SOME FAMILIES OF q-VECTOR FIELDS ON PATH SPACES*

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Some families of H-valued vector fields with calculable Lie brackets are given. These provide examples of vector fields on path spaces with a divergence and we show that versions of Bismut type formulae for forms on a compact Riemannian manifold arise as projections of the infinite dimensional theory.

Keywords: Path space; differential forms; Riemannian manifolds; heat equation; integration by parts; Bismut formula; Malliavin calculus.

Let M be a compact Riemannian manifold and $\{P_t : t \geq 0\}$ its heat semigroup acting on differential forms. Thus if ϕ is a bounded measurable (or square integrable) form on M we have

$$\begin{cases} \dfrac{\partial}{\partial t}(P_t\phi) = \dfrac{1}{2}\Delta(P_t\phi)\,, & t > 0 \\ P_0\phi \qquad = \phi \end{cases}$$

where we use the sign convention

$$\Delta = -(d\delta + \delta d)\,.$$

The formula for the exterior derivative of $P_t\phi$

$$d(P_t\phi)(V_0) = \frac{1}{t}\mathbb{E}\phi\left(W_t^{(q)}\int_0^t (W_s^{(q)})^{-1}\iota_{\langle -,dx_s\rangle}W_s^{(q+1)}(V_0)\right)\,, \tag{0.1}$$

*Research partially supported by NSF research grant DMS 0072387, EPSRC GR/NOO 845, the Alexander von Humboldt stiftung.

where ϕ is a bounded measurable q-form and $V_0 \in \wedge^{q+1}T_{x_0}M$ was given in [9], extending Bismut's formula for the special case of $q=0$, [3]

$$d(P_t f)(V_0) = \frac{1}{t}\mathbb{E}f(x_t)\int_0^t \langle W_s^{(1)}(V_0), dx_s\rangle \tag{0.2}$$

for $f: M \to \mathbb{R}$ bounded and measurable and $V_0 \in T_{x_0}M$. In these formulae the expectation $\mathbb{E}$ is with respect to a Brownian motion $\{x_t : 0 \le t < \infty\}$ on M starting from x_0 and for $q = 1, 2, \ldots$ we use the "damped" parallel translations $W_s^{(q)} : \wedge^q T_{x_0}M \to \wedge^q T_{x_s}M$, $0 \le s < \infty$ of q-vectors along the sample paths of $(x.)$ given by the covariant equation along $(x.)$:

$$\begin{cases} \dfrac{D}{\partial s}(W_s^{(q)}(V_0)) = -\dfrac{1}{2}\mathcal{R}^q_{x_0}(W_s^{(q)}(V_0)) \\ \qquad W_0^{(q)}(V_0) = V_0\,, \end{cases} \tag{0.3}$$

where $\mathcal{R}^q_x : \wedge^q T_xM \to \wedge^q T_xM$ is the qth Weitzenbock curvature defined by

$$\Delta\phi = \nabla^*\nabla\phi - \phi(\mathcal{R}^q_{\cdot}-)$$

for ϕ a smooth q-form. See, for example, [1, 13, 14, 18]. In particular $\mathcal{R}^1$ is the Ricci curvature $Ric^{\#} : TM \to TM$, i.e. $\langle \mathcal{R}^1_x v, u\rangle = Ric(v,u)$, u, $v \in T_xM$.

Formula (0.1) has been refined and extended by Driver and Thalmaier, [5], giving analogous results on various operators on vector bundles over M, e.g. the square of the Dirac operator. Rather different types of Bismut type formulae for covariant derivatives of operators on vector bundles were obtained by Norris in [25] by different methods. The original proof of (0.1) in [9] was to derive it by applying the method of conditional expectation from [16] to the earlier, non-intrinsic, formula of Li, [7, 24], see also [15],

$$d(P_t\phi)_{x_0} = \frac{1}{t}\mathbb{E}\int_0^t \langle T\xi_s -, dx_s\rangle \wedge \xi_t^*(\phi) \tag{0.4}$$

for ϕ a bounded measurable q-form. Here $\{\xi_s : 0 \le s < \infty\}$ is a gradient Brownian flow on M (see below), $x_t = \xi_t(x_0)$ for $x_0 \in M$, $T\xi_t : TM \to TM$ is the (random) derivative of ξ_t, and $\xi_t^*\phi$ is the pull back of ϕ:

$$\begin{aligned} \xi_t^*\phi(V) &:= \phi\left(\wedge^q(T\xi_t)(V)\right) \\ &= \phi\left(T\xi_t(v^1)\wedge\ldots\wedge T\xi_t(v^q)\right), \qquad V = v^1\wedge\ldots\wedge v^q \in \wedge^q TM\,. \end{aligned}$$

In fact the same proof allows the use of any Brownian flow which has Levi–Cività connection as its LeJan–Watanabe connection in the sense of [12]. All the formulae can be extended to obtain differentiation formulae for the corresponding heat kernels and this is done in the references cited.

Bismut's formula (0.2) can be obtained from infinite dimensional integration by parts formulae by considering the cylindrical functions $F(\sigma.) = f(\sigma_t)$ on the space of paths over M and indeed Driver's integration by parts formula can be derived from it, as described in [8] following comments by Nualart. Since integration by parts theorems for forms on path spaces are not yet well understood it is interesting to ask if analogous results hold for (0.1) and (0.4) in the context of the L^2 theory of differential forms being developed in [6, 10] and to derive (0.1) and (0.4) by the methods used there. Our approach is very much in the spirit of Bismut's approach to Malliavin Calculus [2] and of Fang–Franchi for Lie groups [17]. We do not attack the two challenges of : (i) deriving integration by parts formulae for forms on path spaces from (0.1) and (0.4), as done for functions in [8]; and (ii) deriving the more general results of Driver and Thalmaier by similar methods.

In Sec. 1 below we define the class of flows used in (0.1). In Sec. 2 the infinite dimensional theory in [6, 10] is briefly described, and in Sec. 3 the two parts are combined to give both a proof of (0.1) and (0.2) and some interesting examples of q-vector fields on path spaces which have explicitly calculable divergences. On the way we obtain, from our stochastic differential equation (1.1), a family of H-vector fields on Wiener space, which for gradient stochastic differential equations form a commuting family and in certain other cases have easily computable Lie brackets. This gives a $(q+1)$-vector field $\tilde{V}_0$ on the path space $C_{x_0}M$ of M which has a divergence with respect to the Wiener measure μ_{x_0} in the sense that the divergence $\mathrm{div}\tilde{V}_0$ is a vector field on $C_{x_0}M$ satisfying equation (2.3) below and such that (0.1) can be written:

$$\int_{C_{x_0}M} d\phi^t(\tilde{V}_0)\, d\mu_{x_0} = -\int_{C_{x_0}M} \phi^t(\mathrm{div}\tilde{V}_0)\, d\mu_{x_0}$$

for ϕ^t the cylindrical q-form on $C_{x_0}M$ obtained from a differential form ϕ on the manifold M:

$$\phi^t(V) = \phi(x_t)(V_{t,\dots,t}), \qquad \text{for } V \text{ any } q\text{-vector field.}$$

Various versions of formula (0.2) are derived from this. For invariant stochastic differential equations on Lie groups no conditioning is needed in (0.4), cf. [17], and we obtain (3.28). Using gradient systems we give in (3.23) the minor extension to (0.4) mentioned in [9] to the case of semigroups on forms with generators of the form $\frac{1}{2}\Delta + \mathcal{L}_A$. We also consider the connections used by Ikeda and Watanabe in [18], sometimes known as Riemann–Cartan–Weyl connections [26], where the formulae (0.2) and (0.4) have a pleasant form (3.29), although the infinitesimal generators of the semigroup on forms are rather complicated. However, the main aims of the article are to study some interesting classes of q-vector fields on $C_{x_0}M$ and to show that some finite dimensional formulae can be considered as projections of the infinite dimensional theory.

1. Stochastic Flows and LeJan–Watanabe Connections

Consider a Stratonovich stochastic differential equation

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt \tag{1.1}$$

on M driven by an m-dimensional Brownian motion $\{B_t, 0 \le t < \infty\}$. Here $X : M \times \mathbb{R}^m \to TM$ and $A : M \to TM$ are assumed to be smooth and $X(x) := X(x, -) : \mathbb{R}^m \to T_xM$ is assumed to be linear, surjective for each x and to induce the inner product $\langle , \rangle_x$ on T_xM given by the Riemannian structure of M. This implies the Markov process of solutions to (1.1) has infinitesimal generator of the form

$$\mathcal{A} = \frac{1}{2}\Delta + \mathcal{L}_Z\,, \tag{1.2}$$

where $\mathcal{L}_Z$ is Lie differentiation in the direction of some vector field Z:

$$\mathcal{L}_Z(f)(x) = df(Z(x)), \qquad x \in M.$$

The solution to (1.1) from x shall be denoted by $\{\xi_t(x), 0 \le t < \infty\}$. Write $x_t = \xi_t(x_0)$. We can, and will, take versions which makes $\xi_t(x)$ continuous in t and a C^∞ diffeomorphism of M in x.

Let $Y : TM \to \mathbb{R}^m$ be the adjoint of X, i.e. $Y_x = X(x)^*$, so that Y is the right inverse to X. As described in [11], or more generally in [12], X induces a metric connection $\breve{\nabla}$ on M defined by

$$\breve{\nabla}_v U = X(x)d[x \mapsto Y_x(U(x))](v), \qquad v \in T_xM \tag{1.3}$$

for any smooth vector field U on M. In general this, which we call the *LeJan–Watanabe connection* of our stochastic differential equation, has torsion. The torsion is given by

$$\breve{T}(u, v) = X(x)dY(u, v), \qquad u, v \in T_xM \tag{1.4}$$

where dY refers to the exterior derivative of Y considered as an $\mathbb{R}^m$-valued differential one-form on M, see Proposition 2.2.3 in [12]. When the torsion vanishes, $\breve{\nabla}$ is just the Levi–Cività connection, which we will always refer to using ∇, $\frac{D}{\partial t}$ etc. There are two principal classes of examples, Examples 1.1 and 1.2 below, for which this holds.

Example 1.1. Gradient system

Here we consider an isometric immersion $\alpha : M \to \mathbb{R}^m$ for some m, e.g. by using Nash's theorem. Let $X(x): \mathbb{R}^m \to T_xM$ be the orthogonal projection identifying T_xM with its image in $\mathbb{R}^m$ under $d\alpha$. Then $Y_x : T_xM \to \mathbb{R}^m$ is just the inclusion $(d\alpha)_x$ and (1.3) is the classical formula for the Levi–Cività connection of a submanifold of $\mathbb{R}^m$.

Example 1.2. Riemannian symmetric spaces

Let M be a symmetric space (K, H, σ). In particular K is a Lie group acting transitively on M and H can be identified with the isotropy subgroup fixing the point x_0 of M, so that $k \mapsto kx_0$ gives a diffeomorphism of K/H with M. Assume that the Lie algebra $\mathfrak{k}$ of K has an inner product $\langle,\rangle_{\mathfrak{k}}$, say, invariant under the adjoint action of K on $\mathfrak{k}$. The action induces a stochastic differential equation

$$X : M \times \mathfrak{k} \to TM$$

with $X(x)e = \frac{d}{dt}[(\exp(te)) \cdot x]_{t=0}$. It is surjective and so induces a Riemannian structure $(\langle,\rangle_x, x \in M)$ on M which is K-invariant. The LeJan–Watanabe connection $\breve{\nabla}$ is the Levi–Cività connection for the Riemannian structure, as is proved in Corollary 1.4.9 of [12].

