

A Concrete Estimate for the Weak Poincaré Inequality on Loop Space

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Abstract

The aim of the paper is to study the pinned Wiener measure on the loop space over a simply connected compact Riemannian manifold together with a Hilbert space structure and the Ornstein-Uhlenbeck operator d^*d . We give a concrete estimate for the weak Poincaré inequality, assuming positivity of the Ricci curvature of the underlying manifold. The rate function is of order $s^{-\alpha}$ for any $\alpha > 0$.

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1 Introduction

A. Let M be a compact Riemannian manifold. For $a, b \in M$, we consider the pinned path space $\Omega_{a,b}$ over M ,

$$\Omega_{a,b} = \{\omega \in C([0, 1], M); \omega(0) = a, \omega(1) = b\},$$

which is a smooth Finsler manifold with compatible distance function

$$d_\infty(\omega, \gamma) := \sup_{s \in [0, 1]} d(\omega(s), \gamma(s)).$$

When $a = b$, we have the loop space over M , based at a . Let $\mathcal{FC}_b^\infty(\Omega_{a,b})$ be the collection of smooth cylinder function on $\Omega_{a,b}$. Each $F \in \mathcal{FC}_b^\infty(\Omega_{a,b})$ is

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determined by a smooth function f on M^n and a partition $0 < s_1 < \dots < s_n < 1$ of $[0, 1]$:

$$(1.1) \quad F(\omega) = (ev_{s_1, \dots, s_n})_* f = f(\omega(s_1), \dots, \omega(s_n)).$$

For each $T > 0$, endow $\Omega_{a,b}$ with the pinned Wiener measure $\mathbf{P}_{a,b}^T$ (also called the Brownian bridge measure), which is derived by pushing forward the standard Brownian bridge measure on the space of the pinned curves of $C([0, T]; M)$ with starting point a and ending point b , to $C([0, 1]; M)$ through the rescaling map $\omega(t) \mapsto \omega(\frac{t}{T})$. The measure $\mathbf{P}_{a,b}^T$ can be equally defined through its integration over smooth cylindrical functions of type (1.1):

$$\begin{aligned} & \int F(\omega) \mathbf{P}_{a,b}^T(d\omega) \\ &= \frac{1}{p_T(a, b)} \int_{M^n} f(x_1, x_2, \dots, x_n) \cdot p_{s_1 T}(a, x_1) p_{(s_2 - s_1)T}(x_1, x_2) \dots p_{(1 - s_n)T}(x_n, b) \prod_{i=1}^n dx_i. \end{aligned}$$

where $p_t(x, y)$ is the heat kernel on M . Write $\mathbf{P}_{a,b}$ for $\mathbf{P}_{a,b}^1$ for simplicity with corresponding expectation denoted by $\mathbf{E}_{a,b}$.

B. Let $\omega(s)$ be the canonical process on $\Omega_{a,b}$, \mathcal{F}_s be the natural filtration and $\mathcal{F} = \mathcal{F}_1$. Then $\omega(s)$ is a semi-martingale with $(\Omega_{a,b}, \mathcal{F}, \mathcal{F}_s, \mathbf{P}_{a,b}^T)$, see [8]. Denote by $\parallel_{s,t}(\omega) : T_{\omega(s)}M \rightarrow T_{\omega(t)}M$ the stochastic parallel translation along the continuous path $\omega(\cdot)$, which is $\mathbf{P}_{a,b}^T$ a.s defined. Write $\parallel_s = \parallel_{0,s}$. Let \mathbf{H} be the space of finite energy curves in \mathbb{R}^n ,

$$\mathbf{H} = \left\{ h : [0, 1] \rightarrow \mathbb{R}^n \mid \int_0^1 |\dot{h}(s)|^2 ds < \infty, h \text{ is absolutely continuous} \right\}.$$

We identify $T_a M$ with \mathbb{R}^n and define the Bismut's tangent space \mathbf{H}_ω^0 in $\Omega_{a,b}$:

$$\mathbf{H}_\omega^0 = \{ \parallel_s(\omega)h. \mid h \in H, h_0 = 0, h_1 = 0 \},$$

which is a Hilbert space with the inner product:

$$(X, Y)_{\mathbf{H}_\omega^0} = \int_0^1 \left\langle \frac{d}{dt}(\parallel_{t,0}(\omega)X_t), \frac{d}{dt}(\parallel_{t,0}(\omega)Y_t) \right\rangle_{T_a M} dt$$

and corresponding norm $\| \cdot \|_{\mathbf{H}_\omega^0}$.

Consider the differential operator d which sends a differentiable function on $\Omega_{a,b}$, viewed as a Finsler manifold, to a differential 1-form. For F a smooth cylindrical function, dF as a bounded linear map on $T\Omega_{a,b}$ can be considered as a

bounded linear map on Bismut tangent space. By the Riesz representation theorem, there is the H^1 gradient $\mathbf{D}_0 F(\omega) \in \mathbf{H}_\omega^0$, given by

$$(\mathbf{D}_0 F(\omega), X_h)_{\mathbf{H}_\omega^0} = dF(X_h),$$

for all vectors X_h of the form $X_h(s) = //_s(\omega)h_s$ in \mathbf{H}_ω^0 . In particular, for cylindrical function $F = (ev_{s_1, \dots, s_n})_* f$ of the form (1.1),

$$\mathbf{D}_0 F(\omega)(t) = \sum_{i=1}^n //_{s_i, t}(\omega) \nabla_i f(\omega(s_1), \omega(s_2), \dots, \omega(s_n)) \cdot G_0(s_i, t)$$

where $\nabla_i f \in T_{\omega(s_i)}M$ is the value at $\omega(s_i)$ of the gradient of f as a function of the i th variable at the point $(\omega(s_1), \omega(s_2), \dots, \omega(s_n))$ and $G_0(s, t) = s \wedge t - s \cdot t$, $0 \leq s, t \leq 1$, is the Green function of the Gaussian measure on \mathbb{R}^n . Also

(1.2)

$$|\mathbf{D}_0 F|_{\mathbf{H}_\omega^0}^2 = \sum_{i, j=1}^n G_0(s_i, s_j) \cdot \langle //_{s_i, s_j}(\omega) \nabla_i f(\omega(s_1), \dots, \omega(s_n)), \nabla_j f(\omega(s_1), \dots, \omega(s_n)) \rangle_{\omega(s_j)}$$

For each $T > 0$, the quadratic form defined on smooth cylinder function by

$$\tilde{\mathcal{E}}_{a,b}^T(F, F) := \int_{\Omega_{a,b}} |\mathbf{D}_0 F|_{\mathbf{H}_\omega^0}^2 \mathbf{P}_{a,b}^T(d\omega),$$

can be extended to a Dirichlet form $\mathcal{E}_{a,b}^T$, which is due to an integration by parts formula, see [8]. The domain of the Dirichlet form is $\mathcal{D}(\mathcal{E}_{a,b}^T)$ is the the same as the domain of the closure of the gradient operator \mathbf{D}_0 . Follow the custom, we call this Dirichlet form the O-U Dirichlet form. And we denote $\mathcal{E}_{a,b}$ for $\mathcal{E}_{a,b}^1$ for simplicity.

If μ is a probability measure, we denote by $\mathbf{E}[F; \mu]$ the average of a function $F \in L^2(\mu)$ with respect to this measure and $\mathbf{Var}(F; \mu) = \mathbf{E}(F^2; \mu) - [\mathbf{E}(F; \mu)]^2$ the corresponding variance. The main theorem of the paper is:

Theorem 5.1. Let M be a simply connected compact manifold with strict positive Ricci curvature. For any small $\alpha > 0$, there exists a constant $s_0 > 0$ such that the following weak Poincaré inequality holds, i.e.

$$\mathbf{Var}(F; \mathbf{P}_{a,a}) \leq \frac{1}{s^\alpha} \mathcal{E}_{a,a}(F, F) + s \|F\|_\infty^2, \quad s \in (0, s_0), \quad F \in \mathcal{D}(\mathcal{E}_{a,a}).$$

And the constant s_0 does not depend on the starting point $a \in M$.

C. Historical Remark. The Ornstein-Uhlenbeck operator and the Ornstein-Uhlenbeck process plays an important role in the development of the L^2 theory on

loop spaces, c.f. [21]. The study of the functional inequalities for O-U Dirichlet form with respect to the Wiener measure (on path space) and to the pinned Wiener measure (on loop space) goes back a long way. For the Wiener measure on path space over a compact manifold, it turns out that there is no fundamental topological or geometrical obstruction to the validity of the Poincaré inequality. See e.g. the work of Fang [14] for the existence of a Poincaré inequality for O-U Dirichlet form and that of Aida-Elworthy [6] and Hsu [18] for the existence of a logarithmic Sobolev inequality. There is also the approach of gradient stochastic differential equations, e.g. Elworthy-Li[13], Elworthy-LeJan-Li[12], where the Hilbert norms for the Bismut tangent spaces differ in the case of non-flat curvature.

But for the case of loop space over a compact manifold M , the problem seems much more complicated. We limit ourselves here to the case of the pinned Wiener measure. Gross [16] pointed out that Logarithmic Sobolev inequality does not hold when $M = S^1$ and he proved instead a Logarithmic Sobolev inequality plus a potential term when M was a compact Lie group. In general the geometry and the topology of the manifold will play a significant role. In particular a Poincaré inequality does not hold for the Dirichlet form with respect to pinned Wiener measure if the underlying manifold is not simply connected, as the indicator function of each connected component of the loop space is in the domain of the O-U Dirichlet form, see Aida [3]. Furthermore, in [9], Eberle constructed a simply connected compact Riemannian manifold on the loop space over which the Poincaré inequality for O-U Dirichlet form did not hold. As transpired in his proof, the validity of the Poincaré inequality may depend on the starting point of the based loop space. A Clark-Ocone formula with a potential was deduced by Gong and Ma [15], which led to their discovery of a Logarithmic Sobolev inequality with a potential on loop space over general compact manifold. See also Aida [1]. In their results, the simply connected condition is not needed for the underlying manifold. Aida [4], on the other hand, deduced a Clark-Ocone formula which led to a Logarithmic Sobolev inequality for a modified Dirichlet form, under suitable conditions on the small time asymptotics of the Hessian of the logarithm of the heat kernel of the underlying manifold. Built on that, a Poincaré inequality is shown to hold for the O-U Dirichlet form on the loop space over hyperbolic space, see Chen-Li-Wu [7].

Another development in the positive direction comes from Eberle [2], where it was shown that a local Poincaré inequality hold for the O-U Dirichlet form on loop space over compact manifold. A parallel result was given by Aida [2]: when M was simply connected, the O-U Dirichlet form had the weak spectral gap property. By the weak spectral gap property for a Dirichlet form \mathcal{E} in $L^2(\mathbf{P})$ it is meant that $F_n \rightarrow 0$ in probability for any sequence of functions $\{F_n\}_{n=1}^\infty \subset \mathcal{D}(\mathcal{E})$ satisfying

the following conditions,

$$\sup_n \|F_n\|_{L^2} \leq 1, \quad E(F) = 0, \quad \lim_{n \rightarrow \infty} \mathcal{E}(F, F) = 0,$$

see also Kusuoka [19]. Although we do not know the relation between Eberle's local Poincaré inequality and Aida's weak spectral gap property, it was noted in Röckner and Wang in [23], the weak spectral gap property was equivalent to the following weak Poincaré inequality:

$$\mathbf{Var}_\mu(f) \leq \beta(s)\mathcal{E}(f, f) + s\|f\|_\infty^2, \quad s \in (0, s_0) \quad f \in \mathcal{D}(\mathcal{E}) \cap L^\infty(\mu)$$

Here $\beta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a non-increasing function and $s_0 > 0$ is a constant. And in [5], Aida used such weak Poincaré inequality to give an estimate on the spectral gap of a Schrödinger operator on the loop space. We refer the reader to Wang [24] for analysis, development and historical references on such inequalities. Our contribution here is the concrete estimate of $\beta(s)$ in the inequality above. The main difficulty here is to find suitable exhausting local sets replacing the role played by geodesic balls in the proof of weak Poincaré inequality on finite dimensional manifolds (see [24]). The local sets which Eberle takes in [10] are not suitable for our proof. So in our approach, we use a different collection of local sets. On such local sets, we do not derive the exact local Poincaré inequality, some additional term of the L^∞ norm will appear in the estimate, but finally we can control such terms to get a global weak Poincaré inequality.

The paper is organised as follows. In Section 2, we introduce notation and state some results, especially that of Eberle [10, 11] on which our proof is based on. In Section 3, we give some variance estimate for small time. In Section 4, A weak Poincaré type inequality for the distribution of the Brownian bridge evaluated at N equal time intervals is given. We use a combination of small time asymptotics and Poincaré inequality for the Wiener measure to control the growth of the constants with N . In particular, some of the methods in this section are inspired by [11] and [17]. In section 5, the main theorem is proved by reducing the variance of a function on the loop space to the variance of a function on a product manifold which is localized to subsets which are chains of small geodesic balls, and the variance of functions on some sub-path with respect to pinned Wiener measure with small time parameter.

