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Bismut type formulae for differential forms

K. David ELWORTHY [a] (*), Xue-Mei LI [b] (**)

[a] Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

[b] Department of Mathematics University of Connecticut, Storrs, USA and Mathematical Sciences Research Institute, Berkeley, California, USA

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Abstract. Formulae are given $\mathrm{d}P_t\phi$, $\mathrm{d}^*P_t\phi$, and $\Delta P_t\phi$ for P_t the heat semigroup acting on a q-form ϕ. The formulae are Brownian motion expectations of ϕ composed with random translations determined by Weitzenböck curvature terms. Derivatives of the curvature are not involved. © Académie des Sciences/Elsevier, Paris

Formules de Bismut pour les formes différentielles

Résumé. *Nous donnons des expression pour* $\mathrm{d}P_t\phi$, $\mathrm{d}^*P_t\phi$ *et* $\Delta P_t\phi$, *où* P_t *désigne le semi-groupe de la chaleur agissant sur les formes différentielles d'ordre q. Elles s'écrivent comme l'espérance de la fonctionnelle* ϕ *composée avec les transports parallèles aléatoires associés aux courbures de Weitzenböck. Les dérivées de la courbure n'interviennent pas dans ces formules.* © Académie des Sciences/Elsevier, Paris

Version française abrégée

1. Soient M une variété riemanienne compacte, Δ le laplacien avec la convention de signe $\Delta = -(\mathrm{d}\delta + \mathrm{d}\delta)$ et $\mathcal{R} : \wedge^*\mathrm{T}M \to \wedge^*\mathrm{T}M$ est la courbure de Weitzenböck. On a la formule de Weitzenböck :

$$\Delta\phi = \mathrm{trace}\nabla^2\phi - \phi \circ \mathcal{R}. \tag{1}$$

On note $\mathcal{R}^q$, etc. les restrictions des opérateurs sur les formes différentielles d'ordre q et sur les q-vecteurs. Rappelons que $\mathcal{R}^1 = \mathrm{Ric}^\sharp$, le tenseur de Ricci.

On désigne par $\{P_t : t \geq 0\}$ le semi-groupe $\mathrm{e}^{\frac{1}{2}t\Delta}$ agissant sur les formes différentielles de L^2. Il agit donc sur les formes continues par la formule de Feyman–Kac d'Airault $P_t\phi = \mathbb{E}\phi(V_t)$, où V_t est la solution de l'équation covariante le long des trajectoires browniennes $\{x_t : t \geq 0\}$:

$$\frac{\mathrm{D}V_t}{\partial t} = -\frac{1}{2}\mathcal{R}(V_t), \quad V_0 \in \wedge^*_{x_0}\mathrm{T}M. \tag{2}$$

Note présentée par Jean-Michel BISMUT.

On définit $W_t : \wedge^* TM \to \wedge^* TM$ par $V_t = W_t(V_0)$. Soit $\{\breve{B}_t : t \geq 0\}$ l'anti-développement stochastique du mouvement brownien $\{x_t : t \geq 0\}$, et $//_t$ le transport parallèle de la connexion de Levi-Civita le long des trajectoires.

La formule de Bismut est, pour les fonctions bornées mesurables $f : M \to \mathbb{R}$:

$$\mathrm{d}P_t f(v_0) = \frac{1}{t} \mathbb{E} f(x_t) \int_0^t \langle //_s \mathrm{d}\breve{B}_s, W_s \rangle. \tag{3}$$

On a donc la formule pour le gradient du noyau de la chaleur $p_t(x, y)$ sur les fonctions (*voir* [1]) :

$$\nabla_x \log p_t(x_0, y) = \mathbb{E}\left\{ \frac{1}{t} \int_0^t W_s^* (//_d \mathrm{d}\breve{B}_s) | x_t = y \right\}. \tag{4}$$

Nous généralisons ces formules aux cas des q-formes, $1 \leq q < n$.