An important class of examples which give connections other than the Levi–Cività connections are the left-invariant, and the right-invariant stochastic differential equations on Lie groups. For M a Lie group with left-invariant metric takes the Lie algebra $\mathfrak{g}$ on $\mathbb{R}^m$ and defines

$$X^L : G \times \mathfrak{g} \to TG$$

to be left-invariant with $X^L(e)\alpha = \alpha$, for e the identity element of G, i.e.

$$X^L(g)\alpha = TL_g(\alpha)$$

for $L_g : G \to G$ the left-translation $x \mapsto g \cdot x$. Similarly if R_g denotes right-translation, $x \mapsto x \cdot g$, define $X^R : G \times \mathfrak{g} \to TG$ by $X^R(g)\alpha = TR_g(\alpha)$. It is easy to see that the associated connections in the sense of (1.3) are just the canonical left- and right-invariant connections, ∇^L and ∇^R say, of G, respectively. The flow ξ_t^R for the left-invariant stochastic differential equation $dx_t = X^L(x_t) \circ dB_t$, where B_t is a $\mathfrak{g}$-valued Brownian motion, is given by

$$\xi_t^L(x) = xg_t, \qquad x \in G, t \geq 0.$$

Here g_t is the solution starting from the identity e to $dg_t = X^L(g_t) \circ dB_t$. Similarly the solution $\xi_t^R(x)$, with $\xi_0^R(x) = x$ to the right-invariant stochastic differential equation $dx_t = X^R(x_t) \circ dB_t$ is given by the right translation of h_t, the solution starting from e. If the metric on G is bi-invariant, both stochastic differential equations, with $A \equiv 0$, have Brownian motions as solutions. Also in that case the connections are torsion skew symmetric, with ∇^L and ∇^R adjoint to each other. In particular the torsions T^L and T^R are given by

$$T^L(u,v) \equiv -[U,V](g), \qquad T^R(u,v) = [U,V](g), \tag{1.5}$$

where U, V are left-invariant vector fields with $u = U(g)$ and $v = V(g)$.

In the bi-invariant case we can also treat G as a symmetric space by taking $K = G \times G$ with action $(g_1, g_2)x = g_1 x g_2^{-1}$, with $H = \Delta G \equiv \{(g,g) \in K : g \in G\}$ the isotropy subgroup fixing the identity e of G. The symmetry map $\sigma : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is given by $\sigma(\alpha, \beta) = (\beta, \alpha)$. The stochastic differential equation induced by the action as described in Example 1.2 above is just

$$dx_t = TR_{x_t} \circ dB_t - TL_{x_t} \circ dB_t', \tag{1.6}$$

where (B_t) and (B'_t) are two independent Brownian motions on $\mathfrak{g}$. The solution flow is then given by $\xi_t(x) = g_t x (g'_t)^{-1}$ where g_t and g'_t are independent Brownian motions on G and solutions to the following right-invariant stochastic differential equations respectively:

$$dg_t = TR_{g_t} \circ dB_t, \qquad g_0 = e,$$

$$dg'_t = TR_{g'_t} \circ dB'_t, \qquad g_0 = e\,.$$

The solutions $\xi_t(x)$, when $A \equiv 0$, are also Brownian motions on G.

For more general homogeneous spaces $M = K/H$, if the metric on M is induced from an ad_K-invariant inner product on $\mathfrak{k}$ and the orthogonal complement of the ker $X(x_0)$ is ad_H invariant then $\breve{\nabla}$, the connection associated to the stochastic differential equation defined as in Example 1.2 above, is K-invariant. (see [12], Proposition 1.2.9). Again in that case

$$\xi_t(x) = k_t x, \qquad x \in M$$

for k_t the solution to the equation $dk_t = TR_{k_t} \circ dB_t$ on K with initial point $k_0 = e$.

Stochastic flows and differential forms. For $\{\xi_t : t \geq 0\}$ the flow of our stochastic differential equation (1.1) there are semigroups on differential forms defined by $P_t\phi = \mathbb{E}\xi_t^*\phi$. On q-forms the semigroup has infinitesimal generator $\mathcal{A}^q$ given on smooth forms by

$$\mathcal{A}^q\phi = \frac{1}{2}\sum_{j=1}^m \mathcal{L}_{X^j}\mathcal{L}_{X^j}\phi + \mathcal{L}_A(\phi)$$

for $\mathcal{L}_{X^j}$ denoting the Lie differentiation in the direction X^j, etc. and with X^j denoting the vector fields $X(\cdot)e^j$ for $e^1, \ldots, e^m$ the standard basis of $\mathbb{R}^m$. This comes from Itô's formula, see [15] or [14]. The generator can be written

$$\mathcal{A}^q(\phi) = -(d\hat{\delta} + \hat{\delta} d)\phi + \mathcal{L}_A(\phi)\,, \tag{1.7}$$

where, for $V \in \wedge^{q-1}TM$,

$$\begin{aligned} \hat{\delta}\phi(V) &= -\sum_{j=1}^m (\iota_{X^j}\mathcal{L}_{X^j}\phi)(V) \\ &= -\mathrm{trace}\hat{\nabla}_{-}\phi(-, V), \end{aligned} \tag{1.8}$$

and $\hat{\nabla}$ is the adjoint connection to $\breve{\nabla}$, in the sense of Driver [4] so the torsion of $\hat{\nabla}$ is $-\breve{T}$. See [12]. This shows that $\mathcal{A}^q$ depends only on the connection $\breve{\nabla}$ and the Riemannian metric on TM. In particular if $A \equiv 0$ in (1.1) and the associated connection $\breve{\nabla}$ is the Levi–Civitá connection we get the usual Hodge–Kodaira Laplacian, cf. [20].

2. Spaces of Forms on the Path Space of M

Fix $T > 0$ and $x_0 \in M$ and let $C_{x_0}M$ denote the Banach manifold of continuous paths $\sigma : [0,T] \to M$ with $\sigma(0) = x_0$. Let μ_{x_0} be the measure induced on it by the solution to (1.1) starting from x_0.

We can identify the tangent space $T_\sigma C_{x_0}M$ to $C_{x_0}M$ at σ with the space of continuous $v : [0,T] \to TM$ with $v(0) = 0$ such that $v(t) \in T_{\sigma(t)}M$ for each $0 \le t \le T$. For $q = 1, 2, \ldots$, let $\wedge^q T_\sigma C_{x_0}M$ be the space of antisymmetric q-vectors at x_0 completed using the largest reasonable cross norm, so that its dual space can be identified with the Banach space of continuous alternating q-linear maps $\phi : T_\sigma C_{x_0}M \times \ldots \times T_\sigma C_{x_0}M \to \mathbb{R}$. Thus q-forms on $C_{x_0}M$ are sections of the dual bundle $(\wedge^q T_\sigma C_{x_0}M)^*$, e.g. see [22].

As an example of a q-vector on $C_{x_0}M$ consider $\wedge^q(T_{x_0}\xi)(U)$ evaluated on a fixed $\omega \in \Omega$, where $U \in \wedge^q T_{x_0}M$. It is then a q-vector at $\{x_t(\omega) : 0 \le t \le T\}$. For $q = 1$ it is the tangent vector $\{T_{x_0}\xi_t(\cdot,\omega)(U) : 0 \le t \le T\}$ to $C_{x_0}M$ at $x.(\omega)$. For higher q we identify q-vectors V with certain continuous maps $V : [0,T]^q \to \otimes^q TM$ such that $V_{t_1,t_2,\ldots,t_q} \in T_{\sigma(t_1)} \otimes \ldots \otimes T_{\sigma(t_q)}M$. Then, generalizing to allow U to be in $\wedge^{q+1}C([0,T]; T_{x_0}M)$,

$$\wedge^{q+1}(T_{x_0}\xi)(U)_{t_1,\ldots,t_{q+1}} = \left(T_{x_0}\xi_{t_1} \otimes \ldots \otimes T_{x_0}\xi_{t_{q+1}}\right)\left(U_{t_1,\ldots,t_{q+1}}\right).$$

As ω varies in Ω such $q+1$ vectors do not strictly speaking form $q+1$ vector fields on $C_{x_0}M$. However, we can obtain q-vector fields from them by conditioning, 'filtering out the redundant noise', to give $\sigma \mapsto \mathbb{E}\{\wedge^{q+1}(T_{x_0}\xi)(U)\,|\,\xi.(x_0,\omega) = \sigma\}$. As shorthand, we write:

$$\overline{\wedge^{q+1}(T_{x_0}\xi)}_\sigma(U) = \mathbb{E}\{\wedge^{q+1}(T_{x_0}\xi)(U)|\xi.(x_0,\omega) = \sigma\}. \tag{2.1}$$

See [12, 16]. Below we shall often use $\bar{f}$ to denote the conditional expectation of a random function f with respect to the filtration of $\{x_t : 0 \le t < \infty\}$.

An important class of forms on $C_{x_0}M$ are the cylindrical forms. In particular if ϕ is a q-form on M and $t \in [0,T]$ we have the cylindrical q-form ϕ^t on $C_{x_0}M$ given by $\phi^t(V) = \phi(V_{t,\ldots,t})$.

For ψ a $(q-1)$-form on M we see, with P_t as in the last section,

$$P_t(d\psi) = \mathbb{E}(d\psi)\left(\wedge^q(T_x\xi_t)U\right) = \int_{C_{x_0}M} (d\psi)^t\left(\overline{\wedge^q T_x\xi}\,(U)\right) d\mu_{x_0}(\sigma). \tag{2.2}$$

In general we say that an L^2 q-vector field V *has a divergence* if there is an L^2 $(q-1)$-vector field $divV$ on $C_{x_0}M$ such that for all smooth cylindrical $(q-1)$-forms ϕ on $C_{x_0}M$ we have

$$\int_{C_{x_0}M} d\phi(V)\,d\mu_{x_0} = -\int_{C_{x_0}M} \phi(\operatorname{div} V)\,d\mu_{x_0}. \tag{2.3}$$

For $U = u^1 \wedge \ldots \wedge u^q$, a primitive vector, the non-intrinsic Bismut type formula (0.4) will follow from showing that $\overline{\wedge^q(T_{x_0}\xi)}(U)$ has a divergence with

$$\begin{aligned}
&\text{div}\left(\overline{\wedge^q T_x\xi_\sigma(U)}\right)_{t,\ldots,t} \\
&\quad = \frac{1}{t}\mathbb{E}\left\{\sum_{j=1}^{q}(-1)^j\left(\int_0^t \langle T\xi_s(u^j), dx_s\rangle\right)\wedge^{q-1}(T\xi_t)(u^1\wedge\ldots\widehat{u^j}\ldots\wedge u^q)\right. \\
&\quad\quad \left.\Bigg|\,\xi_\cdot(x_\cdot)=\sigma\right\}
\end{aligned}$$

at least for a gradient stochastic differential equation with no drift. Here we have adopted the convention that $\widehat{u^j}$ means the omission of the vector u^j in the tensor. For a general q-vector U in $\wedge^q T_{x_0}M$ this could be written as

$$div\left(\overline{\wedge^q T_x\xi_\sigma(U)}\right)_{t,\ldots,t} = -\frac{1}{t}\overline{\wedge^{q-1}(T\xi_t)\iota_{\int_0^t\langle T\xi_s -, dx_s\rangle}U}.$$

Equation (0.1) will follow, as in [9], by calculating the conditional expectations. Note that we have used another convention: if ℓ is a linear map on a vector space

$$\iota_\ell(b^1\wedge\ldots\wedge b^q) = \sum_{j=1}^{q}(-1)^{j+1}\ell(b^j)b^1\wedge\ldots\wedge\widehat{b^j}\wedge\ldots\wedge b^q.$$