2 Notations and known results

Let $\{B_s\}$ be the $T_a M$ valued stochastic anti-development of the canonical process $\omega(s)$, which is a semi-martingale with $(\Omega_{a,b}, \mathcal{F}, \mathcal{F}_s, \mathbf{P}_{a,b}^T)$. It is however not a

Brownian motion, see [8]. Denote by $L(\mathbb{R}^n; T_a M)$ the set of all linear maps from \mathbb{R}^n to $T_a M$.

Lemma 2.1 ([10]). *Let $\{A_s(\omega), \omega \in \Omega_{a,b}, 0 \leq s \leq 1\}$ be a $L(\mathbb{R}^n; T_a M)$ valued adapted process such that $s \rightarrow A_s(\omega)$ is C^1 for every ω and*

$$\sup_{\omega \in \Omega_{a,b}} \sup_{s \in [0,1]} |A'_s(\omega)| < \infty.$$

Suppose that $H_s(\omega) = A_s(\omega)h_s$ for some $h \in \mathbf{H}$ with $h_1 = 0$ and $X_u(\omega) = \parallel_s(\omega)A_s(\omega)h_s$. Define

$$\delta_u^T X := \int_0^u [T^{-1}H'_s + \frac{1}{2}\parallel_s^{-1}\text{Ric}_{\omega(s)}^\#(\parallel_s H_s)] dB_s, \quad 0 \leq u < 1$$

Then

$$\delta^T X := \lim_{u \rightarrow 1} \delta_u^T X$$

exists in $L^1(\Omega_{a,b}; \mathbf{P}_{a,b}^T)$ and the limit is in $L^2(\Omega_{a,b}; \mathbf{P}_{a,b}^T)$.

If $a, b \in M$ are not in the cut locus of each other, we take A_s such that $\parallel_s A_s$ is the damped stochastic parallel transport and take H_s to be parallel push back of the Jacobi fields along the unique geodesic connecting a and b with initial vector v . By a result of Malliavin-Stroock, the variance of $\delta^T X$ defined in above lemma with respect to $\mathbf{P}_{a,b}^T$ are uniformly bounded for a, b, v, T in compact sets. In fact, we have the following lemma,

Lemma 2.2 ([10], [20]). *Let $b \in M \setminus \text{Cut}(a)$, $v \in T_a M$, and $T > 0$. There is a vector $X_s^{T,a,b,v} = \parallel_s H_s$, with initial value $X_0^{T,a,b,v} = v$ for H as in Lemma 2.1, such that for every $T_0 > 0$ and $r \in (0, \text{inj}_M)$,*

$$\sup_{T \in (0, T_0]} \rho(T, r) < \infty,$$

where

$$(2.3) \quad \rho(T, r) := \sup_{a,b \in M, d(a,b) \leq r, v \in T_a M, |v|=1} \{\mathbf{Var}(\delta^T X^{T,a,b,v}; \mathbf{P}_{a,b}^T)\}.$$

The next lemma deals with the derivative with starting point of the expectation under pinned Wiener measure,

Lemma 2.3 ([10]). *Let $v \in T_a M$. For each $X_s = \parallel_s(\omega)H_s(\omega)$ with $H_s(\omega)$ as in lemma 2.1, and that $X_0 = v \mathbf{P}_{a,b}^T$ a.s.,*

$$(2.4) \quad d_a(\mathbf{E}_{a,b}^T[F])[v] = \mathbf{E}_{a,b}^T[dF(X)] - \mathbf{Cov}(\delta^T X, F; \mathbf{P}_{a,b}^T)$$

for all smooth cylinder function $F \in \mathcal{FC}_b^\infty(\Omega_{a,b})$.

For two paths ω_1, ω_2 with $\omega_1(1) = \omega_2(0)$, define $\omega_1 \vee \omega_2$ as following:

$$\omega_1 \vee \omega_2(s) = \begin{cases} \omega_1(2s) & \text{if } s \in [0, 1/2], \\ \omega_2(2s - 1) & \text{if } s \in [1/2, 1]. \end{cases}$$

For each ω in $\Omega_{a,b}$, we can find one and only one pair of $\widetilde{\omega}_1, \widetilde{\omega}_2$ to satisfy that $\omega = \widetilde{\omega}_1 \vee \widetilde{\omega}_2$, then for each fixed $T > 0$, $\omega \in \Omega_{a,b}$ with $a, b \notin \text{Cut}(\omega(1/2))$ and $\omega = \widetilde{\omega}_1 \vee \widetilde{\omega}_2$, $v \in T_{\omega(1/2)}M$, let

$$\widehat{X}_s^{T,v}(\omega) = \begin{cases} X_{1-2s}^{T/2, \omega(1/2), \omega(0), v}(\widetilde{\omega}_1^{-1}) & \text{if } s \in [0, 1/2], \\ X_{2s-1}^{T/2, \omega(1/2), \omega(1), v}(\widetilde{\omega}_2) & \text{if } s \in [1/2, 1] \end{cases}$$

where $X_s^{T,a,b,v}$ is as in lemma 2.2 and $\widetilde{\omega}_1^{-1}(s) := \widetilde{\omega}_1(1-s)$, $0 \leq s \leq 1$ is the time inverse of the path $\widetilde{\omega}_1$.

For $F \in \mathcal{FC}_b^\infty(\Omega_{a,b})$ and $\omega \in \Omega_{a,b}$, let

$$(2.5) \quad \mathbf{\Gamma}^T(F)(\omega) = \begin{cases} \sup_{\{v \in T_{\omega(1/2)}M, |v|=1\}} (dF(\widehat{X}_s^{T,v}))^2(\omega), & \text{if } a, b \notin \text{Cut}(\omega(1/2)) \\ 0, & \text{otherwise.} \end{cases}$$

For each smooth cylinder function $F \in \mathcal{FC}_b^\infty(\Omega_{a,b})$, there exists a unique function $\widetilde{F}(\widetilde{\omega}_1, \widetilde{\omega}_2)$, defined on $\bigcup_{z \in M} \Omega_{a,z} \times \Omega_{z,b}$, such that $\widetilde{F}(\widetilde{\omega}_1, \widetilde{\omega}_2) = F(\widetilde{\omega}_1 \vee \widetilde{\omega}_2) = F(\omega)$ for each ω in $\Omega_{a,b}$ with $\omega = \widetilde{\omega}_1 \vee \widetilde{\omega}_2$.

Lemma 2.4 ([10]). *For $a, b \in M$, $T > 0$, and $r > 0$, denote $U_{a,b}^r = B_r(a) \cap B_r(b)$ ($B_r(a)$ means the ball with center a and radius r) and*

$$(2.6) \quad \mu_{a,b}^T(dx) = \frac{p_{T/2}(a, x)p_{T/2}(x, b)}{p_T(a, b)}dx.$$

- *There exists a positive number R_1 , such that when $r \in (0, R_1)$,*

$$(2.7) \quad \begin{aligned} \mathbf{Var}(F; \mathbf{P}_{a,b}^T) &\leq 2q(T, r)\mathbf{E}_{a,b}^T[\mathbf{\Gamma}^T(F)] \\ &+ (1 + 4q(T, r)\rho(T/2, r)) \int_{U_{a,b}^r} \{\mathbf{E}_{a,x}^{T/2,1}[\mathbf{Var}_2(\widetilde{F}(\widetilde{\omega}_1, \widetilde{\omega}_2); \mathbf{P}_{x,b}^{T/2})]\} \\ &+ \mathbf{E}_{x,b}^{T/2,2}[\mathbf{Var}_1(\widetilde{F}(\widetilde{\omega}_1, \widetilde{\omega}_2); \mathbf{P}_{a,x}^{T/2})]\} \mu_{a,b}^T(dx) \end{aligned}$$

for every smooth cylinder function $F : \Omega_{a,b} \mapsto \mathbb{R}$ such that $F(\omega) = 0$ if $\omega(1/2)$ is not in $U_{a,b}^r$. Here $\mathbf{\Gamma}^T(F)$, $\rho(T/2, r)$ are defined by (2.5) and (2.3) respectively. $\mathbf{E}^{T/2,i}$, \mathbf{Var}_i indicates that the corresponding expectation or variance is taken with respect to the i th-subpath $\widetilde{\omega}_i$, $i = 1, 2$,

- The constant $q(T, r)$ in above inequality does not depend on $a, b \in M$ and satisfies

$$(2.8) \quad \overline{\lim}_{T \downarrow 0} T^{-1} q(T, r) \leq \frac{1 + Kr^2}{4} \quad \forall r \in (0, R_1).$$

for some $K > 0$.

3 Some estimate on the variance with small time parameter

The following lemma gives a short time asymptotics of the variance. It is crucial for the proof of main result in this section. For simplicity, in the remaining part of the paper, the constants, e.g. C , T , and l , will change according to different situation but we will clarify which parameter such C depends on. At first, we state a lemma deriving from lemma 2.4 by some cut-off procedures,

Lemma 3.1. *There exists a number $R_1 > 0$ such that for all $a, b \in M$ with $d(a, b) < r < R_1$, the following holds for any number $\eta > 0$ and smooth cylindrical function F on $(\Omega_{a,b}, P_{a,b}^T)$, as soon as $T < T_1(\eta, r)$ for a positive number $T_1(\eta, r)$.*

(3.9)

$$\begin{aligned} & \mathbf{Var}(F; \mathbf{P}_{a,b}^T) \\ & \leq 4q(T, r) \mathbf{E}_{a,b}^T [\mathbf{\Gamma}^T(F) \mathbf{1}_{\{\omega(1/2) \in U_{a,b}^r\}}] \\ & + (1 + 4q(T, r) \rho(T/2, r)) \int_{U_{a,b}^r} \left\{ \mathbf{E}_{a,x}^{T/2,1} [\mathbf{Var}_2(\tilde{F}(\tilde{\omega}_1, \tilde{\omega}_2); \mathbf{P}_{x,b}^{T/2})] \right. \\ & \left. + \mathbf{E}_{x,b}^{T/2,2} [\mathbf{Var}_1(\tilde{F}(\tilde{\omega}_1, \tilde{\omega}_2); \mathbf{P}_{a,x}^{T/2})] \right\} \mu_{a,b}^T(dx) + \left(6 + \frac{128q(T, r)}{\eta^2 r^2} \right) e^{-\frac{(1-4\eta)r^2}{2T}} \|F\|_\infty^2. \end{aligned}$$

Here the measure $\mu_{a,b}^T(dx)$ is the distribution of the mid-point of the Brownian Bridge, given by (2.6), $U_{a,b}^r := B_r(a) \cap B_r(b)$, $q(T, r)$ is some constant satisfying (2.8), and $\mathbf{E}^{T/2,i}$, \mathbf{Var}_i indicates that the expectation or variance is taken with respect to the subpath $\tilde{\omega}_i$.

Remark: The constants R_1 and $q(T, r)$ are the same as that in lemma 2.4. R_1 is smaller than the injectivity radius of M . The constant η can be arbitrary, but we are only interested in small η for exponential decay rate.