THÉORÈME 1. – *Soit ϕ une forme différentielle, bornée mesurable sur la variété M.*

$$\mathrm{d}(P_t\phi)(V_0) = \frac{1}{t} \mathbb{E}\, \phi \left(W_t^q \int_0^t (W_s^q)^{-1} \iota_{//_s \mathrm{d}\breve{B}_s} W_s^{q+1}(V_0) \right), \quad V_0 \in \wedge^* \mathrm{T}_{x_0} M, \tag{5}$$

où $\iota_{//_s \mathrm{d}\breve{B}_s}$ désigne le produit intérieur (l'opérateur d'annihilation) :

$$\iota_{//_s \mathrm{d}\breve{B}_s} v^1 \wedge \cdots \wedge v^{q+1} = \sum_{j=1}^{q+1} (-1)^{j+1} \langle //_s \mathrm{d}\breve{B}_s, v^j \rangle v^1 \wedge \cdots \wedge \widehat{v^j} \cdots v^{q+1}.$$

Nous avons un corollaire analogue à (4) :

COROLLAIRE 2. – *Pour le noyau de la chaleur $p_t^q(x_0, y)$ défini sur les q-formes :*

$$\mathrm{d}_x p_t^q(x_0, y)(V_0) = \frac{1}{t} p_t^q(x_0, y) \left(\mathbb{E}\left\{ W_t^q \int_0^t (W_s^q)^{-1} \iota_{//_s \mathrm{d}\breve{B}_s} W_s^{q+1}(V_0) | x_t = y_0 \right\} \right). \tag{6}$$

Nous remarquons que ces formules ne font pas intervenir les dérivées du tenseur de courbure. Dans [10], Norris donne une expression des dérivées covariantes de tel semi-groupes de la chaleur. Toutefois, ses formules font intervenir les dérivées du tenseur de courbure.

La formule (5) provient d'une formule non-intrinsèque de Li [9] :

$$\mathrm{d}(P_t^q \phi) = \frac{1}{t} \mathbb{E} \int_0^t \langle //_s \mathrm{d}\breve{B}_s, \mathrm{T}\xi_s(-) \rangle \wedge \phi(\wedge^q \mathrm{T}\xi_t(-)), \tag{7}$$

voir aussi Elworthy–Li [7], en utilisant les techniques de Elworthy–Yor [5], comme dans Elworthy–Li [8]. Pour les détails voir la version anglaise.

On peut déduire le théorème 1 d'une formule d'intégration par partie plus générale sur les espaces de chemins pour les q-formes (*voir* [6]). Quand on remplace le mouvement brownien par le mouvement brownien avec dérive, la même démonstration fournit des formules avec des changements évidents (*voir* [7], [8]). De même pour le changement de Thalmaier, où $\frac{1}{t}\int_0^t$ est remplacé par $\frac{1}{h}\int_\delta^{\delta+h}$, $0 \leq \delta < \delta + h \leq t$ (*voir* [7]).

2. *Les formules pour* $\mathrm{d}^* P_t \phi$. – Nous donnons un autre corollaire, suite à une conversation avec Bruce Driver. Driver [3] a démontré le cas particulier de $q = 1$ par une méthode différente. Il a également prouvé une formule intrigante faisant intervenir une intégrale stochastique rétrograde (*voir* [2]).

COROLLAIRE 3. – *Soit ϕ une forme différentielle d'ordre q de* L^2. *On a :*

$$\mathrm{d}^* P_t \phi = -\frac{1}{t} \mathbb{E}\phi \left(W_t^q \int_0^t (W_s^q)^{-1} (//_s \mathrm{d}\breve{B}_s \wedge W_s^{q-1}(-)) \right). \tag{8}$$

Voir la version anglaise pour une preuve.

3. *Les formules pour* $\Delta P_t\phi$. – La méthode utilisée dans [7] pour obtenir les dérivées d'ordre supérieur du semi-groupe de la chaleur agissant sur les fonctions peut être utilisée ici pour obtenir des résultats plus élégants :

COROLLAIRE 4

$$\Delta P_t\phi(V) = \frac{4}{t^2}\mathbb{E}\phi\left(W_t\int_{t/2}^t (W_s)^{-1}\mathrm{d}M_s(V)\right),$$

où

$$\mathrm{d}M_s(V) = \iota_{//_s\mathrm{d}\breve{B}_s}W_s\left(\int_0^{t/2}(W_r)^{-1}(//_r\mathrm{d}\breve{B}_r\wedge W_r(V))\right) + \iota_{//_s\mathrm{d}\breve{B}_s}W_s\left(\int_0^{t/2}(W_r)^{-1}(\iota_{//_r\mathrm{d}\breve{B}_r}W_r(V))\right). \tag{9}$$