In [6], see also [10], a general theory for forms on path spaces is given. There we define Hilbert spaces $\mathcal{H}^{(q)}$, densely included in $\wedge^q T_\sigma C_{x_0}M$ and show that suitably regular sections of $\mathcal{H}^{(q)}$ do have a divergence and give evidence to suggest that the divergence is a section of $\mathcal{H}^{(q-1)}$, with proofs for $q = 2, 3$. This is the major step in the construction of an L^2 deRham complex on $C_{x_0}M$, with an L^2 Hodge decomposition and analogue of the Hodge–Kodaira Laplacian. For this construction we take a stochastic differential equation (1.1) on M whose solutions are Brownian motions and with its associated connection $\breve{\nabla}$ the Levi–Cività connection, for example by taking an isometric embedding in $\mathbb{R}^m$ and $A \equiv 0$. We take $(B_\cdot)$ to be the canonical Brownian motion defined up to time T so that $C_0\mathbb{R}^m$ with Wiener measure $\mathbb{P}$ is the underlying probability space. For fixed $x_0 \in M$ there is the Itô map

$$\begin{aligned}
\mathcal{I} : C_0\mathbb{R}^m &\to C_{x_0}M \\
\mathcal{I}(\omega)_t &= \xi_t(x_0,\omega).
\end{aligned}$$

It is smooth in the sense of Malliavin calculus with H-derivative

$$T_\omega\mathcal{I} : H \to TC_{x_0}M$$

defined for almost all $\omega \in C_0\mathbb{R}^m$. Here H is the Cameron–Martin space; $H = L_0^{2,1}([0,T];\mathbb{R}^m)$. We have the formula of Bismut

$$T\mathcal{I}(h)_t = T\xi_t\int_0^t (T\xi_s)^{-1}X(x_s)\dot{h}_s\,ds. \tag{2.4}$$

By an L^2 H-q-vector field on $C_0\mathbb{R}^m$ we mean an element $U \in L^2(C_0\mathbb{R}^m;\wedge^q H)$ where $\wedge^q H$ refers to the completed tensor product using the usual Hilbert space cross norm. Given such a vector field U we can define

$$(\wedge^q T\mathcal{I})(U) : C_0\mathbb{R}^m \to \wedge^q TC_{x_0}M.$$

At $\omega \in C_0\mathbb{R}^m$ this is in $\wedge^q T_{x_\cdot(\omega)}C_{x_0}M$.

To obtain a q-vector field on $C_{x_0}M$ define, for $\sigma \in C_{x_0}M$

$$\overline{(\wedge^q T\mathcal{I})(U)}_\sigma = \mathbb{E}\left\{(\wedge^q T\mathcal{I})(U)\,|x_\cdot = \sigma\right\}.$$

Taking constant U, that is $U(\omega) \equiv h$, some $h \in H$ we obtain continuous linear maps

$$\overline{(\wedge^q T\mathcal{I})}_\sigma : \wedge^q H \to \wedge^q T_\sigma C_{x_0}M$$

defined by

$$\overline{(\wedge^q T\mathcal{I})}_\sigma(h) = \overline{(\wedge^q T\mathcal{I})(h)}_\sigma.$$

We can then define $\mathcal{H}^q_\sigma = \text{Image}(\overline{\wedge^q T\mathcal{I}_\sigma})$ with its induced Hilbert space structure, inner product $\langle , \rangle_\sigma$, so that $(\overline{\wedge^q T\mathcal{I}})_\sigma$ becomes an isometry. These spaces are defined for almost all $\sigma \in C_{x_0}M$. It is shown in [6] that they depend only on the Riemannian structure of M. The fact that suitably regular L^2 sections of $\mathcal{H}^q$ have a divergence can be deduced from the theory for Wiener spaces given by Shigekawa [27].

The following is a specialization of his basic result to primitive adapted q-vector fields.

Theorem 2.1. (Shigekawa [27]) *For $q > 1$, let $\phi : C_0\mathbb{R}^m \to (\wedge^q H)^*$ be an H-differential q-form on Wiener space and $h^i : C_0\mathbb{R}^m \to H$, $i = 1$ to q adapted vector fields such that $h := h^1(\cdot)\wedge\ldots\wedge h^{q+1}(\cdot)$ is in $\mathbb{D}^{1,2}(\wedge^{q+1}H)$. If also $\phi \in \mathbb{D}^{1,2}\left((\wedge^q H)^*\right)$ then*

$$\int_{C_0\mathbb{R}^m} d\phi(h)\, d\mathbb{P} = -\int_{C_0\mathbb{R}^m} \phi\,(\text{div}\, h)\, d\mathbb{P}\,,$$

where

$$\begin{aligned}\text{div}\, h = &\sum_{j=1}^q (-1)^j \int_0^T \langle \dot{h}^j_s, dB_s\rangle \left(h^1 \wedge \ldots \wedge \widehat{h^j} \wedge \ldots \wedge h^q\right)\\ &- \sum_{1\le i<j\le q+1} (-1)^{i+j}[h^i, h^j] \wedge h^1 \wedge \ldots \wedge \widehat{h^i} \wedge \ldots \wedge \widehat{h^j} \wedge \ldots \wedge h^{q+1}.\end{aligned}$$

The assumptions of primitivity of h or adaptness of the components could have been omitted with Skorohod integrals replacing Itô integrals, see [27].

The extension to forms on $C_{x_0}M$ in [6, 10] and the proof we give below for formulae (1) and (2) is based on the following observation:

Theorem 2.2. *Let ψ be a smooth cylindrical q-form on $C_{x_0}M$. Then for $h \in \mathbb{D}^{1,2}(\wedge^{q+1}H)$ of the form $h = h^1 \wedge \ldots \wedge h^{q+1}$ with each h^i adapted, the $(q+1)$-vector field $\overline{\wedge^{q+1}T\mathcal{I}(h)}$ on $C_{x_0}M$ has a divergence*:

$$\int_{C_{x_0}M} d\psi\left(\overline{\wedge^{q+1}T\mathcal{I}(h)}\right) d\mu_{x_0} = -\int_{C_{x_0}M} \psi\left(\text{div}\overline{\wedge^{q+1}T\mathcal{I}(h)}\right) d\mu_{x_0} \tag{2.5}$$

and

$$\operatorname{div}\left(\overline{\wedge^{q+1}T\mathcal{I}(h)}\right) = \overline{\wedge^{q}T\mathcal{I}(\operatorname{div} h)}. \tag{2.6}$$

Proof. The pull back form $\mathcal{I}_*(\psi)$ is in $\mathbb{D}^{1,2}\left((\wedge^q H)^*\right)$. The integration by parts formula of Shigekawa, Theorem 2.1, applies to give:

$$\begin{aligned}\int_{C_{x_0}M} d\psi\left(\overline{\wedge^{q+1}T\mathcal{I}(h)}\right) d\mu_{x_0} &= \int_{C_0\mathbb{R}^m} d(\mathcal{I}_*\psi)(h)\, d\mathbb{P}\\ = -\int_{C_0\mathbb{R}^m} \mathcal{I}_*\psi\,(\operatorname{div} h)\, d\mathbb{P} &= -\int_{C_{x_0}M} \psi\left(\overline{\wedge^{q}T\mathcal{I}(\operatorname{div} h)}\right) d\mu_{x_0}. \qquad \square\end{aligned}$$

For other discussions of differential forms on path spaces see [19, 21, 23] together with the references therein.

3. Some Families of H-Valued Vector Fields

3.1. *The Lie brackets*

Below we identify a family of vector fields whose Lie brackets behave particularly well.

Proposition 3.1. *Given tangent vectors* $b_i \in T_{x_0}M, i = 1, 2$ *and* $\rho : [0, T] \to \mathbb{R}$, *a non-random* L^1 *function, set*:

$$h^i_\cdot = \int_0^\cdot \rho(r)Y_{x_r}\left(T\xi_r(b^i)\right)\, dr. \tag{3.1}$$

Then the Lie bracket of the two vector fields h^i *on the Wiener space* $C_0(\mathbb{R}^m)$ *is given by*:

$$\begin{aligned}\frac{d}{dt}\left[h^1, h^2\right]_t &= \rho(t)\left(\int_0^t \rho(r)\, dr\right)(dY)\left(T\xi_t(b^1), T\xi_t(b^2)\right)\\ &\quad + \rho(t)Y_{x_t}\left(T\xi_t\int_0^t \rho(r)(T\xi_r)^{-1}X(x_r)dY\left(T\xi_r(b^1), T\xi_r(b^2)\right)\, dr\right),\end{aligned} \tag{3.2}$$

and so

$$T\mathcal{I}_t([h^1, h^2]) = \left(\int_0^t \rho(s)ds\right)T\xi_t\left(\int_0^t \rho(s)T\xi_s^{-1}X(x_s)dY\left(T\xi_s(b^1), T\xi_s(b^2)\right)ds\right). \tag{3.3}$$

Proof. Denote by D differentiation on Euclidean spaces, D^H the H-derivatives of Wiener functionals, ∇ covariant differentiation using the Levi–Cività connection on M, and ∇^H the corresponding covariant H-derivatives. Below we regularly abuse notation by omitting ω in the argument, especially with ξ_t and $T\xi_t$. First observe that

$$\begin{aligned} T\mathcal{I}_t\left(h^i\right)(\omega) &= T\xi_t \int_0^t (T\xi_r)^{-1} X(x_r(\omega))(\dot{h}^i_r)\,dr \\ &= \left(\int_0^t \rho(r)dr\right) T\xi_t(b^i) \end{aligned} \tag{3.4}$$

and that

$$\begin{aligned} \frac{d}{dt}\left[h^1, h^2\right]_t(\omega) &= \frac{d}{dt}\left(D^H h^2_t(\omega)(h^1(\omega)) - D^H h^1_t(\omega)(h^2(\omega))\right) \\ &= D^H \dot{h}^2_t(\omega)(h^1(\omega)) - D^H \dot{h}^1_t(\omega)(h^2(\omega)). \end{aligned} \tag{3.5}$$

Now

$$\begin{aligned} &D^H \dot{h}^2_t(\omega)(h^1(\omega)) \\ &\quad = D^H\left(Y_{x_t}\left(T\xi_t(\rho(t)b^2)\right)\right)(\omega)(h^1(\omega)) \\ &\quad = \rho(t)\left(\nabla^H_{h^1(\omega)} Y_{x_t}\right)\left(T\xi_t(b^2)\right) + \rho(t) Y_{x_t(\omega)}\left(\nabla^H_{h^1(\omega)}\left(T\xi_t(b^2)\right)\right) \\ &\quad = \rho(t)\left(\nabla_{T_\omega \mathcal{I}_t(h^1)} Y\right)\left(T\xi_t(b^2)\right) + \rho(t) Y_{x_t(\omega)}\left(\nabla^H_{h^1(\omega)}\left(T\xi_t(b^2)\right)\right). \end{aligned}$$

By the expression (3.4) for $T\mathcal{I}_t(h^1)$, we have

$$\begin{aligned} D^H \dot{h}^2_t(\omega)(h^1(\omega)) &= \rho(t)\left(\int_0^t \rho(r)dr\right) \nabla_{T\xi_t(b^1)} Y\left(T\xi_t(b^2)\right) \\ &\quad + \rho(t) Y_{x_t(\omega)}\left(\nabla^H_{h^1(\omega)}\left(T\xi_t(b^2)\right)\right). \end{aligned} \tag{3.6}$$