Proof. Step 1. For a positive r smaller than the injectivity radius of M , $a, b \in M$ with $d(a, b) < r$, define a function $\Psi_{a,b} : \Omega_{a,b} \rightarrow \mathbb{R}$ by

$$(3.10) \quad \Psi_{a,b}(\omega) := \varphi(d(a, \omega(1/2))) \cdot \varphi(d(b, \omega(1/2))).$$

Here ϕ is a smooth function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfying

$$(3.11) \quad \varphi(s) = \begin{cases} 1, & \text{if } s \leq (1-\eta)r, \\ 0, & \text{if } s \geq r \end{cases} \quad \text{and } |\varphi'| \leq \frac{2}{\eta r}.$$

Then the function $\Psi_{a,b}$ is in $\mathcal{D}(\mathcal{E}_{a,b}^T)$, the domain of the O-U Dirichlet form, and $\sup_{\omega \in \Omega_{a,b}} |\mathbf{D}_0 \Psi_{a,b}(\omega)|_{\mathbf{H}_w^0} \leq \frac{4}{\eta r}$. Furthermore we show below that for all small $\eta > 0$ there is constant $T_1(\eta, r)$ such that if $T < T_1(\eta, r)$,

$$(3.12) \quad \mathbf{P}_{a,b}^T(\Psi_{a,b} \neq 1) \leq 2e^{-\frac{(1-4\eta)r^2}{2T}}.$$

We begin with estimating the probability

$$\mathbf{P}_{a,b}^T(d(a, \omega(1/2)) > (1-\eta)r) = \mu_{a,b}^T(B_{(1-\eta)r}(a)).$$

By Varadhan's estimate [22],

$$\lim_{T \downarrow 0} T \log p_T(a, b) = -\frac{d^2(a, b)}{2} \quad \text{uniformly on } M \times M.$$

Hence for any $\eta > 0$ small, there exists a constant $T_1(\eta, r) > 0$, such that for every $0 < T < T_1(\eta, r)$,

$$(3.13) \quad -\frac{d^2(a, b)}{2T} - \frac{\eta^2 r^2}{4T} \leq \log p_T(a, b) \leq -\frac{d^2(a, b)}{2T} + \frac{\eta^2 r^2}{4T}.$$

In the calculations that follows we assume that $0 < T < T_1(\eta, r)$. Note that $d(a, b) < r$,

$$\begin{aligned} & \mathbf{P}_{a,b}^T(d(a, \omega(1/2)) > (1-\eta)r) \\ &= \frac{1}{p_T(a, b)} \int_{\{d(a,x) > (1-\eta)r\}} p_{T/2}(a, x) p_{T/2}(x, b) dx \\ &\leq e^{\frac{d^2(a,b)}{2T} + \frac{\eta^2 r^2}{4T}} \int_{\{d(a,x) > (1-\eta)r\}} e^{-\frac{d^2(a,x)}{T} + \frac{\eta^2 r^2}{2T}} p_{T/2}(x, b) dx \\ &\leq e^{\frac{r^2}{2T} + \frac{\eta^2 r^2}{4T}} e^{-\frac{(1-\eta)^2 r^2}{T} + \frac{\eta^2 r^2}{2T}} \leq e^{-\frac{(1-4\eta)r^2}{2T}}. \end{aligned}$$

Similarly,

$$\mathbf{P}_{a,b}^T(d(b, \omega(1/2)) > (1 - \eta)r) \leq e^{-\frac{(1-4\eta)r^2}{2T}}.$$

Hence

$$\begin{aligned} \mathbf{P}_{a,b}^T(\Psi_{a,b} \neq 1) &\leq \mathbf{P}_{a,b}^T(d(a, \omega(1/2)) > (1 - \eta)r) + \mathbf{P}_{a,b}^T(d(b, \omega(1/2)) > (1 - \eta)) \\ &\leq 2e^{-\frac{(1-4\eta)r^2}{2T}}. \end{aligned}$$

Step 2. Let R_1 be the constant in Lemma 2.4. Assume that $r < R_1$ and we first observe that

$$\begin{aligned} \text{Var}(F; \mathbf{P}_{a,b}^T) &= \mathbf{E}_{a,b}^T F^2 - (\mathbf{E}_{a,b}^T F)^2 \\ (3.14) \quad &\leq \mathbf{E}_{a,b}^T (F\Psi_{a,b})^2 + \|F\|_\infty^2 \mathbf{P}_{a,b}^T(\Psi_{a,b} \neq 1) - (\mathbf{E}_{a,b}^T [F - F\Psi_{a,b} + F\Psi_{a,b}])^2 \\ &\leq \text{Var}(F\Psi_{a,b}; \mathbf{P}_{a,b}^T) + 3\|F\|_\infty^2 \mathbf{P}_{a,b}^T(\Psi_{a,b} \neq 1) \\ &\leq \text{Var}(F\Psi_{a,b}; \mathbf{P}_{a,b}^T) + 6e^{-\frac{(1-4\eta)r^2}{2T}} \|F\|_\infty^2 \end{aligned}$$

Since $\Psi_{a,b}(\omega) = 0$ when $\omega(1/2) \notin U_{a,b}^r$, Lemma 2.4 applies to $F\Psi_{a,b}$ and we have,

$$\begin{aligned} \text{Var}(F\Psi_{a,b}; \mathbf{P}_{a,b}^T) &\leq 2q(T, r) \mathbf{E}_{a,b}^T [\Gamma^T(F\Psi_{a,b})] \\ &+ (1 + 4q(T, r)\rho(T/2, r)) \int_{U_{a,b}^r} \left\{ \mathbf{E}_{a,x}^{T/2,1} [\text{Var}_2(\widetilde{F\Psi_{a,b}}(\widetilde{\omega}_1, \widetilde{\omega}_2); \mathbf{P}_{x,b}^{T/2})] \right. \\ &\left. + \mathbf{E}_{x,b}^{T/2,2} [\text{Var}_1(\widetilde{F\Psi_{a,b}}(\widetilde{\omega}_1, \widetilde{\omega}_2); \mathbf{P}_{a,x}^{T/2})] \right\} \mu_{a,b}^T(dx), \end{aligned}$$

We next deal with the terms $\text{Var}_i(\widetilde{F\Psi_{a,b}}(\widetilde{\omega}_1, \widetilde{\omega}_2); \mathbf{P}_{x,b}^{T/2})$. Since $\Psi_{a,b}(\omega) = \varphi(d(a, \omega(1/2))) \cdot \varphi(d(b, \omega(1/2)))$ is determined by $\omega(1/2)$, for $i = 1, 2$.

$$\begin{aligned} \text{Var}_i(\widetilde{F\Psi_{a,b}}(\widetilde{\omega}_1, \widetilde{\omega}_2); \mathbf{P}_{x,b}^{T/2}) &= \varphi(d(a, x))^2 \cdot \varphi(d(x, b))^2 \cdot \text{Var}_i(\widetilde{F}(\widetilde{\omega}_1, \widetilde{\omega}_2); \mathbf{P}_{x,b}^{T/2}) \\ &\leq \text{Var}_i(\widetilde{F}(\widetilde{\omega}_1, \widetilde{\omega}_2); \mathbf{P}_{x,b}^{T/2}) I_{\{x \in U_{a,b}^r\}}. \end{aligned}$$

Consequently

$$\begin{aligned} (3.15) \quad \text{Var}(F\Psi_{a,b}; \mathbf{P}_{a,b}^T) &\leq 2q(T, r) \mathbf{E}_{a,b}^T [\Gamma^T(F\Psi_{a,b})] \\ &+ (1 + 4q(T, r)\rho(T/2, r)) \int_{U_{a,b}^r} \left\{ \mathbf{E}_{a,x}^{T/2,1} [\text{Var}_2(\widetilde{F}(\widetilde{\omega}_1, \widetilde{\omega}_2); \mathbf{P}_{x,b}^{T/2}) I_{\{x \in U_{a,b}^r\}}] \right. \\ &\left. + \mathbf{E}_{x,b}^{T/2,2} [\text{Var}_1(\widetilde{F}(\widetilde{\omega}_1, \widetilde{\omega}_2); \mathbf{P}_{x,b}^{T/2}) I_{\{x \in U_{a,b}^r\}}] \right\} \mu_{a,b}^T(dx), \end{aligned}$$

Let $S_a M := \{v \in T_a M, |v| = 1\}$. For each $\omega \in \Omega_{a,b}$, by the definition of $\mathbf{\Gamma}^T$ in (2.5),

$$\begin{aligned}
(3.16) \quad & \mathbf{\Gamma}^T(F\Psi_{a,b})(\omega) = \sup \{[d(F\Psi_{a,b})(\widehat{X}^{T,v})]^2(\omega); v \in S_{\omega(1/2)}M\} \\
& \leq 2 \sup \{[dF(\widehat{X}^{T,v})]^2\Psi_{a,b}^2; v \in S_{\omega(1/2)}M\} + 2 \sup \{F^2[d\Psi_{a,b}(\widehat{X}^{T,v})]^2(\omega); v \in S_{\omega(1/2)}M\} \\
& \leq 2\mathbf{\Gamma}^T(F)(\omega)I_{\{\omega(1/2) \in U_{a,b}^r\}} + 2\|F\|_\infty^2 \sup \{\langle v, \nabla_x[\varphi(d(a,x)) \cdot \varphi(d(x,b))] \rangle^2; v \in S_x M\} \\
& \leq 2\mathbf{\Gamma}^T(F)(\omega)I_{\{\omega(1/2) \in U_{a,b}^r\}} + \frac{32}{\eta^2 r^2} \|F\|_\infty^2 I_{\{\Psi_{a,b}(\omega) \neq 1\}}.
\end{aligned}$$

The required inequality (3.9) follows from (3.14), (3.15) and (3.16). \square

Proposition 3.2. *There is a constant R_0 such that for each $0 < \eta < 1/4$ the following holds on $(\Omega_{a,b}, \mathbf{P}_{a,b}^T)$ provided that $d(a,b) < r < R_0$ and $0 < T < T_0(\eta, r)$ for some $T_0(\eta, r) > 0$:*

$$\mathbf{Var}(F; \mathbf{P}_{a,b}^T) \leq TC(r)\mathbf{E}_{a,b}^T(|\mathbf{D}_0 F|_{\mathbf{H}_0^0}^2) + C(\eta, r)\|F\|_\infty^2 e^{-\frac{(1-4\eta)r^2}{2T}}, \quad F \in \mathcal{FC}_b^\infty(\Omega_{a,b})$$

Here $C(\eta, r), C(r)$ are independent of T .

Proof. By approximation, it suffices to show the inequality holds for all smooth cylindrical functions on dyadic partitions, e.g. of the form

$$(3.17) \quad F(\omega) = f\left(\omega\left(\frac{1}{2^m}\right), \omega\left(\frac{2}{2^m}\right), \dots, \omega\left(\frac{2^m-1}{2^m}\right)\right), \quad f \in C^\infty(M^{2^m}), \quad m \in \mathbb{N}^+.$$

For any $\omega \in \Omega_{a,b}$, let

$$\omega_i(s) = \omega\left(\frac{i-1+s}{2^k}\right), \quad s \in [0, 1], \quad k \in \mathbb{N}^+, \quad 1 \leq i \leq 2^k$$

For simplicity, we did not reflect the index k in the definition of the new path ω_i . For each smooth cylinder function F and positive integer k we define a unique function $F^{[k]}$ of 2^k sub-paths. It is defined on $\bigcup_{z=(x_1, \dots, x_{2^k-1}) \in M^{2^k-1}} \prod_{i=1}^{2^k} \Omega_{x_{i-1}, x_i}$ (here $x_0 = a$ and $x_{2^k} = b$), such that for each $\omega \in \Omega_{a,b}$

$$F^{[k]}(\omega_1, \dots, \omega_{2^k}) = F(\omega).$$

In fact, $\int F^{[k]}(\omega_1, \dots, \omega_{2^k}) \prod_{i=1}^{2^k} \mathbf{P}_{x_{i-1}, x_i}^{T/2^k}(d\omega_i)$ is a smooth version of the conditional expectation $\mathbf{E}_{a,b}^T[F|\omega(1/2^k) = x_1, \dots, \omega(1-1/2^k) = x_{2^k-1}]$ and $F^{[1]}$ is the same as \widetilde{F} in Lemma 2.4 and Lemma 3.1.

For $N \geq 1$ and $T > 0$ we define the probability measure $\mu_{a,b}^{N,T}$ in M^{N-1} as,

$$(3.18) \quad \mu_{a,b}^{N,T}(\mathbf{d}x) := \frac{p_{\frac{T}{N}}(b, x_{N-1}) p_{\frac{T}{N}}(x_{N-1}, x_{N-2}) \cdots p_{\frac{T}{N}}(x_1, a)}{p_T(a, b)} \mathbf{d}x_{N-1} \cdots \mathbf{d}x_1.$$

Fix a number $0 < r < R_1$ for R_1 as in Lemma 3.1, $\eta > 0$ and a positive number $T < T_1(\eta, r)$. For the variance terms for \tilde{F} as a function of any of the two subpaths $\tilde{\omega}_1, \tilde{\omega}_2$ on the right side of inequality (3.9), we apply (3.9) from lemma 3.1, on each sub-path while keeping the other fixed, to obtain an estimate on the variance of F in terms of the variances and the operation $\Gamma^{T/2}$ for $F^{[2]}$ as a function of any of the four subpaths (note that $x \in U_{a,b}^r$, so we can use lemma 3.1 here). Repeat with this procedure by mid-dividing the path and applying (3.9). The variance terms will finally vanish after a repetition of m times for the smooth cylinder function of type (3.17), and we have,

$$(3.19) \quad \begin{aligned} \mathbf{Var}(F; \mathbf{P}_{a,b}^T) &\leq 4 \sum_{k=0}^{m-1} G(k, T, r) q(T/2^k, r) \times \\ &\left(\sum_{j=1}^{2^k} \int_{U_j} \left\{ \mathbf{E}_{x_{j-1}, x_j}^{T/2^k, j} \left[\Gamma^{T/2^k, j}(F^{[k]})(\omega_1, \dots, \omega_{2^k}) I_{\{\omega_j(1/2) \in U_{x_{j-1}, x_j}^r\}} \right] \right. \right. \\ &\times \left. \left. \prod_{i \neq j} \mathbf{P}_{x_{i-1}, x_i}^{T/2^k}(d\omega_i) \right\} \mu_{a,b}^{2^k, T}(dx) \right) \\ &+ \sum_{k=0}^{m-1} G(k, T, r) \left(6 + \frac{128q(T/2^k, r)}{\eta^2 r^2} \right) 2^k e^{-\frac{2^k(1-4\eta)r^2}{2T}} \|F\|_\infty^2 \end{aligned}$$

where $G(0, T, r) = 1$, $G(k, T, r) = \prod_{i=1}^k (1 + 4q(T/2^{i-1}, r)\rho(T/2^i, r))$ for each $k > 0$, $U_j = \{x = (x_1, \dots, x_{2^k-1}) \in M^{2^k-1} : d(x_{j-1}, x_j) < r\}$ for $j = 1, 2, \dots, 2^k$ ($x_0 = a$ and $x_{2^k} = b$). We denote by $\mathbf{E}_{x_{j-1}, x_j}^{T/2^k, j}$ and $\Gamma^{T/2^k, j}(F^{[k]})$ taking the corresponding expectation and the operation $\Gamma^{T/2^k}$ (defined in (2.5)) with respect to the j th sub-path for function $F^{[k]}$.