Nous pouvons itérer cette méthode afin d'obtenir les formules pour $(\Delta)^r P_t\phi$, $r = 1, 2, \ldots$. Elles permettent d'établir les formules pour $\mathrm{d}_x p_t^q(\cdot, y)$, $\mathrm{d}_x^* p_t^q(\cdot, y)$, $(\Delta_x)^r p_t^q(\cdot, y)$. En particulier, on déduit que $\mathrm{d}_x p_t^q(\cdot, y)$, $\mathrm{d}_x^* p_t^q(\cdot, y)$, $(\Delta_x)^r p_t^q(\cdot, y)$ et $\mathrm{d}P_t\phi$, $\mathrm{d}^*P_t\phi$, $(\Delta)^r P_t\phi$, $r = 1, 2, \ldots$, sont uniformément bornés par une constante dépendant uniquement de t, des courbures de Weitzenböck $\mathcal{R}^q$, $\mathcal{R}^{q\pm 1}$ et du maximum de $|\phi|$.

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1. Let M be a compact Riemannian manifold with Hodge–de Rham Laplacian Δ given the sign convention $\Delta = -(\mathrm{d}\delta + \mathrm{d}\delta)$. There is then the Weitzenböck formula:

$$\Delta\phi = \operatorname{trace}\nabla^2\phi - \phi\circ\mathcal{R}, \tag{1}$$

where $\mathcal{R} : \wedge^*\mathrm{T}M \to \wedge^*\mathrm{T}M$ is the Weitzenböck curvature term. Let $\mathcal{R}^q$, etc. denote the restriction to q-forms and q-vectors. Recall $\mathcal{R}^1 = \mathrm{Ric}^\sharp$, the Ricci tensor.

Let $\{P_t : t \geq 0\}$ be the semigroup $\mathrm{e}^{\frac{1}{2}t\Delta}$ acting on L^2-forms (by the essentially self-adjointness of Δ) and on bounded measurable forms via Airault's version of the Feyman–Kac formula:

$$P_t\phi(V_0) = \mathbb{E}\phi(V_t), \tag{2}$$

where V_t satisfies the covariant equation along Brownian paths $\{x_t : t \geq 0\}$:

$$\frac{\mathrm{D}V_t}{\partial t} = -\frac{1}{2}\mathcal{R}(V_t), \quad V_0 \in \wedge^*\mathrm{T}_{x_0}M. \tag{3}$$

We will write $V_t = W_t(V_0)$ to give $W_t : \wedge^*\mathrm{T}M \to \wedge^*\mathrm{T}M$. Let $\{\breve{B}_t : t \geq 0\}$ be the stochastic anti-development of $\{x_t : t \geq 0\}$, a Brownian motion on $\mathrm{T}_{x_0}M$, and denote by $//_t$ the parallel translation for the Levi-Civita connection along our paths.

Bismut's formula is for bounded measurable function $f : M \to \mathbb{R}$. It is

$$\mathrm{d}P_t f(v_0) = \frac{1}{t}\mathbb{E}f(x_t)\int_0^t \langle //_s\mathrm{d}\breve{B}_s, W_s\rangle \tag{4}$$

which leads to his gradient formula for the heat kernel $p_t(x, y)$ on functions (*see* [1]):

$$\nabla_x \log p_t(x_0, y) = \mathbb{E}\left\{\frac{1}{t}\int_0^t W_s^*(//_s\mathrm{d}\breve{B}_s)|x_t = y\right\}. \tag{5}$$

Here we extend these formulae to the case of q-forms, $1 \leq q < n$, to show:

THEOREM 1.1. – *For ϕ a bounded measurable q-form on M and $V_0 \in \wedge^* \mathrm{T}_{x_0} M$*

$$\mathrm{d}(P_t\phi)(V_0) = \frac{1}{t}\,\mathbb{E}\,\phi\left(W_t^q \int_0^t (W_s^q)^{-1} \iota_{//_s \mathrm{d}\breve{B}_s} W_s^{q+1}(V_0)\right), \tag{6}$$

where $\iota_{//_s \mathrm{d}\breve{B}_s}$ denotes the interior product (annihilation operator) given by:

$$\iota_{//_s \mathrm{d}\breve{B}_s} v^1 \wedge \cdots \wedge v^{q+1} = \sum_{j=1}^{q+1} (-1)^{j+1} \langle //_s \mathrm{d}\breve{B}_s, v^j \rangle v^1 \wedge \cdots \wedge \widehat{v^j} \cdots v^{q+1}.$$

As for (5) there is the corollary:

COROLLARY 1.2. – *For the heat kernel $p_t^q(x_0, y)$ on q-forms:*

$$\mathrm{d}_x p_t^q(x_0, y)(V_0) = \frac{1}{t}\, p_t^q(x_0, y)\left(\mathbb{E}\left\{W_t^q \int_0^t (W_s^q)^{-1} \iota_{//_s \mathrm{d}\breve{B}_s} W_s^{q+1}(V_0) | x_t = y_0\right\}\right). \tag{7}$$

Note these formulae do not involve derivatives of the curvature tensor. Expressions for covariant derivatives of such heat semigroups have been obtained by Norris [10], however these do involve such derivatives.

Formula (6) is derived from a non-intrinsic formula in Li [9]:

$$\mathrm{d}(P_t^q \phi) = \frac{1}{t}\,\mathbb{E} \int_0^t \langle //_s \mathrm{d}\breve{B}_s, \mathrm{T}\xi_s(-)\rangle \wedge \phi(\wedge^q \mathrm{T}\xi_t(-)), \tag{8}$$

see also Elworthy–Li [7], by techniques of Elworthy–Yor [5], as used in Elworthy–Li [8]. The details follow:

Proof of Theorem 1.1. – Taken an isometric embedding $\alpha : M \to \mathbb{R}^m$, some m, using Nash's Theorem. Let $X(x) : \mathbb{R}^m \to \mathrm{T}_x N$ be the orthogonal projection, identifying $\mathrm{T}_x M$ with its image in $\mathbb{R}^m$ under $\mathrm{d}\alpha$. Let $\{B_t : t \geq 0\}$ be a Brownian motion on $\mathbb{R}^m$. The *gradient Brownian system*

$$\mathrm{d}x_t = X(x_t) \circ \mathrm{d}B_t \tag{9}$$

has a solution flow $\{\xi_t(x) : t \geq 0,\, x \in M\}$ consisting of random diffeomorphisms $\xi_t : M \to M$ with derivatives denoted by $\mathrm{T}_{x_0}\xi_t : \mathrm{T}_{x_0} M \to \mathrm{T}_{x_t} M$ for $x_t = \xi_t(x_0)$. Each solution $(x_t : t \geq 0)$ is a Brownian motion on M.

Define $J_t : \wedge^{q+1}\mathrm{T}M \to \wedge^q \mathrm{T}M$ by:

$$J_t(V) = \int_0^t \langle \mathrm{T}\xi_t -, //_r \mathrm{d}\breve{B}_r \rangle \wedge (\wedge^q \mathrm{T}\xi_t(-)), \quad t > 0. \tag{10}$$

We see that formula (8) can be written as:

$$\mathrm{d}(P_t\phi)(V) = \frac{1}{t} E_\phi(J_t(V)), \quad t > 0,$$

and J_t satisfies

$$\mathrm{D}J_t = \iota_{X(x_t)\,\mathrm{d}B_t} \wedge^{q+1} \mathrm{T}\xi_t + \mathrm{d} \wedge^q (\nabla X(-)\,\mathrm{d}B_t) - \frac{1}{2}\mathcal{R}^q(J_t)\,\mathrm{d}t,$$

where the term involving $\mathcal{R}^q$ comes from the Stratonovich correction. (*See* Elworthy–Yor [5].) Next set