To calculate the second term, consider the map

$$\begin{aligned} \tilde{\mathcal{I}}_t :& \quad M \times C_0(R^m) \to M \\ \tilde{\mathcal{I}}_t(x,\omega) =& \quad \xi_t(x)(\omega) \end{aligned}$$

with $T_{(x,\omega)}\tilde{\mathcal{I}}_t : T_x M \times L^{2,1}_0(\mathbb{R}^m) \to T_{\xi_t(x)(\omega)} M$ its derivative in the sense of Malliavin calculus. Then

$$T_{(x,\omega)}\tilde{\mathcal{I}}_t(u,h) = T_x\xi_t \int_0^t \left(T_x\xi_r\right)^{-1} X(\xi_r(x))(\dot{h}_r)\,dr + T\xi_t(u),$$

and $T_{(x_0,\omega)}\tilde{\mathcal{I}}_t(u, h(\omega)) = T\mathcal{I}_t(h(\omega)) + T\xi(u)$. Since b^2 is non-random,

$$\begin{aligned} Y_{x_t(\omega)}\left(\nabla^H_{h^1(\omega)}\left(T\xi_t(b^2)\right)\right) &= Y_{x_t(\omega)}\left(\nabla^H_{h^1(\omega)}(T\xi_t)\right)(b^2) \\ &= Y_{x_t(\omega))}\nabla_{b^2}\left[T_{(-,\omega)}\tilde{\mathcal{I}}\left(0, h^1(\omega)\right)\right]_t. \end{aligned}$$

But

$$
\begin{aligned}
&\nabla_{b^2}\left[T_{(-,\omega)}\tilde{\mathcal{I}}\left(0,h^1(\omega)\right)\right]_t \\
&\quad = \nabla_{b^2}\left[T_-\xi_t\int_0^t\left(T_-\xi_r\right)^{-1}X(\xi_r(-,\omega))e\left(\dot{h}_r^1(\omega)\right)\,dr\right] \\
&\quad = \nabla_{b^2}\left[T_-\xi_t\int_0^t\left(T_-\xi_r\right)^{-1}X(\xi_r(-))\rho(r)Y_{x_r}\left(T\xi_r(b^1)\right)\,dr\right].
\end{aligned}
$$

Since $\left(T_-\xi_r\right)^{-1}X(\xi_r(-,\omega))Y_{\xi_r(-)}\left(T_-\xi_r\right)=\mathrm{Id}$,

$$
\begin{aligned}
&\nabla_{b^2}\left[T_{(-,\omega)}\tilde{\mathcal{I}}\left(0,h^1(\omega)\right)\right]_t \\
&\quad = \left(\int_0^t\rho(r)dr\right)\nabla_{b^2}\left(T\xi_t\right)(b^1) \\
&\qquad - T_{x_0}\xi_t\int_0^t\left(T_{x_0}\xi_r\right)^{-1}X(x_r)\nabla_{b^2}\left(\rho(r)Y_{\xi_r(-)}\left(T_-\xi_r(b^1)\right)\right) \\
&\quad = \left(\int_0^t\rho(r)dr\right)\nabla_{b^2}\left(T\xi_t\right)(b^1) - T_{x_0}\xi_t\int_0^t\rho(r)\left(T_{x_0}\xi_r\right)^{-1}\nabla_{b^2}\left(T_-\xi_r\right)(b^1) \\
&\qquad - T_{x_0}\xi_t\int_0^t\rho(r)\left(T_{x_0}\xi_r\right)^{-1}X(x_r)\nabla_{b^2}\left(Y_{\xi_r(-)}\right)\left(T_{x_0}\xi_r(b^1)\right) \\
&\quad = \left(\int_0^t\rho(r)dr\right)\nabla_{b^2}\left(T\xi_t\right)\left(b^1\right) - T\xi_t\int_0^t\rho(r)\left(T\xi_r\right)^{-1}\nabla_{b^2}\left(T\xi_r\right)(b^1)\,dr \\
&\qquad - T\xi_t\int_0^t\rho(r)\left(T\xi_r\right)^{-1}X(x_r)\left(\nabla_{T\xi_r(b^2)}Y\right)\left(T\xi_r(b^1)\right)\,dr,
\end{aligned}
$$

and so

$$
\begin{aligned}
&\nabla_{b^2}\left[T_{(-,\omega)}\tilde{\mathcal{I}}\left(0,h^1(\omega)\right)\right]_t - \nabla_{b^1}\left[T_{(-,\omega)}\tilde{\mathcal{I}}\left(0,h^2(\omega)\right)\right]_t \\
&\quad = \left(\int_0^t\rho(r)dr\right)\left(\nabla_{b^2}\left(T\xi_t\right)\left(b^1\right) - \nabla_{b^1}\left(T\xi_t\right)\left(b^2\right)\right) \\
&\qquad - T\xi_t\int_0^t\rho(r)\left(T\xi_r\right)^{-1}\left(\nabla_{b^2}\left(T\xi_r\right)(b^1) - \nabla_{b^1}\left(T\xi_r\right)(b^2)\right)\,dr \\
&\qquad - T\xi_t\int_0^t\rho(r)\left(T\xi_r\right)^{-1}X(x_r)dY\left(T\xi_r(b^2),T\xi_r(b^1)\right)\,dr \\
&\quad = -T\xi_t\int_0^t\rho(r)\left(T\xi_r\right)^{-1}X(x_r)dY\left(T\xi_r(b^2),T\xi_r(b^1)\right)\,dr.
\end{aligned}
$$

Finally combining (3.5), (3.6) and the last identity, we have

$$
\begin{aligned}
&\frac{d}{dt}\left[h^1,h^2\right]_t(\omega) \\
&\quad = \rho(t)\left(\int_0^t\rho(r)dr\right)\left(\nabla_{T\xi_t(b^1)}Y\right)\left(T\xi_t(b^2)\right) \\
&\qquad - \rho(t)\left(\int_0^t\rho(r)dr\right)\left(\nabla_{T\xi_t(b^2)}Y\right)\left(T\xi_t(b^1)\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \rho(t) Y_{x_t}\left(T\xi_t \int_0^t \rho(r)\left(T\xi_r\right)^{-1} X(x_r) dY\left(T\xi_r(b^2), T\xi_r(b^1)\right)\, dr\right) \\
&= \rho(t)\left(\int_0^t \rho(r) dr\right) dY\left(T\xi_t(b^1), T\xi_t(b^2)\right) \\
&+ \rho(t) Y_{x_t}\left(T\xi_t \int_0^t \rho(r)\left(T\xi_r\right)^{-1} X(x_r) dY\left(T\xi_r(b^1), T\xi_r(b^2)\right)\, dr\right).
\end{aligned}
$$

To prove (3.3), simply do an integration by parts:

$$
\begin{aligned}
T\mathcal{I}_t([h^1, h^2]) &= T\xi_t \int_0^t \rho(s)\left(\int_0^s \rho(r) dr\right)(T\xi_s)^{-1} X dY\left(T\xi_s(b^1), T\xi_s(b^2)\right) ds \\
&\quad + T\xi_t \int_0^t \rho(s) \int_0^s \rho(r)(T\xi_r)^{-1} X dY\left(T\xi_r(b^1), T\xi_r(b^2)\right)\, dr ds \\
&= T\xi_t \int_0^t \rho(s)\left(\int_0^s \rho(r) dr\right)(T\xi_s)^{-1} X dY\left(T\xi_s(b^1), T\xi_s(b^2)\right) ds \\
&\quad + T\xi_t\left(\int_0^t \rho(s) ds\right) \int_0^t \rho(r)(T\xi_r)^{-1} X dY\left(T\xi_r(b^1), T\xi_r(b^2)\right)\, dr ds \\
&\quad - T\xi_t \int_0^t\left(\int_0^s \rho(r) dr\right) \rho(s)(T\xi_s)^{-1} X dY\left(T\xi_s(b^1), T\xi_s(b^2)\right)\, ds \\
&= \left(\int_0^t \rho(s) ds\right) T\xi_t\left(\int_0^t \rho(s)(T\xi_s)^{-1} X dY\left(T\xi_s(b^1), T\xi_s(b^2)\right) ds\right). \quad \square
\end{aligned}
$$

Note that if $\breve{T}$ is the torsion of the connection $\breve{\nabla}$, then $X(dY) = \breve{T}$ by (1.4). Consequently,

Corollary 3.2. *For h^i defined by* (1.4) *in Proposition* 3.1,

$$
\begin{aligned}
X(x_t)\frac{d}{dt}[h^1, h^2]_t &= \rho(t)\left(\int_0^t \rho(r)\, dr\right) \breve{T}\left(T\xi_t(b^1), T\xi_t(b^2)\right) \\
&\quad + \rho(t) T\xi_t \int_0^t \rho(r)(T\xi_r)^{-1} \breve{T}\left(T\xi_r(b^1), T\xi_r(b^2)\right)\, dr \qquad (3.7)
\end{aligned}
$$

and

$$
T\mathcal{I}_t([h^1, h^2]) = \left(\int_0^t \rho(s) ds\right) T\xi_t\left(\int_0^t \rho(s) T\xi_s^{-1} \breve{T}\left(T\xi_s(b^1), T\xi_s(b^2)\right) ds\right). \qquad (3.8)
$$

We shall now return to our examples.

Example 3.1. Gradient system

In Example 1.1, where the stochastic differential equation is a gradient system, we have $dY \equiv 0$. Consequently $\frac{d}{dt}\left[h^1, h^2\right]_t(\omega)$ vanishes and so does $\left[h^1, h^2\right]_t(\omega)$. Indeed in this case if we set

$$
h^v = \int_0^{\cdot} Y_{x_r}\left(T\xi_r(v)\right)\, dr
$$

for $v \in T_{x_0}M$, we have a commuting family $\{h^v : v \in T_{x_0}M\}$ of $\mathcal{H}$-valued vector fields on $C_0\mathbb{R}^m$. Note that if $\alpha : M \to \mathbb{R}^m$ is the immersion defining our stochastic differential equation, then $\alpha \circ \xi_t$ is a random map of M into $\mathbb{R}^m$ and $\dot{h}^v_t = d(\alpha \circ \xi_t)(v) = (\xi_t)_*(d\alpha)(v)$.

Example 3.2. The Lie group case

For M a Lie group with left-invariant stochastic differential equations corresponding to a bi-invariant metric as discussed in Sec. 1, if $v \in \mathfrak{g}$ and $x_0 = e$ the identity element, we have

$$\begin{aligned}
h^i_\cdot &:= \int_0^\cdot \rho(r) Y^L_{x_r}\left(T\xi^L_r(b^i)\right)\, dr \\
&= \int_0^\cdot \rho(r) TL^{-1}_{x_r} TR_{x_r}(b^i)\, dr \\
&= \int_0^\cdot \rho(r) ad(x_r^{-1})(b^i)\, dr\,.
\end{aligned}$$

In this case

$$T\mathcal{I}_t(h^i) = TR_{x_t}\int_0^t ad(x_r)\dot{h}^i_r\, dr = TR_{x_t}\left(\int_0^t \rho(r)dr b^i\right).$$

Let T^R and T^L be the torsions for the left- and right-invariant connections respectively. From above, (1.5), the fact that $T\xi^L_t(u) = TR_{x_t}(u)$, and by the right invariance of TR,

$$\begin{aligned}
\frac{d}{dt}[h^1, h^2]_t &= -\rho(t)\left(\int_0^t \rho(r)\, dr\right) TL^{-1}_{x_t} TR_{x_t}[b^1, b^2] \\
&\quad - \rho(t) TL^{-1}_{x_t} TR_{x_t}\int_0^t \rho(r)[b^1, b^2]\, dr \\
&= -2\rho(t)\left(\int_0^t \rho(r)\, dr\right) ad(x_t^{-1})[b^1, b^2] \\
&= -\frac{d}{dt}\left(\int_0^t \rho(r) dr\right)^2 ad(x_t^{-1})[b^1, b^2] \\
&= -ad(x_t^{-1})\frac{d}{dt}\left[\int_0^t \rho(r) dr b^1, \int_0^t \rho(r) dr b^2\right].
\end{aligned}$$