By (5.8) in the proof of lemma 5.1 in [10] if T is small enough,

$$\sup_{k \in \mathbb{N}} G(k, T, r) < C(r).$$

By this and Lemma 3.3 below we can find a positive number $R_0 < R_1$, such that for each $0 < r < R_0$, there is a $T_2(r) > 0$, when $T < T_2(r)$ the following holds

for all positive integer m :

$$\begin{aligned}
(3.20) \quad & \sum_{k=0}^{m-1} G(k, T, r) q(T/2^k, r) \cdot \left(\sum_{j=1}^{2^k} \int_{U_j} \left\{ \mathbf{E}_{x_{j-1}, x_j}^{T/2^k, j} [\mathbf{\Gamma}^{T/2^k, j}(F^{[k]})(\omega_1, \dots, \omega_{2^k}) \mathbf{1}_{\{\omega_j(1/2) \in U_{x_{j-1}, x_j}^r\}}] \right. \right. \\
& \times \left. \left. \prod_{i \neq j} \mathbf{P}_{x_{i-1}, x_i}^{T/2^k}(d\omega_i) \right\} \mu_{a,b}^{2^k, T}(dx) \right) \\
& \leq TC(r) \mathbf{E}_{a,b}^T |\mathbf{D}_0 F|_{\mathbf{H}^0}^2
\end{aligned}$$

Note that by part 2 of Lemma 2.4 there is $T_0(\eta, r) < \min(T_2(r), T(\eta, r))$ such that if $T < T_0(\eta, r)$, then $|q(T, r)| \leq C(r)$ for some constant C depending only on r . Using this bound and the bound on $\sup_k G(k, T, r)$ we see that for $T < T_0$,

$$\sup_{k \in \mathbb{N}} G(k, T, r) \left(6 + \frac{128q(T/2^k, r)}{\eta^2 r^2} \right) \leq C(r, \eta)$$

and

$$\begin{aligned}
(3.21) \quad & \sum_{k=0}^{m-1} G(k, T, r) \left(6 + \frac{128q(T/2^k, r)}{\eta^2 r^2} \right) 2^k e^{-\frac{2^k(1-4\eta)r^2}{2T}} \leq C(r, \eta) \sum_{k=0}^{\infty} 2^k e^{-\frac{2^k(1-4\eta)r^2}{2T}} \\
& \leq C(r, \eta) e^{-\frac{(1-4\eta)r^2}{2T}}
\end{aligned}$$

We conclude the proof from (3.19-3.21). \square

Finally we complete the proof of the Proposition by proving the lemma below.

Lemma 3.3. *Let $U_j = \{x = (x_1, \dots, x_{2^k-1}) \in M^{2^k-1} : d(x_{j-1}, x_j) < r, \}$ for $j = 1, 2, \dots, 2^k$ ($x_0 = a$ and $x_{2^k} = b$). We can find a $R_2 > 0$, for each $0 < r < R_2$, there is a number $T(r) > 0$, when $T < T(r)$, we have*

$$\begin{aligned}
& \sum_{k=0}^{m-1} q(T/2^k, r) \cdot \left(\sum_{j=1}^{2^k} \int_{U_j} \left\{ \mathbf{E}_{x_{j-1}, x_j}^{T/2^k, j} [\mathbf{\Gamma}^{T/2^k, j}(F^{[k]})(\omega_1, \dots, \omega_{2^k}) \mathbf{1}_{\{\omega_j(1/2) \in U_{x_{j-1}, x_j}^r\}}] \right. \right. \\
& \times \left. \left. \prod_{i \neq j} \mathbf{P}_{x_{i-1}, x_i}^{T/2^k}(d\omega_i) \right\} \mu_{a,b}^{2^k, T}(dx) \right) \\
& \leq TC(r) \mathbf{E}_{a,b}^T |\mathbf{D}_0 F|_{\mathbf{H}^0}^2
\end{aligned}$$

Here $C(r)$ is independent of T .

Proof. Following the notation from [10], let $\{h_{k,j}; k \geq 0, 1 \leq j \leq 2^k\}$ be the orthonormal basis of $H_0^{1,2}([0, 1]; \mathbb{R})$ consisting of Schauder functions, i.e. $h_{0,1}(s) = s \wedge (1 - s)$,

$$\begin{cases} h_{k,j}(s) = 2^{-k/2} h_{0,1}(2^k s - (j-1)) & \text{if } s \in [(j-1)2^{-k}, j2^{-k}], \\ h_{k,j}(s) = 0 & \text{otherwise} \end{cases}$$

for $k \geq 1$ and $1 \leq j \leq 2^k$. Let $d = \dim(M)$. We choose $\{e_i, 1 \leq i \leq d\}$, a family of measurable vector fields on M with $\{e_i(z); 1 \leq i \leq d\}$ an orthonormal basis on $T_z M$ for every $z \in M$. These give rise to an orthonormal basis of \mathbf{H}_ω^0 :

$$Z_s^{k,j,i}(\omega) = h_{k,j}(s) /_{1/2,s}(\omega) e_i(\omega(1/2)), \quad s \in [0, 1], k \geq 0, 1 \leq j \leq 2^k, 1 \leq i \leq d.$$

For each $F \in \mathcal{F}C_b^\infty(\Omega_{a,b})$, let

$$\Lambda_{k,j}(F)(\omega) = \sum_{i=1}^d [dF(Z^{k,j,i})]^2 = \sum_{i=1}^d (\mathbf{D}_0 F, Z^{k,j,i})_{\mathbf{H}_\omega^0}^2.$$

Then we have

$$|\mathbf{D}_0 F(\omega)|_{\mathbf{H}_\omega^0}^2 = \sum_{k=0}^{\infty} \sum_{j=1}^{2^k} \Lambda_{k,j}(F)(\omega), \quad \forall \omega \in \Omega_{a,b}.$$

By Lemma 4.3 in [10], there exist constants $R_2 > 0$, such that for each $r \in (0, R_2)$, there is a $\tilde{T}(r) > 0$ such that when $T < \tilde{T}(r)$

$$\mathbf{\Gamma}^T(F)(\omega) \mathbf{1}_{\{\omega(1/2) \in U_{a,b}^r\}} \leq C(r) \Lambda_{0,1}(F)(\omega) + \sum_{l=0}^{\infty} C(r) (T + 2^{-l}) \sum_{n=1}^{2^l} \Lambda_{l,n}(F)(\omega),$$

for each smooth cylinder function F and $\omega \in \Omega_{a,b}$. Thus, we obtain

(3.22)

$$\begin{aligned} & \mathbf{\Gamma}^{T/2^k, j}(F^{[k]})(\omega_1, \dots, \omega_{2^k}) \mathbf{1}_{\{\omega_j(1/2) \in U_{x_{j-1}, x_j}\}} \\ & \leq C(r) \Lambda_{0,1}^{k,j}(F^{[k]})(\omega_1, \dots, \omega_{2^k}) + \sum_{l=0}^{\infty} C(r) (T/2^k + 2^{-l}) \sum_{n=1}^{2^l} \Lambda_{l,n}^{k,j}(F^{[k]})(\omega_1, \dots, \omega_{2^k}). \end{aligned}$$

Here $\Lambda_{l,n}^{k,j}(F^{[k]})$ means the corresponding operation $\Lambda_{l,n}$ is taken with respect to

the j -th subpath for function $F^{[k]}$. Note that

(3.23)

$$\begin{aligned}\Lambda_{l,n}^{k,j}(F^{[k]})(\omega_1, \dots, \omega_{2^k}) &= \sum_{i=1}^d [dF^{[k]}(Z^{l,n,i})]^2(\omega_1, \dots, \omega_{j-1}, \bullet, \omega_{j+1}, \dots, \omega_{2^k}) \\ &= \sum_{i=1}^d 2^k [dF(Z^{l+k, (j-1)2^l+n, i})]^2(\omega) = 2^k \Lambda_{l+k, (j-1)2^l+n}(F)(\omega).\end{aligned}$$

The second equality above is due to the definition of $Z^{l,n,i}$ and some time rescaling procedure. By (3.22) and (3.23) we obtain,

(3.24)

$$\begin{aligned}& \sum_{k=0}^{m-1} q(T/2^k, r) \sum_{j=1}^{2^k} \Gamma^{T/2^k, j}(F^{[k]})(\omega_1, \dots, \omega_{2^k}) \mathbf{1}_{\{\omega_j(1/2) \in U_{x_{j-1}, x_j}^r\}} \\ & \leq \sum_{k=0}^{m-1} \left\{ C(r) q(T/2^k, r) 2^k + \sum_{l=0}^k C(r) (2^{k-2l} + T) q(T/2^{k-l}, r) \right\} \sum_{j=1}^{2^k} \Lambda_{k,j}(F)(\omega).\end{aligned}$$

Let $g(k, T, r) := C(r) q(T/2^k, r) 2^k + \sum_{l=0}^k C(r) (2^{k-2l} + T) q(T/2^{k-l}, r)$, by the estimate (2.8) for $q(T, r)$ we can find a $T(r) < \tilde{T}(r)$, such that for each $T < T(r)$, $\sup_{k \in \mathbb{N}} g(k, T, r) \leq TC(r)$, where $C(r)$ is a constant independent of T and k .

So by (3.24), for $T < T(r)$, we have,

$$\begin{aligned}& \sum_{k=0}^{m-1} q(T/2^k, r) \cdot \left(\sum_{j=1}^{2^k} \int_{U_j} \left\{ \mathbf{E}_{x_{j-1}, x_j}^{T/2^k, j} [\Gamma^{T/2^k, j}(F^{[k]})(\omega_1, \dots, \omega_{2^k}) \mathbf{1}_{\{\omega_j(1/2) \in U_{x_{j-1}, x_j}^r\}}] \right\} \right. \\ & \quad \left. \times \prod_{i \neq j} \mathbf{P}_{x_{i-1}, x_i}^{T/2^k}(d\omega_i) \right\} \mu_{a,b}^{2^k, T}(dx) \\ & \leq \sum_{k=0}^{m-1} g(k, T, r) \sum_{j=1}^{2^k} \mathbf{E}_{a,b}^T[\Lambda_{k,j}(F)] \\ & \leq TC(r) \mathbf{E}_{a,b}^T |\mathbf{D}_0 F|_{\mathbf{H}_0^0}^2\end{aligned}$$

□

4 An estimate over discretized loop space

For each $r \in \mathbb{R}^+$ and integer $N \geq 1$, define the subset $U_{a,b}^{r,N}$ of M^{N-1} as,

$$(4.25) \quad U_{a,b}^{r,N} := \{(x_1, \dots, x_{N-1}) \in M^{N-1}; d(x_{i-1}, x_i) < r, 1 \leq i \leq N, x_0 = a, x_N = b\}.$$

And recall that $\mu_{a,b}^{N,T}$ is the probability measure on M^{N-1} defined in (3.18), which is also the joint distribution of $(\omega(i/N), i = 1, 2, \dots, N-1)$ under $\mathbf{P}_{a,b}^T$. We have the following weak estimates of the variance with respect to $\mu_{a,b}^{N,1}$.

Proposition 4.1. *Let M be a compact simply connected manifold with strict positive Ricci curvature. For any $0 < \eta < 1/8$, $0 < r < R_0$, there exists an integer $N_1(\eta, r) > 0$, such that for $l > N_1(\eta, r)$ there exists an integer $N_2(l, \eta, r)$ with the property that if $N > N_2(l, \eta, r)$*

$$\begin{aligned} & \mathbf{Var}(f; \mu_{a,a}^{N,1}) \\ & \leq C(l, r) N^{C(l,r)} e^{2N\eta r^2} \sum_{i=1}^{N-1} \int_{M^{N-1}} |\nabla_i f|^2 d\mu_{a,a}^{N,1} + C(l, \eta, r) N^{C(l,r)} e^{-\frac{N(1-8\eta)r^2}{2}} \|f\|_\infty^2 \end{aligned}$$

for all $f \in C^\infty(M^{N-1})$ with $\text{supp}(f) \subset \bar{U}_{a,a}^{r,N}$.