$$\overline{J}_t = E\{J_t | \mathcal{F}_t^{x_0}\},$$

where $\mathcal{F}^{x_0}$ is the filtration generated by $\{x_.\}$ and the conditional expectation is understood to be $//_t E\{//_t^{-1} J^t | \mathcal{F}_t^{x_0}\}$. Using the decomposition of B_t into relevant and "redundant" noise as in Elworthy–Yor [5], we have:

$$\mathrm{d} \wedge^q \nabla X(-)\, \mathrm{d}B_t = \mathrm{d} \wedge^q \nabla X(-) \tilde{//}_t \mathrm{d}\beta_t$$

where $\tilde{//}_t$, $t \geq 0$, consists of random isometries of $\mathbb{R}^m$ and $\{\beta_t : t \geq 0\}$ is a Brownian motion on $\operatorname{Ker} X(x_0)$ independent of $\mathcal{F}_t^{x_0}$. From this $\overline{J}_t$ satisfies

$$D\overline{J}_t = \iota_{X(x_t)\,\mathrm{d}B_t} W_t^{q+1} - \frac{1}{2} \mathcal{R}^q(\overline{J}_t)\, \mathrm{d}t. \tag{11}$$

For a general discussion, *see* Elworthy–Le Jan–Li [4]. Solve the equation (11) to get:

$$\overline{J}_t = W_t^q \int_0^t (W_s^q)^{-1} \left(\iota_{X(x_s)\,\mathrm{d}B_s}\right) W_s^{q+1}.$$

The required result follows. □

Theorem 1.1 can also be deduced from a more general integration by parts formula for q-forms on path space (*see* [6]). The obvious variation when the Brownian motion is replaced by one with a drift Z also holds with the same proof: here $\{W_.\}$ is replaced by $\{W_.^Z\}$ as in [7], [8]. So also does Thalmaier's variant when $\frac{1}{t}\int_0^t$ is replaced by $\frac{1}{h}\int_\delta^{\delta+h}$ for $0 \leq \delta < \delta + h \leq t$ in (17), (15), and (16).

2. *Formulae for* $\mathrm{d}^* P_t\phi$. – We give the next corollary following conversations with Bruce Driver. Driver [3] proved it in the special case $q = 1$, by a different method. Also in [2], he gave an intriguing different formula in that case involving a backward stochastic integral.

COROLLARY 2.1. – *Let ϕ be a* L^2 *q-form. Then:*

$$\mathrm{d}^* P_t \phi = -\frac{1}{t} \mathbb{E}\phi \left(W_t^q \int_0^t (W_s^q)^{-1} (//_s \mathrm{d}\breve{B}_s \wedge W_s^{q-1}(-)) \right). \tag{12}$$

Proof. – We can assume that ϕ is smooth. Also by going to the double cover if necessary we can assume that M is oriented with volume element $d\mathrm{vol}$ considered as a n-form. The Hodge star construction then gives $* : \wedge \mathrm{T}^* M \to \wedge \mathrm{T}^* M$ and $* : \wedge \mathrm{T}M \to \wedge \mathrm{T}M$ such that $\alpha \wedge *\beta = \langle \alpha, \beta \rangle \,\mathrm{d\,vol}$ and $v_1 \wedge *v_2 = \langle v_1, v_2 \rangle (\mathrm{d\,vol})^\sharp$. Then $*\phi(V) = (-1)^{q(n-q)} \phi(*V)$, for $V \in \wedge^q \mathrm{T}_x M$, $** = (-1)^{q(n-q)} \mathrm{Id}$, and $\mathrm{d}^*\phi = (-1)^{n(q-1)+1} * \mathrm{d} * \phi$.