Note that

$$\begin{aligned}
T\mathcal{I}_t([h^1, h^2]) &= -TR_{x_t}\int_0^t ad(x_r)\left(\frac{d}{dr}[h^1, h^2]_r\right) dr \\
&= -TR_{x_t}\int_0^t 2\rho(r)\left(\int_0^r \rho(s) ds\right) dr [b^1, b^2] \\
&= -TR_{x_t}\left[\int_0^t \rho(r) dr b^1, \int_0^t \rho(r) dr b^2\right].
\end{aligned}$$

Example 3.3. Lie group as a symmetric space

We now consider the stochastic differential equation (1.6):

$$dx_t = TR_{x_t} \circ dB_t - TL_{x_t} \circ dB'_t$$

on the Lie group G with bi-invariant metric. The derivative flow of the solution ξ_t is given by $T\xi_t = TR_{(g'_t)^{-1}} TL_{g_t}$. Note that in our notation the map, giving the equation (1.6), $X(g) : \mathfrak{g} \times \mathfrak{g} \to T_x M$ is given by $X(g)(\alpha_1, \alpha_2) = TR_g(\alpha_1) - TL_g(\alpha_2)$ with inverse $Y : TG \to \mathfrak{g} \times \mathfrak{g}$ given, for $u \in T_g G$, by

$$\begin{aligned} Y(u) &= (TR_g^{-1}(u), -TL_g^{-1}(u)) \\ &= (Y^R u, -Y^L u). \end{aligned}$$

Thus if $u, v \in T_g G$, for T^R and T^L the torsion tensors as in (1.5),

$$\begin{aligned} dY(u,v) &= \left(TR_g^{-1} T^R(u,v), TL_g^{-1} T^L(u,v)\right) \\ &= \left([Y^R(u), Y^R(v)],\ [Y^L(u), Y^L(v)]\right) \end{aligned}$$

so

$$\begin{aligned} \frac{d}{dt}[h^1, h^2]_t = \rho(t) \left(\int_0^t \rho(r)dr\right) \Big(& [Y^R \left(TL_{g_t} TR_{g'_t}^{-1} b^1\right), Y^R(TL_{g_t} TR_{g'_t}^{-1} b^2)], \\ & [Y^L(TL_{g_t} TR_{g'_t}^{-1} b^2), Y^L(TL_{g_t} TR_{g'_t}^{-1} b^1])\Big). \end{aligned}$$

Now

$$\begin{aligned} Y^R \left(TL_{g_t} TR_{(g'_t)}^{-1} b^i\right) &= \left(TR_{g_t}(g'_t)^{-1}\right)^{-1} TL_{g_t} TR_{g'_t}^{-1} b^i \\ &= \left(TR_{g'_t}^{-1} \circ TR_{g_t}\right)^{-1} TL_{g_t} TR_{g'_t}^{-1} b^i \\ &= ad(g_t) b^i \end{aligned}$$

while

$$\begin{aligned} Y^L \left(TL_{g_t} TR_{(g'_t)}^{-1} b^i\right) &= \left(TL_{g_t}(g'_t)^{-1}\right)^{-1} TL_{g_t} TR_{g'_t}^{-1} b^i \\ &= TL_{g'_t} TR_{g'_t}^{-1} b^i = ad(g'_t) b^i, \end{aligned}$$

so

$$\begin{aligned} \frac{d}{dt}[h^1, h^2]_t &= \rho(t) \left(\int_0^t \rho(r)dr\right) \left(ad(g_t)[b^1, b^2], ad(g'_t)[b^1, b^2]\right) \\ &\in \mathfrak{g} \times \mathfrak{g}. \end{aligned}$$

In this case $h_t^i = \left(\int_0^t \rho(r) ad(g_r) b^i\, dr, -\int_0^t \rho(r) ad(g'_r) b^i\, dr\right)$.

3.2. A family of $\mathcal{H}^2$-valued vector fields

In general when the associated connection $\breve{\nabla}$ for (1.1) is the Levi–Cività connection ∇ and $A \equiv 0$ we expect that $\overline{\wedge^q T\mathcal{I}(h^1 \wedge \ldots \wedge h^q)}_\sigma \in \mathcal{H}^q_\sigma$ for almost all $\sigma \in C_{x_0}M$. It follows from [10] that this holds for $q = 1$ and from [6] for $q = 2$, and a proof for $q = 2$ for our special h^i's is given below in Proposition 3.3 as an illustrative example. For general q we have,

$$\overline{\wedge^q (T\mathcal{I})(h^1 \wedge \ldots \wedge h^q)} = \mathbb{E}\left\{\wedge^q (T\xi_\cdot)\left(b^{\rho,1}_\cdot \wedge \ldots \wedge b^{\rho,q}_\cdot\right) | x_s : 0 \le s \le T\right\},$$

where $b^{\rho,j}_t = \left(\int_0^t \rho(r)dr\right) b^j$ and $(T\xi_\cdot)(b^{\rho,j}_\cdot)$ is just the vector field $\{(T\xi_t)(b^{\rho,j}_t) : 0 \le t \le T\}$ along $\xi_\cdot(x_0)$. From [12] or [16] for gradient systems, we can deduce that for $0 < t_1 < \ldots < t_q \le T$

$$\begin{aligned}
&\overline{\wedge^q T\mathcal{I}(h^1 \wedge \ldots \wedge h^q)}_{t_1,\ldots,t_q} \\
&\quad = \int_0^{t_1} \rho(r)dr \int_0^{t_2} \rho(r)dr \ldots \int_0^{t_q} \rho(r)dr \cdot \left(\mathbf{1} \otimes \ldots \otimes \mathbf{W}^{\mathbf{t_{q-1}}}_{\mathbf{t,q}}\right) \\
&\quad\quad \cdot \left(\mathbf{1} \otimes \ldots \otimes \mathbf{W^{(2)}}^{\mathbf{t_{q-2}}}_{\mathbf{t_{q-1}}}\right) \ldots W^{(q)}_{t_1}(b^1 \wedge \ldots \wedge b^q)\,.
\end{aligned} \tag{3.9}$$

Here $W^{(q)}{}^s_t = W^{(q)}_t \circ (W^{(q)}_s)^{-1} : \wedge^q T_{x_s}M \to \wedge^q T_{x_t}M$ with $W^s_t = W^{(1)}{}^s_t$, for $W^{(q)}$ the damped parallel transport defined by (0.3).

Let us recall the characterization of $\mathcal{H}^1$ and $\mathcal{H}^2$. For $q = 2$ it is shown in [6], that

$$\mathcal{H}^2_\sigma = \left\{U + Q_\sigma(U) | U \in \wedge^2 \mathcal{H}^1_\sigma\right\} \subset \wedge^2 T_\sigma C_{x_0}M$$

where $Q_\sigma : \wedge^2\mathcal{H}^1_\sigma \to \wedge^2 T_\sigma C_{x_0}M$ is defined by

$$Q(U)_{s,t} = (\mathbf{1} \otimes \mathbf{W^s_t})\mathbf{W^{(2)}_s} \int_{\mathbf{0}}^{\mathbf{s}} (\mathbf{W^{(2)}_r})^{-\mathbf{1}} \mathcal{R}(\mathbf{U_{r,r}})\,\mathbf{dr}, \qquad \mathbf{0 \le s \le t \le T} \tag{3.10}$$

for $\mathcal{R} : \wedge^2 TM \to \wedge^2 TM$ the curvature operator.

The space $\mathcal{H}^1$ is the 'Bismut tangent space', [19], but with a different Hilbert space structure. In fact elements of $\mathcal{H}^1$ are exactly vector fields v in M along σ such that $\|v\|_\sigma := \int_0^T \left|\frac{\mathbb{D}}{dt} v_t\right|^2_{\sigma(t)} dt < \infty$ where

$$\frac{\mathbb{D}}{dt} v_t := \frac{D}{dt} v_t + \frac{1}{2} Ric^{\#}(v_t) = W_t^{-1} \frac{d}{dt} W_t^{-1} V_t. \tag{3.11}$$

From this it follows that $\wedge^2\mathcal{H}^1_\sigma$ consists of those U in $\wedge^2 T_\sigma C_{x_0}M$ for which

$$\left(\wedge^2 (W_\cdot)^{-1}(U)\right)_{s,t} = \int_0^s \int_0^t H(r_1, r_2)\, dr_1 dr_2$$

for some H in $L^2\left([0,T] \times [0,T]; T_{x_0}M \otimes T_{x_0}M\right)$.

Proposition 3.3. *Let $\lambda \in L^{2,1}_0([0,T];\mathbb{R})$. Take V in $\wedge^2 T_{x_0}M$ and $V^\lambda \in \wedge^2 L^{2,1}_0 T_{x_0}M$ defined by $V^\lambda_{s,t} = \lambda(s)\lambda(t)V$. If $A \equiv 0$ in (1.1) and the associated connection is the Levi–Cività connection, the vector field Z, where*

$$Z = \mathbb{E}\left\{\wedge^2(T\xi_\cdot)(V^\lambda) | x_s, 0 \le s \le T\right\}$$

forms a section of $\mathcal{H}^2$. *It is equal to* $U + Q(U)$ *for* Q *as above and* U *given by*

$$\begin{aligned} U_{s,t} &= (\mathbf{1} \otimes \mathbf{W_t^s})\lambda(\mathbf{s})\lambda(\mathbf{t})\mathbf{W_s^{(2)}}(\mathbf{V}) \\ &\quad - (W_s \otimes W_t) \int_0^s \lambda^2(r) \wedge^2 (W_r)^{-1} \mathcal{R}\left(W_r^{(2)}(V)\right) dr, \end{aligned} \tag{3.12}$$

for all $0 \le s \le t \le T$. *It has divergence*

$$\begin{aligned} (\operatorname{div} Z)_t &= -\int_0^T \left(\langle \frac{\mathbb{D}}{\partial r} -, dx_r\rangle \otimes \mathbf{1}\right) Z_{r,t} \\ &\quad + W_t \int_0^t \left(\langle -, dx_r\rangle \otimes (W_r)^{-1})\mathcal{R}(\mathbf{1} \otimes \mathbf{W_t^r})\right)^{-1} Z_{r,t}. \end{aligned} \tag{3.13}$$

(*For an interpretation of the, apparently adapted, integrals, see the proof below.*)

Proof. From (3.9), for $0 \le s \le t \le T$,

$$Z_{s,t} = \lambda(s)\lambda(t)(\mathbf{1} \otimes \mathbf{W_t^s})\mathbf{W_s^{(2)}}(\mathbf{V}). \tag{3.14}$$

So

$$\left[(\wedge^2 W_\cdot)^{-1} Z\right]_{s,t} = \lambda(s)\lambda(t)(\wedge^2 W_s)^{-1} W_s^{(2)}(V). \tag{3.15}$$

Setting

$$k(r) = \wedge^2(W_r^{-1}) W_r^{(2)}(V) \in \wedge^2 T_{x_0} M \tag{3.16}$$

we see that

$$\left[(\wedge^2 W_\cdot)^{-1} Z\right]_{s,t} = \lambda(s)\lambda(t)k(s \wedge t), \qquad \text{for all } s, t \in [0, T].$$

Set

$$\tilde{U} = \wedge^2(W^{-1})(U)\,.$$

We only need to show that

$$\tilde{U} + (\wedge^2 W_\cdot^{-1}) Q(U) = \lambda(s)\lambda(t)k(s \wedge t) \tag{3.17}$$

and that U belongs to $\wedge^2\mathcal{H}^1$.