Proof. Step 1. First choose an integer $N \gg l \gg 1$ and consider a function $f \in C^\infty(M^{N-1})$. Define a smooth cylinder function $\widehat{F} : \Omega_{a,a} \mapsto \mathbb{R}$ as,

$$(4.26) \quad \widehat{F}(\omega) := f(\omega(1/N), \dots, \omega(1 - 1/N)).$$

For $x = (x_1, \dots, x_{N-l}) \in M^{N-l}$ we define a function $f_l : M^{N-l} \rightarrow \mathbb{R}$ as following,

$$(4.27) \quad f_l(x_1, \dots, x_{N-l}) = \int_{M^{l-1}} f(x_1, \dots, x_{N-l}, y_1, \dots, y_{l-1}) \mu_{x_{N-l}, a}^{l, \frac{1}{N}}(dy_1 \dots dy_{l-1})$$

In fact

$$f_l(x_1, \dots, x_{N-l}) = \mathbf{E}_{a,a} \left[\widehat{F}(\omega) \mid \omega(1/N) = x_1, \dots, \omega\left(\frac{N-l}{N}\right) = x_{N-l} \right].$$

For such x define a function on $\Omega_{x_{N-l}, a}$ by

$$(4.28) \quad \widetilde{F}_l(x_1, \dots, x_{N-l}, \omega) := f(x_1, \dots, x_{N-l}, \omega(1/l), \dots, \omega(1 - 1/l)), \quad \omega \in \Omega_{x_{N-l}, a}.$$

Clearly

$$f_l(x_1, \dots, x_{N-l}) = \mathbf{E}_{x_{N-l}, a}^{l, \frac{1}{N}} [\widetilde{F}_l(x_1, \dots, x_{N-l}, \bullet)].$$

Let $\wp_l = \sigma\{\omega(i/N), 1 \leq i \leq N-l\}$, an σ -algebra on $\Omega_{a,a}$ and $\mu_{a,b}^{N,l,T}$ be the probability measure on M^{N-l} :

$$(4.29) \quad \mu_{a,b}^{N,l,T}(dx_1, \dots, dx_{N-l}) = \frac{p_{\frac{lT}{N}}(b, x_{N-l})p_{\frac{T}{N}}(x_{N-l}, x_{N-l-1}) \cdots p_{\frac{T}{N}}(x_1, a)}{p_T(a, a)} dx_{N-l} \dots dx_1.$$

We have

$$(4.30) \quad \begin{aligned} \mathbf{Var}(f; \mu_{a,a}^{N,l,1}) &= \mathbf{Var}(\widehat{F}; \mathbf{P}_{a,a}) \\ &= \mathbf{E}_{a,a} \left[(\widehat{F} - \mathbf{E}_{a,a}[\widehat{F}|\wp_l])^2 \right] + \mathbf{E}_{a,a} \left[(\mathbf{E}_{a,a}[\widehat{F}|\wp_l] - \mathbf{E}_{a,a}[\widehat{F}])^2 \right] \\ &= \int_{M^{N-l}} \mathbf{Var}(\widetilde{F}_l; \mathbf{P}_{\frac{l}{x_{N-l},a}}^{\frac{l}{N}}) \mu_{a,a}^{N,l,1}(dx) + \mathbf{Var}(f_l, \mu_{a,a}^{N,l,1}) \end{aligned}$$

Step 2. Now we are going to estimate $\mathbf{Var}(f_l, \mu_{a,a}^{N,l,1})$. Let \mathbf{P}_a^1 be the distribution of a standard Brownian motion on compact manifold M starting from a with time parameter 1, which is a probability measure on the path space Ω_a over M with starting point a and time 1. Let

$$\gamma_a^{N,l,1}(dx_1, \dots, dx_{N-l}) := p_{\frac{1}{N}}(a, x_1), \dots, p_{\frac{1}{N}}(x_{N-l-1}, x_{N-l}) dx_1, \dots, dx_{N-l}$$

be a probability measure on M^{N-l} , which is the joint distribution of $(\omega(i/N), \omega \in \Omega_a, i = 1, 2, \dots, N-l)$ under \mathbf{P}_a^1 . By the Poincaré inequality for \mathbf{P}_a^1 on the path space over compact manifold [14] we get,

$$\mathbf{Var}(f_l; \gamma_a^{N,l,1}) = \mathbf{Var}(\overline{F}_l; \mathbf{P}_a^1) \leq C \mathbf{E}_a^1 |\mathbf{D}\overline{F}_l|_{\mathbf{H}_\omega}^2 \leq CN \sum_{i=1}^{N-l} \int_{M^{l-1}} |\nabla_i f_l|^2 d\gamma_a^{N,l,1}$$

where $\overline{F}_l(\omega) := f_l(\omega(1/N), \dots, \omega(1-l/N))$ for $\omega \in \Omega_a$ and \mathbf{D} is the gradient operator related to Bismut tangent norm $|\cdot|_{\mathbf{H}_\omega}$ in path space over compact manifold M . We also use the relation

$$|\mathbf{D}\overline{F}_l(\omega)|_{\mathbf{H}_\omega}^2 \leq (N-l) \sum_{i=1}^{N-l} |\nabla_i f_l(\omega(1/N), \dots, \omega(1-l/N))|^2, \quad \omega \in \Omega_a,$$

in above inequality which can be checked by direct computation. Thus, we have

$$(4.31) \quad \begin{aligned} \mathbf{Var}(f_l; \mu_{a,a}^{N,l,1}) &= \mathbf{Var}\left(f_l; \frac{p_{\frac{l}{N}}(a, x_{N-l})}{p_1(a, a)} \gamma_a^{N,l,1}\right) \\ &\leq C \cdot \text{osc}\left(p_{\frac{l}{N}}(a, \cdot)\right) \cdot N \cdot \sum_{i=1}^{N-l} \int_{M^{l-1}} |\nabla_i f_l|^2 d\mu_{a,a}^{N,l,1}. \end{aligned}$$

Here for g a function on M , $\text{osc}(g(\cdot)) := \frac{\sup_{x \in M} g(x)}{\inf_{x \in M} g(x)}$ is the oscillation of g . By Varadhan's estimate (3.13), if $\frac{l}{N} < T(\eta, r)$ then

$$\text{osc}\left(p_{\frac{l}{N}}(a, \cdot)\right) \leq e^{\frac{N}{l}(\eta^2 r^2 + \frac{D^2}{2})}$$

for D the diameter of the compact manifold M . So by this and (4.31) when $\frac{l}{N} < T(\eta, r)$, we have

$$(4.32) \quad \text{Var}(f_l; \mu_{a,a}^{N,l,1}) \leq C N e^{\frac{N}{l}(\eta^2 r^2 + \frac{D^2}{2})} \sum_{i=1}^{N-l} \int_{M^{l-1}} |\nabla_i f_l|^2 d\mu_{a,a}^{N,l,1}.$$

Step 3. Now we are going to estimate $|\nabla_i f_l|$ in terms of f . It is easy to see that for $i < N - l$,

$$(4.33) \quad |\nabla_i f_l|^2(x_1, \dots, x_{N-l}) \leq \int_{M^{l-1}} |\nabla_i f|^2(x_1, \dots, x_{N-l}, y_1, \dots, y_{l-1}) \mu_{x_{N-l}, a}^{l, \frac{l}{N}}(dy)$$

and for $i = N - l$,

$$(4.34) \quad \begin{aligned} & |\nabla_{N-l} f_l|^2(x_1, \dots, x_{N-l}) = \sup_{|v|=1} \mathbf{d}_{N-l} f_l(v) \\ & \leq \int_{M^{l-1}} |\nabla_{N-l} f|^2(x_1, \dots, x_{N-l}, y_1, \dots, y_{l-1}) \mu_{x_{N-l}, a}^{l, \frac{l}{N}}(dy) \\ & \quad + \sup_{|v|=1} \left| \mathbf{d}_z \left(\mathbf{E}_{z,a}^{\frac{l}{N}}(\tilde{F}_l) \right) \Big|_{z=x_{N-l}}(v) \right|^2. \end{aligned}$$

Here $\tilde{F}_l(\omega) := f(x_1, \dots, x_{N-l}, \omega(1/l), \dots, \omega(1 - 1/l))$ is as defined in (4.28). As $d(a, x_{N-l}) \leq r$ may not hold, Lemmas 2.3 does not apply in its present form. To estimate the differentiation of the expectation with respect to the starting point we choose another vector field $X^{l,v}(s) := \int_s^1 (1-ls)^+ v$, $0 \leq s \leq 1$. Recall that (B_s) is the stochastic anti-development, in the expression for $\delta^{\frac{l}{N}} X$ in lemma 2.1, and

$$B_s = \beta_s + \int_0^s \left(\int_u^{-1} \nabla \log p_{\frac{(1-u)l}{N}}(\omega(u), a) \right) du, \quad 0 \leq s < 1$$

for some process (β_s) whose distribution is the Brownian motion with time parameter $\frac{l}{N}$ under the probability measure $\mathbf{P}_{x_{N-l}, a}^{\frac{l}{N}}$ (see [8]). We get,

$$\begin{aligned} \text{Var}\left(\delta^{\frac{l}{N}} X^{l,v}; \mathbf{P}_{x_{N-l}, a}^{\frac{l}{N}}\right) & \leq \mathbf{E}_{x_{N-l}, a}^{\frac{l}{N}} \left(\delta^{\frac{l}{N}} X^{l,v} \right)^2 \\ & \leq \mathbf{E}_{x_{N-l}, a}^{\frac{l}{N}} \left[\int_0^1 \left(-Nv + \frac{1}{2} \text{Ric}_{\omega(s)}(\int_s^1 (1-ls)v) \right) \left(d\beta_s + \int_s^{-1} \nabla \log p_{\frac{(1-s)l}{N}}(\omega(s), a) ds \right) \right]^2 \\ & \leq C(l)N^4, \quad v \in S_{x_{N-l}} M. \end{aligned}$$

In the last step we used the estimate $|\nabla \log p_s(x, a)| \leq C \left[\frac{d(x, a)}{s} + \frac{1}{\sqrt{s}} \right]$, for compact manifolds. Also note that $X^{l, v}(\frac{i}{l}) = 0$ for $1 \leq i \leq l$. By lemma 2.3, we have,

$$(4.35) \quad \begin{aligned} & \sup_{|v|=1} \left| \mathbf{d}_{N-l} \left(\mathbf{E}_{x_{N-l}, a}^{\frac{l}{N}}(\tilde{F}_l) \right) (v) \right|^2 \\ & \leq \sup_{|v|=1} \left\{ \left| \mathbf{E}_{x_{N-l}, a}^{\frac{l}{N}}[d\tilde{F}_l(X^{l, v})] \right| + \left[\mathbf{Var}(\delta^{\frac{l}{N}} X^{l, v}; \mathbf{P}_{x_{N-l}, a}^{\frac{l}{N}}) \right]^{1/2} \left[\mathbf{Var}(\tilde{F}_l; \mathbf{P}_{x_{N-l}, a}^{\frac{l}{N}}) \right]^{1/2} \right\}^2 \\ & \leq C(l) N^4 \mathbf{Var}(\tilde{F}_l; \mathbf{P}_{x_{N-l}, a}^{\frac{l}{N}}). \end{aligned}$$

Using (4.33-4.35) we derive estimates

$$\sum_{i=1}^{N-l} |\nabla_i f|^2(-) \leq \sum_{i=1}^{N-l} \int_{M^{l-1}} |\nabla_i f|^2(-, y) \mu_{x_{N-l}, a}^{l, \frac{N}{l}}(dy) + C(l) N^4 \mathbf{Var}(\tilde{F}_l; \mathbf{P}_{x_{N-l}, a}^{\frac{l}{N}}).$$

then from that Combining this with (4.30) and (4.32), we obtain the following,

$$(4.36) \quad \begin{aligned} \mathbf{Var}(f; \mu_{a, a}^{N, 1}) & \leq C(l) N \exp\left(\frac{N}{l}(\eta^2 r^2 + \frac{D^2}{2})\right) \sum_{i=1}^{N-l} \int_{M^{N-1}} |\nabla_i f|^2 \mu_{a, a}^{N, 1}(dx) \\ & + \left[1 + C(l) N^5 \exp\left(\frac{N}{l}(\eta^2 r^2 + \frac{D^2}{2})\right) \right] \int_{M^{N-l}} \mathbf{Var}(\tilde{F}_l; \mathbf{P}_{x_{N-l}, a}^{\frac{l}{N}}) \mu_{a, a}^{N, 1}(dx) \end{aligned}$$

Step 4. We estimate the variance $\mathbf{Var}(\tilde{F}_l; \mathbf{P}_{x_{N-l}, a}^{\frac{l}{N}})$ by the variance with respect to the Brownian bridge measure. Note that

$$(4.37) \quad \mathbf{Var}(\tilde{F}_l; \mathbf{P}_{x_{N-l}, a}^{\frac{l}{N}}) = \mathbf{Var}\left(f(x_1, \dots, x_{N-l}, \bullet, \dots, \bullet); \mu_{x_{N-l}, a}^{l, \frac{l}{N}}\right).$$

Let $\bar{\mu}_{x_{N-l}, a}^{l, \frac{l}{N}}$ be normalization of $\mu_{x_{N-l}, a}^{l, \frac{l}{N}}$ in the subset $U_{x_{N-l}, a}^{r, l}$ of M^{l-1} , i.e.