Furthermore, $*P_t = P_t *$, $*W_t = W_t *$, and $*\iota_u V = (-1)^{q-1} u \wedge (*V)$ any $u \in \mathrm{T}_x M$. Thus $\mathrm{d}^* P_t \phi(-) = (-1)^{n(q-1)+1} * \mathrm{d} * (P_t\phi)(-) = (-1)^q \mathrm{d} * (P_t \phi)(*-) = (-1)^q \mathrm{d}(P_t * \phi)(*-)$. Consequently,

$$\begin{aligned}
\mathrm{d}^* P_t \phi(-) &= (-1)^q \frac{1}{t} \mathbb{E}(*\phi) \left(W_t^{n-q} \int_0^t (W_s^{n-q})^{-1} \iota_{X(x_s)\,\mathrm{d}B_s} W_s^{n-q+1}(*-) \right) \\
&= (-1)^q \frac{1}{t} (-1)^{q(n-q)} \mathbb{E}(\phi) \left(W_t^q \int_0^t (W_s^q)^{-1} \left[(-1)^{n-q} X(x_s)\, \mathrm{d}B_s \wedge W_s^{q-1}(**-) \right] \right) \\
&= -\frac{1}{t} \mathbb{E}\,\phi \left(W_t^q \int_0^t (W_s^q)^{-1} (X(x_s)\, \mathrm{d}B_s \wedge W_s^{q-1}(-)) \right)
\end{aligned} \tag{9}$$

□

giving (12).

3. *Formulae for* $\Delta P_t\phi$. – The method used in [7] to obtain higher order derivatives fo heat semigroups on functions can be applied to the present situation, with neater results. First, note that for a $(q+1)$-form ψ equation (6) at time $t/2$ is equivalent to

$$\mathrm{d}P_{t/2}\psi(V) = \frac{2}{t}\,\mathbb{E}\psi_{\xi_t^{t/2}(x_0)}\left(W_t^{\frac{t}{2}}\int_{t/2}^{t}(W_s^{t/2})^{-1}\iota_{X(\xi_s^{t/2}(x_0))\,\mathrm{d}B_s}W_s^{t/2}(V)\right), \tag{13}$$

where $\xi_s^{t/2}$ and $W_s^{t/2}$, $t/2 \le s < \infty$, refer to the respective flows starting at time $t/2$ and evaluated at time s. Since $\mathrm{d}^*\mathrm{d}P_t\phi(V) = \mathrm{d}P_{t/2}^*(\mathrm{d}P_{t/2})\,\phi(V)$ equations (13), (8), the Markov property, and the semigroup property of stochastic flows and the development map yield

$$\mathrm{d}^*\mathrm{d}P_t\phi(V) = -\frac{4}{t^2}\,\mathbb{E}\phi_{x_t}\left(W_t\int_{t/2}^{t}(W_s)^{-1}\iota_{//_s\mathrm{d}\breve{B}_s}W_s\int_0^{t/2}(W_r)^{-1}//_r\mathrm{d}\breve{B}_r\wedge W_r(V)\right).$$

Adding on the corresponding formula for $\mathrm{d}^*\mathrm{d}P_t\phi(V)$ we obtain:

COROLLARY 3.1. – *We have* $\Delta P_t\phi(V) = \frac{4}{t^2}\,\mathbb{E}\phi\left(W_t\int_{t/2}^{t}(W_s)^{-1}\mathrm{d}M_s(V)\right)$, *where*

$$\mathrm{d}M_s(V) = \iota_{//_s d\breve{B}_s}W_s\left(\int_0^{t/2}(W_r)^{-1}(//_r\mathrm{d}\breve{B}_r\wedge W_r(V))\right) + //_s d\breve{B}_s\wedge W_s\left(\int_0^{t/2}(W_r)^{-1}(\iota_{//_r\mathrm{d}\breve{B}_r}W_r(V))\right).$$

This can be iterated to give formulas for $(\Delta)^r P_t\phi$, $r = 1, 2, \ldots$. These together with the previous ones lead to formulae for the corresponding operators acting on the heat kernels; as for (7), and show that $\mathrm{d}_x p_t^q(\cdot, y)$, $\mathrm{d}_x^* p_t^q(\cdot, y)$, $(\Delta_x)^r p_t^q(\cdot, y)$ and $\mathrm{d}P_t\phi$, $\mathrm{d}^*P_t\phi$, $(\Delta)^r P_t\phi$, $r = 1, 2, \ldots$, all have uniform bounds for fixed positive t, depending only on t and bounds for the Weitzenböck curvatures $\mathcal{R}^q$, $\mathcal{R}^{q\pm 1}$ (and the uniform bound of ϕ).

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