Now for any $K\colon [0,T] \times \Omega \to \wedge^2 TM$ over $\{x_t : 0 \le t \le T\}$

$$\wedge^2(W_r)\frac{d}{dr}\left(\wedge^2(W_r)^{-1}K_r\right) = \mathcal{R}(K_r) + W_r^{(2)}\frac{d}{dr}[(W_r^{(2)})^{-1}K_r] \tag{3.18}$$

when the derivative exists, so $k'(s) = \wedge^2(W_s^{-1})\mathcal{R}\left(W_s^{(2)}(V)\right)$ and consequently by (3.12)

$$\tilde{U}_{s,t} = \lambda(s)\lambda(t)k(s \wedge t) - \int_0^{s \wedge t} \lambda^2(r) k'(r)\, dr, \tag{3.19}$$

for $s, t \in [0, T]$. Integrating by parts,

$$\begin{aligned}\tilde{U}_{s,t} &= \lambda(s)\lambda(t)k(s\wedge t) - \lambda^2(s\wedge t)k(s\wedge t) + 2\int_0^{s\wedge t}\lambda'(r)\lambda(r)k(r)\,dr\\ &= 2\int_0^{s\wedge t}\lambda'(r)\lambda(r)k(r)\,dr + (\lambda(s)\lambda(t)-\lambda^2(s\wedge t))k(s\wedge t). \end{aligned}\tag{3.20}$$

From this it is easy to see that

$$\tilde{U}_{\bar{s}\bar{t}} = \int_0^{\bar{s}} ds \int_0^{\bar{t}} dt\,(\lambda'(s)\lambda'(t)k(s\wedge t) + \lambda(s\wedge t)\lambda'(s\vee t)k'(s\wedge t))\tag{3.21}$$

since for $\bar{s} \le \bar{t}$ one has

$$\begin{aligned}&\int_0^{\bar{s}} ds \int_0^{\bar{t}} dt\,(\lambda'(s)\lambda'(t)k(s\wedge t) + \lambda(s\wedge t)\lambda'(s\vee t)k'(s\wedge t))\\ &\quad= \int_0^{\bar{s}}\left\{\int_0^{s}(\lambda'(s)\lambda'(t)k(t) + \lambda(t)\lambda'(s)k'(t))\,dt\right.\\ &\qquad\left.+\int_s^{\bar{t}}(\lambda'(s)\lambda'(t)k(s)+\lambda(s)\lambda'(t)k'(s))\,dt\right\}\\ &\quad= \int_0^{\bar{s}}\lambda'(s)\lambda(s)k(s)\,ds + \int_0^{\bar{s}}(\lambda(\bar{t})-\lambda(s))\frac{d}{ds}(\lambda(s)k(s))\,ds\\ &\quad= 2\int_0^{\bar{s}}\lambda'(s)\lambda(s)k(s)\,ds + (\lambda(\bar{t})-\lambda(\bar{s}))\lambda(\bar{s})k(\bar{s}). \end{aligned}\tag{3.22}$$

This shows that $U \in \wedge^2\mathcal{H}^1$. Next note that

$$\tilde{U}_{s,s} = 2\int_0^s \lambda'(r)\lambda(r)k(r)\,dr$$

by (3.20). By the definition, (3.10), of Q, (3.16), and using (3.18) again we see that for $0 \le s \le t \le T$,

$$\begin{aligned}\left[\wedge^2(W_\cdot^{-1})Q(U)\right]_{s,t} &:= \wedge^2(W_s^{-1})W_s^{(2)}\int_0^s (W_r^{(2)})^{-1}\mathcal{R}\,(U_{rr})\,dr\\ &= \wedge^2(W_s^{-1})W_s^{(2)}\int_0^s (W_r^{(2)})^{-1}\wedge^2(W_r)\frac{d}{dr}\tilde{U}_{r,r}\,dr\\ &\quad-\wedge^2(W_s^{-1})W_s^{(2)}\int_0^s \frac{d}{dr}\left[(W_r^{(2)})^{-1}U_{rr}\right]\,dr\\ &= \wedge^2(W_s^{-1})W_s^{(2)}\int_0^s 2(W_r^{(2)})^{-1}\wedge^2(W_r)\lambda'(r)\lambda(r)k(r)\,dr - \tilde{U}_{s,s}\\ &= \wedge^2(W_s^{-1})W_s^{(2)}\int_0^s 2\lambda'(r)\lambda(r)V\,dr - \tilde{U}_{s,s}\end{aligned}$$

$$
\begin{aligned}
&= \lambda^2(s)k(s) - 2\int_0^s \lambda'(r)\lambda(r)k(r)\,dr \\
&= \int_0^s [\lambda(r)]^2 k'(r)\,dr.
\end{aligned}
$$

Finally add (3.19) to the above to obtain the required identity (3.17).

To obtain (3.13) we can assume V to be primitive, $V = b^1 \wedge b^2$ say. By (2.6) the divergence of Z is just the conditional expectation $\bar{J}$ of J for $J = T\mathcal{I}(div(h^1 \wedge h^2))$ where $\dot{h}_t^j = \dot{\lambda}(t) Y_{x_t} T\xi_t(b^j)$. By Theorem 2.1 and Proposition 3.1

$$
\begin{aligned}
J_t &= T\mathcal{I}_t\left(-\int_0^T \langle \dot{h}_r^1, dB_r\rangle h_\cdot^2 + \int_0^T \langle \dot{h}_r^2, dB_r\rangle h_\cdot^1\right) \\
&= J_t^0 + J_t^1\,,
\end{aligned}
$$

where

$$
\begin{aligned}
J_t^0 &= T\mathcal{I}_t\left(-\int_t^T \langle \dot{h}_r^1, dB_r\rangle h_\cdot^2 + \int_t^T \langle \dot{h}_r^2, dB_r\rangle h_\cdot^1\right) \\
&= -\int_t^T \left\langle \dot{\lambda}(r) T\xi_r^t(T\xi_t(b^1)), dx_r\right\rangle \lambda(t) T\xi_t(b^2) \\
&\quad + \int_t^T \left\langle \dot{\lambda}(r) T\xi_r^t(T\xi_t(b^2)), dx_r\right\rangle \lambda(t) T\xi_t(b^1).
\end{aligned}
$$

Therefore

$$
\mathbb{E}\left\{J_t^0 \middle|\, \mathcal{F}^{x_0} \vee \mathcal{F}_t\right\} = -\lambda(t)\int_t^T \dot{\lambda}(r)\iota_{\langle W_r^t -, dx_r\rangle} \wedge^2 (T\xi_t)(b^1 \wedge b^2)
$$

giving

$$
\bar{J}_t^0 = \int_t^T \left(\mathbf{1} \otimes \left\langle \frac{\mathbb{D}}{\partial \mathbf{r}} -, \mathbf{dx_r}\right\rangle\right) Z_{t,r}
$$

by (3.14).

The second term J_t^1 could be treated as in Corollary 3.5 below, but to get the form (3.13) observe

$$
J_t^1 = -\int_0^t \lambda(t)\dot{\lambda}(r)\left(\langle -, dx_r\rangle \otimes \mathbf{1}\right)\left(\wedge^2(T\xi)(b^1 \wedge b^2)\right)_{rt}
$$

and so by (3.9)

$$
\bar{J}_t^1 = -\int_0^t \lambda(t)\dot{\lambda}(r)\left(\langle -, dx_r\rangle \otimes \mathbf{1}\right)\left(\mathbf{1} \otimes \mathbf{W_t^r})\mathbf{W_r^{(2)}}(\mathbf{b^1} \wedge \mathbf{b^2})\right)_{rt}.
$$

Here we can interpret the, apparently adapted, integral by using $W_t^r = W_t(W_r)^{-1}$ and taking the term $(\mathbf{1} \otimes \mathbf{W_t})$ outside of the integral. Now

$$
\begin{aligned}
\left(\frac{\mathbb{D}}{\partial r} \otimes \mathbf{1}\right) Z_{rt} &= \frac{D}{\partial r}\left[\lambda(r)\lambda(t)(\mathbf{1} \otimes \mathbf{W_t^r})\mathbf{W_r^{(2)}}(\mathbf{b^1} \wedge \mathbf{b^2})\right] \\
&\quad + \frac{1}{2}(Ric^{\#} \otimes \mathbf{1})\mathbf{Z_{rt}}
\end{aligned}
$$

giving

$$
\begin{aligned}
&\dot{\lambda}(r)\lambda(t)(\mathbf{1}\otimes \mathbf{W^r_t})\mathbf{W_r^{(2)}}(\mathbf{b^1}\wedge\mathbf{b^2})\\
&\quad=\left(\frac{\mathbb{D}}{\partial r}\otimes\mathbf{1}\right)Z_{rt}-\frac{1}{2}(Ric^{\#}\otimes\mathbf{1})\mathbf{Z_{rt}}\\
&\qquad-\lambda(r)\lambda(t)(1\otimes W_t)\frac{D}{\partial r}(\mathbf{1}\otimes(\mathbf{W_r})^{-1})\mathbf{W_r^{(2)}}(\mathbf{b^1}\wedge\mathbf{b^2}),
\end{aligned}
$$

but

$$
\begin{aligned}
&\frac{D}{\partial r}(\mathbf{1}\otimes(\mathbf{W_r})^{-1})\mathbf{W_r^{(2)}}(\mathbf{b^1}\wedge\mathbf{b^2})\\
&\quad=(W_r\otimes\mathbf{1})\frac{\mathbf{D}}{\partial\mathbf{r}}(\wedge^{\mathbf{2}}\mathbf{W_r^{-1}})\mathbf{W_r^{(2)}}(\mathbf{b^1}\wedge\mathbf{b^2})\\
&\qquad-\frac{1}{2}(Ric^{\#}\otimes\mathbf{1})(\mathbf{1}\otimes\mathbf{W_r^{-1}})\mathbf{W_r^{(2)}}(\mathbf{b^1}\wedge\mathbf{b^2})\\
&\quad=(\mathbf{1}\otimes\mathbf{W_r^{-1}})\mathcal{R}\mathbf{W_r^{(2)}}(\mathbf{b^1}\wedge\mathbf{b^2})-\frac{\mathbf{1}}{\mathbf{2}}(\mathbf{Ric}^{\#}\otimes\mathbf{1})(\mathbf{1}\otimes\mathbf{W_r^{-1}})\mathbf{W_r^{(2)}}(\mathbf{b^1}\wedge\mathbf{b^2})
\end{aligned}
$$

by (3.18), and so

$$
\dot{\lambda}(r)\lambda(t)(\mathbf{1}\otimes\mathbf{W^r_t})\mathbf{W_r^{(2)}}(\mathbf{b^1}\wedge\mathbf{b^2})=\left(\frac{\mathbb{D}}{\partial\mathbf{r}}\otimes\mathbf{1}\right)\mathbf{Z_{rt}}-(\mathbf{1}\otimes\mathbf{W^r_t})\mathcal{R}(\mathbf{1}\otimes\mathbf{W^r_t})^{-\mathbf{1}}\mathbf{Z_{rt}}.
$$

The result follows noting that $(\mathbf{1}\otimes\mathbf{W^r_t})^{-\mathbf{1}}\mathbf{Z_{rt}}$ is $\mathcal{F}^{x_0}_r$-measurable and the factor $(\mathbf{1}\otimes\mathbf{W_t})$ in $(\mathbf{1}\otimes\mathbf{W^r_t})=(\mathbf{1}\otimes\mathbf{W_t})(\mathbf{1}\otimes(\mathbf{W_r})^{-\mathbf{1}})$ can be taken out of the stochastic integrals to make them Itô integrals. □

Remark 3.3. (1) The formula (3.12) in Proposition 3.3 demonstrates the alternative characterisation of $\mathcal{H}^2$ from [6] which states that V is in $\mathcal{H}^2$ if and only if $V-\mathbb{R}(V)$ is in $\wedge^2\mathcal{H}^1$ where $\mathbb{R}$ is the curvature operator of the damped Markovian connection on $\mathcal{H}^1$.