$$\bar{\mu}_{x_{N-l}, a}^{l, \frac{l}{N}}(A) = \mu_{x_{N-l}, a}^{l, \frac{l}{N}}(A) / \mu_{x_{N-l}, a}^{l, \frac{l}{N}}(U_{x_{N-l}, a}^{r, l}), \quad A \subseteq U_{x_{N-l}, a}^{r, l}.$$

For each smooth function g with support in $\bar{U}_{x_{N-l}, a}^{r, l}$, we have,

$$(4.38) \quad \begin{aligned} & \mathbf{Var}\left(g; \mu_{x_{N-l}, a}^{l, \frac{l}{N}}\right) \\ & \leq \mu_{x_{N-l}, a}^{l, \frac{l}{N}}(U_{x_{N-l}, a}^{r, l}) \mathbf{Var}\left(g; \bar{\mu}_{x_{N-l}, a}^{l, \frac{l}{N}}\right) + \frac{\left(1 - \mu_{x_{N-l}, a}^{l, \frac{l}{N}}(U_{x_{N-l}, a}^{r, l})\right)}{\mu_{x_{N-l}, a}^{l, \frac{l}{N}}(U_{x_{N-l}, a}^{r, l})} \|g\|_{\infty}^2. \end{aligned}$$

By asymptotic property (3.13), when $\frac{l}{N} < T(\eta, r)$, it satisfies that,

$$\begin{aligned}
(4.39) \quad & 1 - \mu_{x_{N-l}, a}^{l, \frac{l}{N}}(U_{x_{N-l}, a}^{r, l}) = \mu_{x_{N-l}, a}^{l, \frac{l}{N}}(\{z : d(z_i, z_{i+1}) > r \text{ for some } 0 \leq i \leq l-1\}) \\
& \leq \sum_{i=0}^{l-1} \frac{\int_{d(z_i, z_{i+1}) > r} p_{\frac{1}{N}}(x_{N-l}, z_1) \cdots p_{\frac{1}{N}}(z_{l-1}, a) dz_1, \dots, dz_{l-1}}{p_{\frac{1}{N}}(x_{N-l}, a)} \\
& \leq l \cdot \frac{\exp(-\frac{(1-4\eta)Nr^2}{2})}{\exp(-\frac{N}{2l}(\eta^2 r^2 + D^2))}.
\end{aligned}$$

If we choose l sufficient large, e.g. $\frac{\eta^2 r^2 + D^2}{l} < 2(1-4\eta)r^2$, there is an integer $\tilde{N}(\eta, l, r)$, such that whenever $N > \tilde{N}(\eta, l, r)$

$$(4.40) \quad \mu_{x_{N-l}, a}^{l, \frac{l}{N}}(U_{x_{N-l}, a}^{r, l}) > \frac{1}{2}$$

Since the support $\text{supp}(f)$ of f is a subset of $\overline{U}_{a, a}^{r, N}$ for fixed x_1, \dots, x_{N-l} ,

$$\text{supp}(f(x_1, \dots, x_{N-l}, \bullet, \dots, \bullet)) \subset \overline{U}_{x_{N-l}, a}^{r, l}$$

Hence by (4.38-4.40), for each integer l sufficiently big, there exists an integer $\tilde{N}(\eta, l, r)$, for each $N > \tilde{N}(\eta, l, r)$, we have,

$$\begin{aligned}
(4.41) \quad & \text{Var}\left(f(x_1, \dots, x_{N-l}, \bullet, \dots, \bullet); \mu_{x_{N-l}, a}^{l, \frac{l}{N}}\right) \\
& \leq \frac{1}{\lambda(U_{x_{N-l}, a}^{r, l}; \overline{\mu}_{x_{N-l}, a}^{l, \frac{l}{N}})} \times \sum_{i=1}^{l-1} \int |\nabla_{N-i+1} f|^2(x_1, \dots, x_{N-l}, z_1, \dots, z_{l-1}) \mu_{x_{N-l}, a}^{l, \frac{l}{N}}(dz) \\
& + 2l \cdot \frac{\exp(-\frac{(1-4\eta)Nr^2}{2})}{\exp(-\frac{N}{2l}(\eta^2 r^2 + D^2))} \|f\|_\infty^2,
\end{aligned}$$

where

$$(4.42) \quad \lambda(U_{x, a}^{r, l}; \overline{\mu}_{x, a}^{l, \frac{l}{N}}) := \inf_{g \in C_0^\infty(U_{x, a}^{r, l})} \frac{\int_{U_{x, a}^{r, l}} |\nabla g|^2 d\overline{\mu}_{x, a}^{l, \frac{l}{N}}}{\text{Var}(g; \overline{\mu}_{x, a}^{l, \frac{l}{N}})}.$$

Therefore, by (4.36) and (4.41), we have for each l big enough, $N > \tilde{N}(\eta, l, r)$

and $\frac{l}{N} < T(\eta, r)$,

$$\begin{aligned}
& \mathbf{Var}(f; \mu_{a,a}^{N,1}) \\
(4.43) \quad & \leq \left[\frac{C(l)N^5 \exp\left(\frac{N}{l}(\eta^2 r^2 + D^2/2)\right)}{\inf_{x \in M} \lambda(U_{x,a}^{r,l}; \bar{\mu}_{x,a}^{l, \frac{l}{N}})} \right] \sum_{i=1}^{N-1} \int_{M^{N-1}} |\nabla_i f|^2 d\mu_{a,a}^{N,1} \\
& \quad + \left[C(l)N^5 \exp\left(N\left(-\frac{(1-4\eta)r^2}{2} + \frac{3\eta^2 r^2 + 2D^2}{l}\right)\right) \right] \|f\|_\infty^2.
\end{aligned}$$

Step 5. Finally, by (4.43) and the uniform estimate of $\lambda(U_{x,a}^{r,l}; \bar{\mu}_{x,a}^{l, \frac{l}{N}})$ in x derived in Lemma 4.2 below, for each integer l sufficiently big, there exists an integer $N(\eta, l, r) > 0$, such that if $N > N(\eta, l, r)$, then

$$\begin{aligned}
(4.44) \quad & \mathbf{Var}(f; \mu_{a,a}^{N,1}) \\
& \leq \left[C(l, r)N^{C(l,r)} \exp\left(N\left(\frac{L(\varepsilon) + \eta^2 r^2 + D^2/2}{l} + 4D\varepsilon\right)\right) \right] \sum_{i=1}^{N-1} \int_{M^{N-1}} |\nabla_i f|^2 d\mu_{a,a}^{N,1} \\
& \quad + \left[C(l, \eta, r)N^{C(l,r)} \exp\left(N\left(-\frac{(1-4\eta)r^2}{2} + \frac{3\eta^2 r^2 + 2D^2}{l}\right)\right) \right] \|f\|_\infty^2.
\end{aligned}$$

Note that the constants C and L in the inequality above do not depend on N , and L does not depend on l and the starting point a . So for any fixed $\eta > 0$, $0 < r < R_0$, we first choose a $\varepsilon = \frac{\eta r^2}{4D}$ to make $4D\varepsilon = \eta r^2$, then take a l big enough such that $\frac{L(\varepsilon) + \eta^2 r^2 + D^2/2}{l} < \eta r^2$ and $\frac{3\eta^2 r^2 + 2D^2}{l} < \eta r^2$ for the chosen $\varepsilon = \frac{\eta r^2}{4D}$ (i.e. $l > N_0(\eta, r)$ for some constant $N_0(\eta, r)$ which only depends on η and r). Hence by (4.44), there is a constants $N_1(\eta, r)$, such that for each integer $l > N_1(\eta, r)$, there exists an integer $N(\eta, l, r) > 0$, such that if $N > N(\eta, l, r)$ then we have, as required,

$$\begin{aligned}
& \mathbf{Var}(f; \mu_{a,a}^{N,1}) \\
& \leq C(l, r)N^{C(l,r)} e^{2N\eta r^2} \sum_{i=1}^{N-1} \int_{M^{N-1}} |\nabla_i f|^2 d\mu_{a,a}^{N,1} + C(l, \eta, r)N^{C(l,r)} e^{-\frac{N(1-8\eta)r^2}{2}} \|f\|_\infty^2.
\end{aligned}$$

□

Lemma 4.2. *Let M be a compact simply connected manifold with strict Ricci curvature. For $a, x \in M$, $r < R_0$ and $N \in \mathbb{N}^+$, $\lambda(U_{x,a}^{r,l}; \bar{\mu}_{x,a}^{l, \frac{l}{N}})$ as defined in (4.42),*

there exists a constant $T(l, r)$, such that when $\frac{l}{N} < T(l, r)$, for each $\varepsilon > 0$ small enough,

$$\inf_{x, a \in M} \lambda(U_{x,a}^{r,l}; \bar{\mu}_{x,a}^{l, \frac{l}{N}}) \geq \frac{C(l, r)}{N^{C(l, r)}} \exp\left(-\left(\frac{L(\varepsilon)}{l} + 4D\varepsilon\right) \cdot N\right).$$

Proof. Step 1. Following [11] define a measure $\nu_{a,b}^{l,T}$ on M^{l-1} that approximates $\mu_{a,b}^T$.

$$\nu_{a,b}^{l,T}(dz) = \exp(-E_{a,b}^l(z)/T) dz,$$

where $z = (z_1, \dots, z_{l-1}) \in M^{l-1}$, $dz = \prod_{j=1}^{l-1} dz_j$ and

$$E_{a,b}^l(z_1, \dots, z_{l-1}) = \frac{l}{2} \sum_{i=0}^{l-1} d(z_i, z_{i+1})^2, \quad z_0 = a, z_l = b,$$

the energy of the piecewise geodesic connecting constrained to pass through z_1, \dots, z_{l-1} in that order. Let $\bar{\nu}_{a,b}^{l,T}(dz)$ be the restriction of $\nu_{a,b}^{l,T}(dz)$ on the subset $U_{a,b}^{r,l}$ of M^{l-1} normalized to have mass 1. From [11, lemma 3.2], for each fixed $l > 0$,

$$\overline{\lim}_{T \downarrow 0} \sup_{a, b \in M} \sup_{U_{a,b}^{r,l}} \text{osc}(d\mu_{a,b}^{l,T}/d\nu_{a,b}^{l,T}) \leq C(l, r),$$

So, there is a $T(l, r) > 0$ such that for any $\frac{l}{N} < T(l, r)$,

$$(4.45) \quad \lambda(U_{x,a}^{r,l}; \bar{\mu}_{x,a}^{l, \frac{l}{N}}) \geq \frac{1}{2C(l, r)} \lambda(U_{x,a}^{r,l}; \bar{\nu}_{x,a}^{l, \frac{l}{N}}).$$

Step 2. As in [11], let $U_{a,b,\Delta}^{r,l} := \bar{U}_{a,b}^{r,l}/\sim$ be the one point compactification of $U_{a,b}^{r,l}$, which is obtained by identifying the boundary $\partial U_{a,b}^{r,l}$ as a single point Δ . And let $\tilde{C}([0, 1]; \bar{U}_{a,b}^{r,l})$ denote the path in $\bar{U}_{a,b}^{r,l}$ which restricted to a continuous path on the space $U_{a,b,\Delta}^{r,l}$. Define

$$(4.46) \quad M_{a,b}^{r,l}(z) := \inf_{p \in \mathbf{I}_{a,b}^{r,l}} \sup_{s \in [0,1]} E_{a,b}^l(p(s)) \quad a, b \in M,$$

where $\mathbf{I}_{a,b}^{r,l} = \{p \in \tilde{C}([0, 1]; \bar{U}_{a,b}^{r,l}); p(0) = z, p(1) = z_0\}$ and z_0 is a minimum point of $E_{a,b}^l$ in $\bar{U}_{a,b}^{r,l}$. And define

$$(4.47) \quad m_{a,b}^{r,l} := \sup_{\bar{U}_{a,b}^{r,l}} (M_{a,b}^{r,l} - E_{a,b}^l)$$

In fact, if we take the supremum only among the local minimum points of $E_{a,b}^l$ on $\overline{U}_{a,b}^{r,l}$ in the above definition, the value of $m_{a,b}^{r,l}$ will not change, see lemma 2.1 in [11].

According to the proof of Theorem 2.2 in [11], for each $x, a \in M$, if $\frac{l}{N}$ is less than some $T(x, a, l)$,

$$\lambda(U_{x,a}^{r,l}; \overline{U}_{x,a}^{r,l}) \geq C(x, a, l) \left(\frac{l}{N}\right)^{3(l-1)d-2} \exp\left(-\frac{Nm_{x,a}^{r,l}}{l}\right), \quad x \in M,$$

where d is the dimension of M .

Now our goal is to confirm that the constants $T(x, a, l)$, $C(x, a, l)$ above can be chosen to be independent of $x, a \in M$. From step by step checking the proof Theorem 2.2 in [11], if the following three conditions are true, then we can find such constants:

- (a) Uniform estimate on the gradient of the energy function: there exists a constant $C(l) > 0$ depending only on l such that

$$\sup_{x,a \in M} \sup_{z \in \overline{U}_{x,a}^{r,l}} |\nabla E_{x,a}^l(z)|^2 \leq C(l).$$

- (b) A lower bound on the size of the tube $U_{x,a}^{r,l}$: there exists a constant $\theta(l) > 0$, such that for all $R < 1$,

$$\sup_{x,a \in M} \sup_{z \in \partial U_{x,a}^{r,l}} \frac{\text{Vol}(B_R(z)/U_{x,a}^{r,l})}{\text{Vol}(B_R(z))} \geq \theta(l),$$

where $\text{Vol}(A)$ denotes the Riemannian volume of a subset A of M^{l-1} .

- (c) For T sufficiently small, say smaller than some $T(l) > 0$, there are finite subsets $\Sigma_T^0(x, a) \subset \partial U_{x,a}^{r,l}$ and $\Sigma_T(x, a) \subset \overline{U}_{x,a}^{r,l}$ such that

- $\Sigma_T^0(x, a) \subset \Sigma_T(x, a)$
- $\Sigma_T(x, a)$ contains a minimum point $z_0(x, a)$ of $E_{x,a}^l$.
- $\partial U_{x,a}^{r,l} \subseteq \bigcup_{z \in \Sigma_T^0(x,a)} B_T(z)$, $\overline{U}_{x,a}^{r,l} \subseteq \bigcup_{z \in \Sigma_T(x,a)} B_T(z)$.
- $\sup_{x,a \in M} \#\Sigma_T(x, a) \leq C(l)T^{-(l-1)d}$ for some constants $C(l)$.

where $\#$ means the number of elements in a finite set.

Since R_0 from proposition 3.2 is less than the injective radius of compact manifold M , when $r \in (0, R_0)$, $E_{x,a}^l$ is differentiable in the domain $U_{x,a}^{r,l}$ and condition 1 can be checked by direct computation. From the proof of Corollary 3.3 in [11], condition 2 is true.

For condition 3, note that there is a $T(l) > 0$, for each $T < T(l)$, due to the compactness of M^{l-1} , we can find a finite subset $\tilde{\Sigma}_T \subseteq M$ such that $M^{l-1} \subseteq \bigcup_{z \in \tilde{\Sigma}_T} B_T(z)$ and $\#\tilde{\Sigma}_T \leq C(l)T^{-(l-1)d}$. Now since $M^{l-1} \subseteq \bigcup_{z \in \tilde{\Sigma}_{T/2}} B_{T/2}(z)$, we start to construct the set $\Sigma_T(x, a)$ as following:

- (i) if $z \in \tilde{\Sigma}_{T/2}$ and $B_{T/2}(z) \subset U_{x,a}^{r,l}$, then add such z into $\Sigma_T(x, a)$;
- (ii) if $z \in \tilde{\Sigma}_{T/2}$ and $B_{T/2}(z) \cap \partial U_{x,a}^{r,l} \neq \emptyset$, then take a point $\tilde{z} \in B_{T/2}(z) \cap \partial U_{x,a}^{r,l}$ and add this point \tilde{z} into $\Sigma_T(x, a)$.
- (iii) add a minimum point $z_0(x, a)$ of $E_{x,a}^l$ on $\overline{U_{x,a}^{r,l}}$ into $\Sigma_T(x, a)$.

Since in (ii), $B_T(\tilde{z}) \supseteq B_{T/2}(z)$, we have

$$\begin{aligned} \bigcup_{\tilde{z} \in \Sigma_T(x,a)} B_T(\tilde{z}) &\supseteq \bigcup_{z \in \tilde{\Sigma}_{T/2}} B_{T/2}(z) \supseteq M^{l-1} \supseteq \overline{U_{x,a}^{r,l}} \\ \bigcup_{\tilde{z} \in \Sigma_T(x,a) \cap \partial U_{x,a}^{r,l}} B_T(\tilde{z}) &\supseteq \bigcup_{z \in \tilde{\Sigma}_{T/2}; B_{T/2}(z) \cap \partial U_{x,a}^{r,l} \neq \emptyset} B_{T/2}(z) \supseteq \partial U_{x,a}^{r,l}, \quad x, a \in M \end{aligned}$$

and $\#\Sigma_T(x, a) \leq \#\tilde{\Sigma}_{T/2} + 1 \leq 2^{(l-1)d} C(l) T^{-(l-1)d}$, so condition 3 is satisfied.

By the above argument, we can find constants $T(l)$ and $C(l)$, which are most importantly independent of x and a , such that if $\frac{l}{N} < T(l)$, then

$$(4.48) \quad \lambda(U_{x,a}^{r,l}; \overline{v_{x,a}^{l, \frac{l}{N}}}) \geq C(l) \left(\frac{l}{N}\right)^{3(l-1)d-2} \exp\left(-\frac{Nm_{x,a}^{r,l}}{l}\right).$$

Step 3. In the following, we try to give some uniform estimate about $m_{x,a}^{r,l}$. As in [11], define the energy of a path $\gamma \in \Omega_{a,b}$ (possibly infinite) as:

$$E(\gamma) := \frac{1}{2} \sup \sum_{i=0}^{k-1} \frac{d(\gamma(s_i), \gamma(s_{i+1}))^2}{s_{i+1} - s_i}$$

where the supremum is obtained over all partitions $0 = s_0 < s_1 < \dots < s_k = 1$. Assume $a, b \in M$ and a is not conjugate to b , let $\Xi_{a,b}$ denote the set of all geodesics (i.e. critical points of E) in $\Omega_{a,b}$, and let $\Xi_{a,b}^{\min}$ denote the subset of all local energy

minimum. Fix a global energy minimum geodesic $\gamma_{a,b} \in \Omega_{a,b}$, then for each geodesic $\gamma \in \Xi_{a,b}$, we define:

$$M_{a,b}(\gamma) := \inf_{H \in \mathbf{I}} \sup_{s \in [0,1]} E \circ H(s)$$

where $\mathbf{I} = \{H \in C([0,1], \Omega_{a,b}); H(0) = \gamma, H(1) = \gamma_{a,b}\}$. And define

$$m_{a,b} := \sup \{M_{a,b}(\gamma) - E(\gamma); \gamma \in \Xi_{a,b}^{\min}\}.$$

The item $m_{a,b}$ can be viewed as an infinite dimensional version of the item (4.47). Furthermore, every point $z \in U_{a,b}^{r,l}$ corresponds to a piecewise geodesic in M , so intuitively we may have more choices to take supremum in defining $M_{a,b}$ than in defining $M_{a,b}^{r,l}$ as (4.46). In fact, according to the proof of Corollary 1.5 in [11], we have, if a is not conjugate to b ,

$$(4.49) \quad m_{a,b}^{r,l} \leq m_{a,b}, \quad r \in (0, \text{inj}M), \quad l \in \mathbb{N}^+.$$

For $0 < r < R_0$, choose a $\varepsilon > 0$, satisfying with $r + \varepsilon < \text{inj}M$. For any $x \in M$, $a \in M$ and $\tilde{x} \in B_\varepsilon(x)$, $\tilde{a} \in B_\varepsilon(a)$, if $z = (z_1, \dots, z_{l-1}) \in U_{\tilde{x}, \tilde{a}}^{r,l}$, then

$$d(z_1, x) \leq d(x, \tilde{x}) + d(z_1, \tilde{x}) < r + \varepsilon \quad d(z_{l-1}, a) \leq d(a, \tilde{a}) + d(z_{l-1}, \tilde{a}) < r + \varepsilon$$

and $d(z_i, z_{i+1}) < r$, $1 \leq i \leq l-2$

which means $z \in U_{x,a}^{r+\varepsilon,l}$, hence we have $\overline{U_{\tilde{x}, \tilde{a}}^{r,l}} \subseteq U_{x,a}^{r+\varepsilon,l}$.

Suppose $z_0(\tilde{x}, \tilde{a})$ be a minimum point of $E_{\tilde{x}, \tilde{a}}^l$ on $\overline{U_{\tilde{x}, \tilde{a}}^{r,l}}$, and $z_0(x, a)$ be a minimum point of $E_{x,a}^l$ on $\overline{U_{x,a}^{r+\varepsilon,l}}$, by the definition of $M_{a,b}^{r,l}$ in (4.46), for each $\delta > 0$ and each $z \in \overline{U_{\tilde{x}, \tilde{a}}^{r,l}} \subseteq U_{x,a}^{r+\varepsilon,l}$, there exists a path $q_1 \in \tilde{C}([0,1]; \overline{U_{x,a}^{r+\varepsilon,l}})$, such that $q_1(0) = z$, $q_1(1) = z_0(x, a)$, and

$$(4.50) \quad E_{x,a}^l \circ q_1(s) \leq E_{x,a}^l(z) + m_{x,a}^{r+\varepsilon,l} + \delta, \quad 0 \leq s \leq 1$$

As the same reason, we can find a path $q_2 \in \tilde{C}([0,1]; \overline{U_{x,a}^{r+\varepsilon,l}})$ with $q_2(0) = z_0(\tilde{x}, \tilde{a})$, $q_2(1) = z_0(x, a)$ and

$$(4.51) \quad E_{x,a}^l \circ q_2(s) \leq E_{x,a}^l(z_0(\tilde{x}, \tilde{a})) + m_{x,a}^{r+\varepsilon,l} + \delta, \quad 0 \leq s \leq 1.$$

Let

$$q(s) = \begin{cases} q_1(2s) & \text{if } 0 < s \leq \frac{1}{2}, \\ q_2(2-2s) & \text{if } \frac{1}{2} < s \leq 1 \end{cases}$$

and $\tau = \inf\{s; q(s) \in \partial U_{\tilde{x}, \tilde{a}}^{r,l}\} \wedge 1$, $\hat{\tau} = \sup\{s; q(s) \in \partial U_{\tilde{x}, \tilde{a}}^{r,l}\} \vee 1$. Define

$$\tilde{q}(s) = \begin{cases} q(s) & \text{if } s \in [0, \tau) \cup (\hat{\tau}, 1], \\ q(\hat{\tau}) & \text{if } s \in [\tau, \hat{\tau}]. \end{cases}$$

Then $\tilde{q} \in \tilde{C}([0, 1]; \overline{U}_{\tilde{x}, \tilde{a}}^{r,l})$ and $\tilde{q}(0) = z$, $\tilde{q}(1) = z_0(\tilde{x}, \tilde{a})$. Note that for each $z \in \overline{U}_{\tilde{x}, \tilde{a}}^{r,l}$,

$$(4.52) \quad \begin{aligned} & |E_{\tilde{x}, \tilde{a}}^l(z) - E_{x,a}^l(z)| \\ &= \left| \frac{l(d(z_1, x)^2 - d(z_1, \tilde{x})^2)}{2} + \frac{l(d(z_{l-1}, a)^2 - d(z_{l-1}, \tilde{a})^2)}{2} \right| \\ &\leq (d(a, \tilde{a}) + d(x, \tilde{x}))Dl \leq 2lD\varepsilon \end{aligned}$$

where D is the diameter of the manifold M . Then, by (4.50), (4.51), (4.52) and the definition of \tilde{q} , we have

$$\begin{aligned} E_{\tilde{x}, \tilde{a}}^l \circ \tilde{q}(s) &\leq E_{x,a}^l \circ \tilde{q}(s) + 2lD\varepsilon \leq \max\{E_{x,a}^l(z), E_{x,a}^l(z_0(\tilde{x}, \tilde{a}))\} + m_{x,a}^{r+\varepsilon,l} + \delta + 2lD\varepsilon \\ &\leq \max\{E_{\tilde{x}, \tilde{a}}^l(z), E_{\tilde{x}, \tilde{a}}^l(z_0(\tilde{x}, \tilde{a}))\} + m_{x,a}^{r+\varepsilon,l} + \delta + 4lD\varepsilon \\ &= E_{\tilde{x}, \tilde{a}}^l(z) + m_{x,a}^{r+\varepsilon,l} + \delta + 4lD\varepsilon, \quad 0 \leq s \leq 1. \end{aligned}$$

The equality in the last step above is due to the fact that $z_0(\tilde{x}, \tilde{a})$ is a minimum point of $E_{\tilde{x}, \tilde{a}}^l$ on $\overline{U}_{\tilde{x}, \tilde{a}}^{r,l}$. Thus, according to the above inequality and the definition of $M_{\tilde{x}, \tilde{a}}^{r,l}$, and by the arbitrary of δ , we obtain $M_{\tilde{x}, \tilde{a}}^{r,l}(z) \leq E_{\tilde{x}, \tilde{a}}^l(z) + m_{x,a}^{r+\varepsilon,l} + 4lD\varepsilon$. Hence, by this (4.49) and the definition of $m_{\tilde{x}, \tilde{a}}^{r,l}$, when $d(x, \tilde{x}) < \varepsilon$ and $d(a, \tilde{a}) < \varepsilon$, we have

$$(4.53) \quad m_{\tilde{x}, \tilde{a}}^{r,l} \leq m_{x,a}^{r+\varepsilon,l} + 4lD\varepsilon \leq m_{x,a} + 4lD\varepsilon.$$