(2) Note that for the proof of (3.12) we could allow $V: C_{x_0}M\to\wedge^2T_{x_0}M$ to be nonconstant provided it is in $L^{1+\varepsilon}$ for some $\varepsilon>0$. Our filtration in calculating (3.9) can be chosen to be $\{\mathcal{F}_t\vee\mathcal{F}^{x_0}: 0\le t\le T\}$ as in [8]. Now let $V:[0,T]\times[0,T]\to\wedge^2T_{x_0}M$ be such that $V_{s,t}=\int_0^s\int_0^t Z_{a,b}\,da\,db$ some $Z\in L^2\left([0,T]\times[0,T]\times C_{x_0}M\to\wedge^2T_{x_0}M\right)$ symmetric in $(s,t)\in[0,T]\times[0,T]$. As a corollary we see, for $0\le s\le t\le T$,

$$
\mathbb{E}\{\wedge^2(T\xi.)(V)|\,x_s:0\le s\le t\le T\}=U+Q(U)
$$

for

$$
U_{s,t}=(\mathbf{1}\otimes\mathbf{W^s_t})\mathbf{W_s^{(2)}}(\mathbf{V_{s,t}})-(\mathbf{W_s}\otimes\mathbf{W_t})\int_{\mathbf{0}}^{\mathbf{s}}\wedge^{\mathbf{2}}(\mathbf{W_r})^{-\mathbf{1}}\mathcal{R}\left(\mathbf{W_r^{(2)}}(\mathbf{V_{rr}})\right)\,\mathbf{dr}.
$$

Moreover U is a section of $\wedge^2\mathcal{H}^1$ and so $U+Q(U)$ is a section of $\mathcal{H}^2$. This follows from the proposition by polarization for $V_{s,t}=\left(\lambda^1(s)\lambda^2(t)+\lambda^2(s)\lambda^1(t)\right)b$, some $b\in L^2(C_{x_0}M;\wedge^2T_{x_0}M)$ and then for general V by continuity since we can consider such V as elements of the completed tensor product $[\odot^2 L_0^{2,1}([0,T];\mathbb{R})] \otimes L^2(C_{x_0}M; \wedge^2 T_{x_0}M)$, where $\odot$ refers to the symmetric tensor product. Note the mapping

$$\begin{aligned}
&\odot^2 L_0^{2,1}([0,T];\mathbb{R}) \otimes L^2(C_{x_0}M; \wedge^2 T_{x_0}M) \\
&\xrightarrow{\Theta} L^2\left(C_{x_0}M; \wedge^2 (L_0^{2,1}(0,T];\mathbb{R}) \otimes T_{x_0}M\right) \\
&\simeq L^2\left(C_{x_0}M; \wedge^2 (L_0^{2,1}(T_{x_0}M)\right)
\end{aligned}$$

given by

$$\begin{aligned}
&\Theta\left((u \odot v) \otimes f(\alpha \wedge \beta)\right)(\sigma) \\
&\quad = \frac{1}{2} f(\sigma)\left[(u \otimes \alpha) \wedge (v \otimes \beta) + (v \otimes \alpha) \wedge (u \otimes \beta)\right], \sigma \in C_{x_0}M,
\end{aligned}$$

where $f(\cdot)(\alpha \wedge \beta) \in L^2\left(C_{x_0}M; \wedge^2 T_{x_0}M\right) \simeq L^2(C_{x_0}M;\mathbb{R}) \otimes \wedge^2 T_{x_0}M$, $\alpha, \beta \in T_{x_0}M$, $f \in L^2(C_{x_0}M;\mathbb{R})$.

In this generality the stochastic integrals in (3.13) will need to be treated more carefully and if V is non-random an additional term involving its H-derivative will be involved in the divergence.

3.3. *From Integration by parts to Bismut type formulae*

First we shall give an extension of the non-intrinsic formula (0.4). For this consider a general non-degenerate stochastic differential equation (1.1) with smooth coefficients. For $\{\xi_t : 0 \le t < \infty\}$, its flow of diffeomorphisms of the manifold, consider the semigroup described in Sec. 1, given by

$$P_t \phi = \mathbb{E}\xi_t^*(\phi).$$

Since M is compact $d(P_t\phi) = P_t(d\phi)$ when ϕ is C^1. The generator $\mathcal{A}$ of $\{P_t\}_{t \ge 0}$ is given on smooth forms by (1.7) above.

Proposition 3.4. *For any bounded measurable form ϕ on M and L^1 function $\rho : [0,t] \to \mathbb{R}$ with $\int_0^t \rho(r)\,dr \neq 0$,*

$$\begin{aligned}
&d(P_t\phi)(b^1 \wedge \ldots \wedge b^{q+1}) \\
&\quad = \left(\int_0^t \rho(r)dr\right)^{-1} \mathbb{E}\sum_{j=1}^{q+1}(-1)^{j+1}\int_0^t \rho(s)\langle T\xi_s(b^j), X(x_s)dB_s\rangle_{x_s} \\
&\qquad \cdot \xi_t^*(\phi)(b^1 \wedge \ldots \wedge \widehat{b^j} \wedge \ldots \wedge b^{q+1})
\end{aligned}$$

$$-\left(\int_0^t \rho(r)dr\right)^{-1} \sum_{1\le i<j\le q+1} (-1)^{i+j+1}\, \mathbb{E}\xi_t^*(\phi)$$
$$\cdot\left(\int_0^t \rho(s)T\xi_s^{-1}\left(\breve{T}\left(T\xi_s(b^i), T\xi_s(b^j)\right)\right) ds \wedge b^1 \wedge \ldots \wedge \widehat{b^i} \ldots \wedge \widehat{b^j} \ldots \wedge b^{q+1}\right), \tag{3.23}$$

where $\breve{T} : TM \oplus TM \to TM$ *is the torsion of the connection* $\breve{\nabla}$ *given by* (1.4).

Proof. Define $h^i, i = 1$ to $q+1$, by (3.1). Set $h = h^1 \wedge \ldots \wedge h^{q+1}$ and $b = b^1 \wedge \ldots \wedge b^{q+1}$. Arguing as in Theorem 2.2, but without taking conditional expectations we see, for ϕ a C^1 q-form,

$$\begin{aligned} d(P_t\phi)(b) &= P_t(d\phi)(b) \\ &= \mathbb{E}\xi_t^*(d\phi)(b) = \left(\int_0^T \rho(r)dr\right)^{-q-1} \mathbb{E}d\phi\left(\wedge^{q+1}(T\mathcal{I})(h)_{t,\ldots,t}\right) \\ &= -\left(\int_0^T \rho(r)dr\right)^{-q-1} \mathbb{E}\phi\left(\wedge^q(T\mathcal{I})(\operatorname{div} h)_{t,\ldots,t}\right). \end{aligned}$$

By Shigekawa's result, Theorem 2.1, formula (3.23) follows using Proposition 3.1, equations (2.4), (3.8), and the fact that $\int_0^t \langle \dot{h}_s^i, dB_s\rangle = \int_0^t \rho(s)\langle T\xi_s(b^i), X(x_s)dB_s\rangle_{x_s}$. By continuity it also holds for bounded measurable ϕ. □

To obtain an intrinsic formula from (3.23), we shall take conditional expectations, which can easily be done if the torsion $\breve{T} = X(dY)$ is invariant under the flow (e.g. for certain homogeneous spaces). In this case formula (3.23) becomes:

$$\begin{aligned} &d(P_t\phi)(b^1 \wedge \ldots \wedge b^{q+1}) \\ &\quad = \left(\int_0^t \rho(r)dr\right)^{-1} \mathbb{E}\sum_{j=1}^{q+1}(-1)^{j+1}\int_0^t \rho(s)\langle T\xi_s(b^j), X(x_s)dB_s\rangle_{x_s} \\ &\qquad \cdot\xi_t^*(\phi)(b^1 \wedge \ldots \wedge \widehat{b^j} \wedge \ldots \wedge b^{q+1}) \\ &\qquad - \sum_{1\le i<j\le q+1} (-1)^{i+j+1}\mathbb{E}\Big(\xi_t^*(\phi)\Big(\breve{T}(b^i, b^j) \wedge b^1 \wedge \ldots \wedge \widehat{b^i} \ldots \wedge \widehat{b^j} \ldots \wedge b^{q+1}\Big)\Big). \end{aligned} \tag{3.24}$$

Let $\breve{W}_t^{A,q} : \wedge^q T_{x_0}M \to \wedge^q T_{x_t}M$, the damped parallel translation of q vectors, defined by:

$$\begin{cases} \dfrac{\hat{D}}{\partial s}(\breve{W}_s^{A,q}(V_0)) = -\dfrac{1}{2}\breve{\mathcal{R}}_{x_0}^q\left(\breve{W}_s^{A,q}(V_0)\right) + d\wedge^q\left(\breve{\nabla}A\right)\left(\breve{W}_s^{A,q}(V_0)\right), \\ \breve{W}_0^{A,q}(V_0) \quad = V_0. \end{cases} \tag{3.25}$$

Here $\frac{\hat{D}}{\partial s}$ refers to covariant differentiation using the connection $\hat{\nabla}$, the adjoint connection of $\breve{\nabla}$. Since the conditional expectation of $\wedge^q T\xi_t$ is given by

$$\overline{\wedge^q T\xi_t}(-) = \breve{W}_t^{A,q}(-),$$

by Theorem 3.3.7 in [12], or Theorem A in [16] for the Levi–Cività connection, we only need to worry about the first term on the right-hand side of equation (3.24). Set

$$U_t = \int_0^t \rho(s) \sum_{j=1}^{q+1} (-1)^{j+1} \langle T\xi_s(b^j), X(x_s)dB_s\rangle \cdot \wedge^q T\xi_t(b^1 \wedge \ldots \wedge \widehat{b^j} \wedge \ldots \wedge b^{q+1})$$

and

$$V_t^j = \wedge^q T\xi_t(b^1 \wedge \ldots \wedge \widehat{b^j} \wedge \ldots \wedge b^{q+1}).$$

Then, e.g. by (3.3.10) in [12],

$$\begin{aligned}
\hat{D}U_t &= \sum_{j=1}^{q+1} (-1)^{j+1} \rho(t) \langle T\xi_t(b^j), X(x_t)dB_t\rangle \cdot V_t^j \\
&\quad + \sum_{j=1}^{q+1} (-1)^{j+1} \int_0^t \rho(s) \langle T\xi_s(b^j), X(x_s)dB_s\rangle \cdot d\wedge^q \left(\breve{\nabla} X(-)dB_t)(V_t^j)\right) \\
&\quad + \sum_{j=1}^{q+1} (-1)^{j+1} \int_0^t \rho(s) \langle T\xi_s(b^j), X(x_s)dB_s\rangle \\
&\quad \cdot \left(-\frac{1}{2}(\breve{R}^q)^*(V_t^j)dt + d\wedge^q \breve{\nabla} A(V_t^j)dt\right) \\
&= \sum_{j=1}^{q+1} (-1)^{j+1} \rho(t) \langle T\xi_t(b^j), X(x_t)dB_t\rangle \cdot V_t^j \\
&\quad + d\wedge^q \left(\breve{\nabla} X(-)dB_t)(U_t)\right) - \frac{1}{2}(\breve{R}^q)^*(U_t)dt + d\wedge^q \breve{\nabla} A(U_t)dt\,.
\end{aligned}$$

Taking the conditional expectation of the above equation, as in the proof of Proposition 3.3.7 in [12] or that of Theorem A in [16], we have

$$\begin{aligned}
\hat{D}\overline{U_t} &= \overline{\sum_{j=1}^{q+1} (-1)^{j+1} \rho(t) \langle T\xi_t(b^j), X(x_t)dB_t\rangle \cdot \wedge^q T\xi_t(b^1 \wedge \ldots \wedge \widehat{b^j} \wedge \ldots \wedge b^{q+1})} \\
&\quad - \frac{1}{2}(\breve{R}^q)^*(\overline{U_t})dt + d\wedge^q \breve{\nabla} A(\overline{U_t})dt \\
&= \rho(t)\iota_{\langle -, X(x_t)dB_t\rangle} \overline{\wedge^{q+1} T\xi_t(b^1 \wedge \ldots \wedge b^{q+1})} - \frac{1}{2}(\breve{R}^q)^*(\overline{U_t})dt + d\wedge^q \breve{\nabla} A(\overline{U_t})dt \\
&= \rho(t)\iota_{\langle -, X(x_t)dB_t\rangle} \breve{W}_t^{A,q+1}(b^1 \wedge \ldots \wedge b^{q+1}) - \frac{1}{2}(\breve{R}^q)^*(\overline{U_t})dt + d\wedge^q \breve{\nabla} A(\overline{U_t})dt \\
&= \rho(t)\iota_{\langle -, \breve{/\!/}_t d\breve{B}_t\rangle} \breve{W}_t^{A,q+1}(b^1 \wedge \ldots \wedge b^{q+1}) - \frac{1}{2}(\breve{R}^q)^*(\overline{U_t})dt + d\wedge^q \breve{\nabla} A(\overline{U_t})dt
\end{aligned}$$

Here $\breve{/\!/}_t$ denotes parallel translation corresponding to the connection $\breve{\nabla}$ and $(\breve{B}_s)$ is the stochastic anti-development Brownian motion on $T_{x_0}M$, i.e. the martingale part of $\int_0^\cdot \breve{/\!/}_s^{-1} \circ dx_s$. Solve the equation to obtain

$$\overline{U_t} = \breve{W}_t^{A,q} \int_0^t \left(\breve{W}_s^{A,q}\right)^{-1} \rho(s) \iota_{\langle -, \breve{/\!/}_s d\breve{B}_s \rangle} \breve{W}_s^{A,q+1}(b^1 \wedge \ldots \wedge b^{q+1})\, ds.$$

Finally we arrive at:

Corollary 3.5. *Suppose that the torsion $\breve{T} \equiv XdY$ is invariant under the flow ξ_t. Then for $b^i \in T_{x_0}M, i = 1, \ldots, q+1$,*

$$\begin{aligned} &d(P_t\phi)(b^1 \wedge \ldots \wedge b^{q+1}) \\ &= \left(\int_0^t \rho(r)dr\right)^{-1} \mathbb{E}\phi \left(\breve{W}_t^{A,q} \int_0^t \rho(s)(\breve{W}_s^{A,q})^{-1} \iota_{\langle -, \breve{/\!/}_s d\breve{B}_s \rangle} \breve{W}_s^{A,q+1}(b^1 \wedge \ldots \wedge b^{q+1}) \right) \\ &\quad -\phi \left(\breve{W}_t^{A,q} \left(\sum_{1 \le i < j \le q+1} (-1)^{i+j+1} \breve{T}(b^i, b^j) \wedge b^1 \wedge \ldots \wedge \widehat{b^i} \ldots \wedge \widehat{b^j} \ldots \wedge b^{q+1} \right)\right)\right). \end{aligned} \tag{3.26}$$

Note that in the non-invariant case, an analogous proof to that of Corollary 3.5 leads to the intrinsic formula below. If V is a $q+1$ vector, we define $\iota_{\breve{T}}V$ to be the operator from $\wedge^{q+1}TM \to \wedge^q TM$ which restricted to primitive vectors is given by:

$$\iota_{\breve{T}}(b^1 \wedge \ldots \wedge b^{q+1}) = \sum_{1 \le i < j \le q+1} (-1)^{i+j+1} \breve{T}(b^i, b^j) \wedge b^1 \wedge \ldots \wedge \widehat{b^i} \ldots \wedge \widehat{b^j} \ldots \wedge b^{q+1}.$$

Corollary 3.6. *Let b be a q vector in $\wedge^{q+1}T_{x_0}M$, then*

$$\begin{aligned} &\left(\int_0^t \rho(r)dr\right) d(P_t\phi)(b) \\ &\quad = \mathbb{E}\phi \left(\breve{W}_t^{A,q} \int_0^t \rho(s)(\breve{W}_s^{A,q})^{-1} \left(\iota_{\langle -, \breve{/\!/}_s d\breve{B}_s \rangle} W_s^{A,q+1}(b) + \iota_{\breve{T}} \breve{W}_s^{A,q+1}(b)\, ds \right)\right). \end{aligned} \tag{3.27}$$

Proof. We only need to worry about the last term of (3.23), since the previous term is as in Corollary 3.5. For this

$$\begin{aligned} &\sum_{1 \le i < j \le q+1} (-1)^{i+j+1} \\ &\qquad \times \mathbb{E}(\xi_t^*)(\phi) \Bigg(\int_0^t \rho(s) T\xi_s^{-1} \left(\breve{T}\left(T\xi_s(b^i), T\xi_s(b^j)\right)\right) \\ &\qquad \times ds \wedge b^1 \wedge \ldots \wedge \widehat{b^i} \ldots \wedge \widehat{b^j} \ldots \wedge b^{q+1} \Bigg) \end{aligned}$$

$$= \mathbb{E}\phi\Bigg(\sum_{1\le i<j\le q+1} (-1)^{i+j+1} T\xi_t \int_0^t \rho(s) T\xi_s^{-1}\left(\breve{T}\left(T\xi_s(b^i), T\xi_s(b^j)\right)\right) ds$$
$$\wedge T\xi_t(b^1)\wedge\ldots\wedge \widehat{T\xi_t(b^i)}\ldots\wedge\widehat{T\xi_t(b^j)}\ldots\wedge T\xi_t(b^{q+1})\Bigg).$$

Set

$$U_t = \sum_{1\le i<j\le q+1} (-1)^{i+j+1} T\xi_t \int_0^t \rho(s) T\xi_s^{-1}\left(\breve{T}\left(T\xi_s(b^i), T\xi_s(b^j)\right)\right) ds$$
$$\wedge T\xi_t(b^1)\wedge\ldots\wedge \widehat{T\xi_t(b^i)}\ldots\wedge\widehat{T\xi_t(b^j)}\ldots\wedge T\xi_t(b^{q+1}).$$

Then after covariant differentiation and filtering we have

$$\begin{aligned}\hat{D}\overline{U_t} &= \sum_{1\le i<j\le q+1} (-1)^{i+j+1}\rho(t) \\ &\quad \overline{\cdot\,\breve{T}\left(T\xi_t(b^i), T\xi_t(b^j)\right)\wedge T\xi_t(b^1)\wedge\ldots\wedge\widehat{T\xi_t(b^i)}\ldots\wedge\widehat{T\xi_t(b^j)}\ldots\wedge T\xi_t(b^{q+1})} \\ &\quad -\frac{1}{2}(\breve{R}^q)^*(\overline{U_t})dt + d\wedge^q \breve{\nabla} A(\overline{U_t})dt \\ &= \rho(t)\overline{\iota_{\breve{T}}\wedge^{q+1} T\xi_t(b^1\wedge\ldots\wedge b^{q+1})} - \frac{1}{2}(\breve{R}^q)^*(\overline{U_t})dt + d\wedge^q\breve{\nabla}A(\overline{U_t})dt \\ &= \rho(t)\iota_{\breve{T}}\breve{W}_t^{A,q+1}(b) - \frac{1}{2}(\breve{R}^q)^*(\overline{U_t})dt + d\wedge^q\breve{\nabla}A(\overline{U_t})dt,\end{aligned}$$

giving

$$\bar{U}_t = \breve{W}_t^{A,q}\int_0^t \rho(s)\left(\breve{W}_s^{A,q}\right)^{-1}\iota_{\breve{T}}\breve{W}_s^{A,q+1}(b)\,ds.$$

The required equation now follows. □

Special cases

(1) When the connection $\breve{\nabla}$ defined by (1.3) is the Levi–Cività connection and $\rho(t)\equiv 1$, formula (3.23) essentially reduces to (0.4), but in this case P_t has generator given by $\frac{1}{2}\Delta + \mathcal{L}_A$ on smooth forms.

(2) For a left-invariant stochastic differential equation on a Lie group G with bi-invariant metric and $A\equiv 0$, formula (3.23) reduces to

$$\begin{aligned} d(P_t\phi)(b^1\wedge\ldots\wedge b^{q+1}) &= \left(\int_0^t \rho(r)dr\right)^{-1}\mathbb{E}\left(\left\langle\int_0^t ad(x_s)dB_s, -\right\rangle\wedge R_{x_t}^*(\phi)\right)(b^1\wedge\ldots\wedge b^{q+1}) \\ &\quad - \sum_{1\le i<j\le q+1}(-1)^{i+j}\mathbb{E}\,R_{x_t}^*(\phi)\left([b^i,b^j]\wedge b^1\wedge\ldots\widehat{b^i}\wedge\ldots\widehat{b^j}\wedge\ldots\wedge b^{q+1}\right),\end{aligned} \tag{3.28}$$

for $x_0 = e$ (so $b^j\in\mathfrak{g}$, each j). In this case the generator of P_t is $\frac{1}{2}\mathrm{trace}\nabla^R\nabla^R$ by (2.4.3) of [12].

Formula (3.28) is intrinsic. It could have been deduced from the path space integration by parts formula of [17]. The $q = 0$ case was given in [7].

(3) Another computable example comes when $A \equiv 0$ and X is chosen so that $\breve{\nabla}$ has torsion $\breve{T}(u, v) = \frac{2}{n-1}(v \wedge u)Z(x)$ for $u, v \in T_xM$ and Z a fixed vector field on M. Here $v \wedge u$ is the operator such that $(v \wedge u)Z(x) = \langle v, Z(x)\rangle_x u - \langle u, Z(x)\rangle_x v$. This connection was used by [18]. In [12], example 2.3.5 (though there is a minor misprint in the formula written there), the generator on q-forms is shown to be $\frac{1}{2}\Delta + \mathcal{L}_{[\frac{2(q-1)}{n-1}-1]Z} - \frac{2}{n-1}\iota_Z d$. Set $Z^{\#} = \langle Z(x), -\rangle_x$. The term involving the torsion in (3.23) reduces to

$$\left(\int_0^t \rho(r)dr\right)^{-1} \frac{4}{n-1}\mathbb{E}\,\xi_t^*(Z^{\#} \wedge \phi)(b^1 \wedge \ldots \wedge b^{q+1})$$
$$= \frac{4}{(n-1)\int_0^t \rho(r)\,dr} P_t(Z^{\#} \wedge \phi).$$

In this case we have:

$$d(P_t\phi)(b^1 \wedge \ldots \wedge b^{q+1})$$
$$= \left(\int_0^t \rho(r)dr\right)^{-1} \mathbb{E}\,\phi\left(\int_0^t \rho(s)\iota_{\langle -, /\!\!/_s d\breve{B}_s\rangle} W_s^{q+1}(b)\right)$$
$$+ \frac{4}{(n-1)\int_0^t \rho(r)\,dr}\left(Z^{\#} \wedge \phi\right)\left(b^1 \wedge \ldots \wedge b^{q+1}\right). \qquad (3.29)$$

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