By [11, Theorem 1.4], when M is a compact simply connected manifold with strict Ricci curvature, we have $m_{a,b} < \infty$ for each pair of $a, b \in M$ if a is not conjugate to b . Since for any $\varepsilon > 0$, $a \in M$, there exists a finite set $\Theta_{\varepsilon,a} \subseteq \{x \in M : x \text{ is not conjugate to } a\}$ such that $\bigcup_{x \in \Theta_{\varepsilon,a}} B_\varepsilon(x) \supseteq M$, then by (4.53), for each $a, b \in M$ with $d(a, b) < \varepsilon$,

$$(4.54) \quad \sup_{y \in M} m_{y,b}^{r,l} \leq \sup_{x \in \Theta_{\varepsilon,a}} m_{x,a} + 4lD\varepsilon.$$

As the same way, there is a finite set Θ_ε , such that $\bigcup_{x \in \Theta_\varepsilon} B_\varepsilon(x) \supseteq M$, by (4.53) and (4.54),

$$(4.55) \quad \sup_{y,b \in M} m_{y,b}^{r,l} \leq \sup_{a \in \Theta_\varepsilon} \sup_{x \in \Theta_{\varepsilon,a}} m_{x,a} + 4lD\varepsilon.$$

Let

$$L(\varepsilon) := \sup_{a \in \Theta_\varepsilon} \sup_{x \in \Theta_{\varepsilon,a}} m_{x,a} < +\infty.$$

So, by (4.45), (4.48) and (4.55), if $\frac{l}{N}$ less than some $T(l, r)$, then

$$(4.56) \quad \inf_{x,a \in M} \lambda(U_{x,a}^{r,l}; \bar{\mu}_{x,a}^{l, \frac{l}{N}}) \geq \frac{C(l, r)}{N^{C(l,r)}} \exp\left(-\left(\frac{L(\varepsilon)}{l} + 4D\varepsilon\right) \cdot N\right).$$

where constant $C(l, r)$ only depends on l, r , by now we have completed the proof. \square

5 The Main Theorem

Theorem 5.1. *Let M be a simply connected compact manifold with strict positive Ricci curvature. For any small $\alpha > 0$, there exists a constant $s_0 > 0$ such that the following weak Poincaré inequality holds, i.e.*

$$(5.57) \quad \mathbf{Var}(F; \mathbf{P}_{a,a}) \leq \frac{1}{s^\alpha} \mathcal{E}_{a,a}(F, F) + s \|F\|_\infty^2, \quad s \in (0, s_0), \quad F \in \mathcal{D}(\mathcal{E}_{a,a}).$$

The constants s_0 does not depend on the starting point $a \in M$.

Proof. It suffices to show that (5.57) holds for $F \in \mathcal{F}C_b^\infty(\Omega_{a,a})$. Let $\omega_i(s) := \omega\left(\frac{i-1+s}{N}\right)$ for each $\omega \in \Omega_{a,a}$. For a function $F \in \mathcal{F}C_b^\infty(\Omega_{a,a})$, as in the proof of proposition 3.2, there is a unique function $F^{[N]}$ defined on $\bigcup_{(x_1, \dots, x_{N-1}) \in M^{N-1}} \prod_{i=1}^N \Omega_{x_{i-1}, x_i}$ such that,

$$F^{[N]}(\omega_1, \omega_2, \dots, \omega_N) = F(\omega), \quad \omega \in \Omega_{a,a},$$

Step 1. We fix $N > N(\eta, r, l)$ with $l > N_1(\eta, r)$, for $N_1(\eta, r)$ and $N(\eta, l, r)$ as given in proposition 4.1. We first assume $F(\omega) = 0$ if $(\omega(1/N), \omega(2/N), \dots, \omega(1-1/N))$ is not in $U_{a,a}^{r,N}$. For $(x_1, \dots, x_{N-1}) \in M^{N-1}$,

$$\begin{aligned} f^{[N]}(x_1, x_2, \dots, x_{N-1}) &:= \int F^{[N]} \prod_{i=1}^N \mathbf{P}_{x_{i-1}, x_i}^{\frac{1}{N}}(d\omega_i) \\ &= \mathbf{E}_{a,a} \left[F(\omega) \mid \omega\left(\frac{1}{N}\right) = x_1, \dots, \omega\left(\frac{N-1}{N}\right) = x_{N-1} \right]. \end{aligned}$$

Let $\mathfrak{S}_N := \sigma\{\omega(i/N), 1 \leq i \leq N-1\}$, an σ -algebra on $\Omega_{a,a}$. Then we have,

$$\begin{aligned}
(5.58) \quad & \mathbf{Var}(F; \mathbf{P}_{a,a}) \\
&= \mathbf{E}_{a,a}[(F - \mathbf{E}_{a,a}[F|\mathfrak{S}_N])^2] + \mathbf{E}_{a,a}[(\mathbf{E}_{a,a}[F|\mathfrak{S}_N] - \mathbf{E}_{a,a}[F])^2] \\
&= \int_{M^{N-1}} \mathbf{Var}(F^{[N]}; \otimes_{i=0}^{N-1} \mathbf{P}_{x_i, x_{i+1}}^{\frac{1}{N}}) d\mu_{a,a}^{N,1} + \mathbf{Var}(f^{[N]}; \mu_{a,a}^{N,1}) \\
&\leq \int_{U_{a,a}^{r,N}} \left\{ \sum_{j=1}^N \int \mathbf{Var}_j(F^{[N]}; \mathbf{P}_{x_{j-1}, x_j}^{\frac{1}{N}}) \prod_{i \neq j} \mathbf{P}_{x_{i-1}, x_i}^{\frac{1}{N}}(d\omega_i) \right\} \mu_{a,a}^{N,1}(dx) + \mathbf{Var}(f^{[N]}; \mu_{a,a}^{N,1}).
\end{aligned}$$

Here \mathbf{Var}_j indicates the variance is with respect to the j th subpath. Note that $f^{[N]}$ is smooth with support in $\bar{U}_{a,a}^{r,N}$ and $\|f^{[N]}\|_\infty \leq \|F\|_\infty$. From Proposition 4.1, if $N > N(\eta, l, r)$, then

$$\begin{aligned}
(5.59) \quad & \mathbf{Var}(f^{[N]}; d\mu_{a,a}^{N,1}) \\
&\leq C(l, r) N^{C(l,r)} e^{2N\eta r^2} \sum_{i=1}^{N-1} \int_{M^{N-1}} |\nabla_i f^{[N]}|^2 d\mu_{a,a}^{N,1} + C(l, \eta, r) N^{C(l,r)} e^{-\frac{N(1-8\eta)r^2}{2}} \|F\|_\infty^2.
\end{aligned}$$

According to the proof of lemma 6.1 and lemma 6.2 in [10] (since the support of $f^{[N]}$ is in $\bar{U}_{a,a}^{r,N}$, we can choose some vector with better asymptotic property in the estimate of the derivative of expectation with pinned Wiener measure), there exists a constant $C(r)$, such that

$$\begin{aligned}
(5.60) \quad & \sum_{i=1}^{N-1} \int |\nabla_i f^{[N]}|^2 d\mu_{a,a}^{N,1} \leq C(r) N \mathbf{E}_{a,a} |\mathbf{D}_0 F|_{\mathbf{H}_\omega^0}^2 \\
&+ C(r) N \int_{U_{a,a}^{r,N}} \left\{ \sum_{j=1}^N \int_{M^{N-1}} \mathbf{Var}_j(F^{[N]}; \mathbf{P}_{x_{j-1}, x_j}^{\frac{1}{N}}) \prod_{i \neq j} \mathbf{P}_{x_{i-1}, x_i}^{\frac{1}{N}}(d\omega_i) \right\} \mu_{a,a}^{N,1}(dx).
\end{aligned}$$

By proposition 3.2, if $\frac{1}{N} < T(\eta, r)$, then

$$\begin{aligned}
(5.61) \quad & \int_{U_{a,a}^{r,N}} \left\{ \sum_{j=1}^N \int \mathbf{Var}_j(F^{[N]}; \mathbf{P}_{x_{j-1}, x_j}^{\frac{1}{N}}) \prod_{i \neq j} \mathbf{P}_{x_{i-1}, x_i}^{\frac{1}{N}}(d\omega_i) \right\} \mu_{a,a}^{N,1}(dx_1, \dots, dx_{N-1}) \\
&\leq \frac{C(r)}{N} \int \sum_{j=1}^N |\mathbf{D}_{0,(j)} F^{[N]}(\omega_1, \dots, \omega_N)|_{\omega_j}^2 \mathbf{P}_{a,a}(d\omega) + NC(\eta, r) e^{-\frac{(1-4\eta)Nr^2}{2}} \|F\|_\infty^2,
\end{aligned}$$

where $\mathbf{D}_{0,(j)}$ means the gradient \mathbf{D}_0 of $F^{[N]}$ with respect to the j th subpath. According to (6.16) in the proof of lemma 6.3 in [10], we have the following relation,

$$(5.62) \quad \sum_{j=1}^N |\mathbf{D}_{0,(j)} F^{[N]}(\omega_1, \dots, \omega_N)|_{\omega_j}^2 \leq N |\mathbf{D}_0 F|_{\mathbf{H}_\omega^0}^2, \quad \omega \in \Omega_{a,a}.$$

By (5.58—5.62), if $N > N(\eta, l, r)$ with $l > N_1(\eta, r)$, then

$$(5.63) \quad \begin{aligned} & \mathbf{Var}(F; \mathbf{P}_{a,a}^1) \\ & \leq C(l, r) N^{C(l,r)} e^{2N\eta r^2} \mathbf{E}_{a,a} |\mathbf{D}_0 F|_{\mathbf{H}_\omega^0}^2 + C(l, \eta, r) N^{C(l,r)} e^{-\frac{N(1-8\eta)r^2}{2}} \|F\|_\infty^2. \end{aligned}$$

Step 2. Consider a general function F from $\mathcal{F}C_b^\infty(\Omega_{a,a})$. Define a smooth cut-off function on $\Omega_{a,a}$ as,

$$\Psi_N(\omega) := \prod_{i=1}^N \varphi \left(d \left(\omega \left(\frac{i-1}{N} \right), \omega \left(\frac{i}{N} \right) \right) \right)$$

where φ is defined as in the proof of proposition 3.2. By the proof of proposition 3.2, if $\frac{1}{N} < T(\eta, r)$,

$$(5.64) \quad \mathbf{P}_{a,a}(\Psi_N \neq 1) \leq N \exp \left(-\frac{(1-4\eta)Nr^2}{2} \right), \quad |\mathbf{D}_0 \Psi_N(\omega)|_{\mathbf{H}_\omega^0} \leq \frac{6N}{\eta r}.$$

Then note that $F\Psi_N(\omega) = 0$ when $(\omega(1/N), \omega(2/N), \dots, \omega(1-1/N))$ is not in $U_{a,a}^{r,N}$, hence by (5.63) and (5.64), if $N > N(\eta, r, l)$ with $l > N_1(\eta, r)$, we obtain

$$(5.65) \quad \begin{aligned} & \mathbf{Var}(F; \mathbf{P}_{a,a}) \leq \mathbf{Var}(F\Psi_N; \mathbf{P}_{a,a}) + 3\mathbf{P}_{a,a}(\Psi_N \neq 1) \|F\|_\infty^2 \\ & \leq C(l, r) N^{C(l,r)} e^{2N\eta r^2} \mathbf{E}_{a,a} |\mathbf{D}_0(F\Psi_N)|_{\mathbf{H}_\omega^0}^2 + C(l, \eta, r) N^{C(l,r)} e^{-\frac{N(1-8\eta)r^2}{2}} \|F\|_\infty^2 \\ & \leq C(l, r) N^{C(l,r)} e^{2N\eta r^2} \mathbf{E}_{a,a} |\mathbf{D}_0 F|_{\mathbf{H}_\omega^0}^2 + C(l, \eta, r) N^{C(l,r)} e^{-\frac{N(1-8\eta)r^2}{2}} \|F\|_\infty^2. \end{aligned}$$

Let $s := C(l, \eta, r) N^{C(l,r)} e^{-\frac{(1-8\eta)Nr^2}{2}}$ in (5.65), then s tends to zero when N tends to infinity. In particular for any small $\alpha > 0$ we can choose a η small enough so that there is a constant $s_0(\eta, r, l, \varepsilon, \alpha)$, s_0 does not depend on the starting point a of the loop space, such that,

$$\mathbf{Var}(F; \mathbf{P}_{a,a}) \leq \frac{1}{s^\alpha} \mathcal{E}_{a,a}(F, F) + s \|F\|_\infty^2, \quad s \in (0, s_0), F \in \mathcal{D}(\mathcal{E}_{a,a}).$$

By now we have completed the proof. \square

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