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ON THE COMPACTNESS OF MANIFOLDS*

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It is believed that the family of Riemannian manifolds with negative curvatures is much richer than that with positive curvatures. In fact there are many results on the obstruction of furnishing a manifold with a Riemannian metric whose curvature is positive. In particular any manifold admitting a Riemannian metric whose Ricci curvature is bounded below by a positive constant must be compact. Here we investigate such obstructions in terms of certain functional inequalities which can be considered as generalized Poincaré or log-Sobolev inequalities. A result of Saloff-Coste is extended.

Keywords: Compactness; Riemannian manifolds; functional inequality; curvature.

1. Introduction

Let M be a complete connected Riemannian manifold of dimension d. A basic topic in Riemannian geometry is the non-existence of Riemannian structures of particular properties on topological manifolds. One of the often studied question is to equip a manifold with certain curvature conditions. A classical result in this direction is Myers's theorem [16] which says that a noncompact manifold does not admit Ricci curvature bounded below by a positive constant, say K. Furthermore an upper bound for the diameter D of the manifold given: $D \leq \pi\sqrt{d-1}/\sqrt{K}$. Some effort have been made to extend Myers' theorem and tounderstand the intrinsic meaning of the conditions imposed. See e.g. Bonnet [4] and Ambrose [1]. In Ambrose [1] it

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was shown that compactness follows if

$$\int_0^\infty \mathrm{Ric}(\dot\gamma(t), \dot\gamma(t))\mathrm{d}t = \infty$$

for each geodesic γ emanating from a fixed point and parameterized by arc length, allowing Ricci curvature being negative. In [10] Galloway showed, by a careful study of equations $x'' + r(t)x = 0$ of the Jacobi type being oscillatory, (1.1) can be replaced by the following

$$\int_0^\infty t^\lambda \mathrm{Ric}(\dot\gamma(t), \dot\gamma(t))\mathrm{d}t = \infty \tag{1.1}$$

for some $\lambda \in [0, 1)$, thus allowing quadratic decay of the Ricci curvature at the infinity. If furthermore the Ricci curvature is nonnegative, the manifold is compact if $\liminf t^2 \mathrm{Ric}(\dot\gamma(t), \dot\gamma(t)) > (d-1)/4$.

Another extension of Myers' result was made by Li [15] using the stochastic positivity of Ricci curvature. More precisely, let $\rho(x)$ denote the Riemannian distance between x and a fixed point p, M is compact provided

$$\kappa(x) := \inf\left\{\mathrm{Ric}(X, X) : X \in T_x M, |X| = 1\right\} \geq \frac{-d}{(d-1)\rho(x)^2}$$

for big $\rho(x)$ and

$$\sup_{x\in K} \int_0^\infty \mathbb{E}\exp\left[-\frac{1}{2}\int_0^t \kappa(x_s)\mathrm{d}s\right] < \infty \tag{1.2}$$

for any compact $K \subset M$, where x_s denotes the Brownian motion on M starting from x. Note that for compact manifolds, see [9], (1.2) is equivalent to the operator $-\Delta + \frac{1}{2}\kappa(x)$ being positive.

Compactness was also studied by Saloff-Coste [18] using the log-Sobolev inequality. He proved that a manifold M of finite volume is compact provided the Ricci curvature is bounded below and that there exists $C_0 > 0$ such that

$$\mu(f^2 \log f^2) \leq \mu(f^2)\log\mu(f^2) + C_0\mu(|\nabla f|^2), \quad f \in C_0^\infty(M), \tag{1.3}$$

where μ denotes the normalized volume measure. Estimates of D are presented by Saloff-Coste [18] and Ledoux [13] in terms of C_0, d and the lower bound of the Ricci curvature.

The compactness of Riemannian manifolds with Ricci curvature bounded from below also follows from the following condition on the heat kernel p_t:

$$\int_M \frac{1}{p_t(x,y)}\mathrm{d}y < \infty,$$

a result proved in Gong and Wang [11] and conjectured in Buler [5].

The purpose of this paper is to investigate the compactness of complete Riemannian manifolds in relations to certain functional inequalities which is in general weaker than the corresponding log-Sobolev inequalities. In some cases the Ricci curvature is allowed not to be bounded from below, see §3.

Let $L := \Delta + \nabla V$ be a C^2 function on the manifold with $Z := \int_M \mathrm{e}^V dx$ finite. Consider the normalized measure $\mu := Z^{-1}\mathrm{e}^V \mathrm{d}x$ and the following functional inequality

$$\mu(f^2) \le r\mu(|\nabla f|^2) + \beta(r)\mu(|f|)^2, \quad r > r_0, f \in C_0^\infty(M), \tag{1.4}$$

where $r_0 \ge 0$ is a constant and $\beta : (r_0, \infty) \to (0, \infty)$ is a decreasing function. This inequality was introduced in [19] and there it was shown that the essential spectrum $\sigma_{ess}(-L)$ of $-L$ satisfies $\sigma_{ess}(-L) \subset [\frac{1}{r_0}, \infty)$ if and only if (1.4) holds for some β. Note that (1.3) holds for some $C_0 > 0$ if and only if (1.4) holds for $r_0 = 0$ and $\beta(r) = \exp[c(1 + r^{-1})]$ for some $c > 0$. In fact (1.4) generalizes the concepts of Poincaré inequality, log-Sobolev inequality, Sobolev inequality, and Nash inequality.

In §2 we show the inequality (1.4) with $r_0 = 0$ together with a curvature-dimension condition implies the manifold is necessarily compact. Our proof is based on a spectrum argument. In section 3 we consider the following question: assume a functional inequality of type (1.4) holds what is the weakest possible condition on the curvature which implies the compactness of M. For example if (1.3) holds then the curvature condition

$$\liminf_{\rho(x)\to\infty} \frac{0 \vee (-\kappa(x))}{\rho(x)^2} < \frac{1}{4(d-1)^2 C_0^2}$$

implies the compactness of M. This curvature condition is much weaker than the one used in Saloff-Coste [18], namely, the Ricci curvature is bounded below.

2. A Spectrum Argument

The basic idea is the following: if $\lambda_{ess} \equiv \lambda_{ess}(-L) := \inf \sigma_{ess}(-L)$ is positive then the first Dirichlet eigenvalue on geodesic balls of certain size is shown to have small uniform upper bound which forces the manifold to be compact. Let D be an open connected open set of M. Denote by $\lambda_0(D)$ the first Drichlet eigenvalue of $L \equiv \Delta + \nabla V$ on D, i.e.

$$\lambda_0(D) \equiv \lambda_0(D, L) := \inf\{\mu(|\nabla f|^2) : \mu(f^2) = 1, f \in C_0^\infty(D)\},$$

where $C_0^\infty(D) := \{f \in C_0^\infty(M), \operatorname{supp} f \subset D\}$, and $\mu(\mathrm{d}x) = \mathrm{e}^{V(x)}\, dx$.

Let $B(x, r)$ denote the open geodesic ball around x with radius r.

Theorem 2.1. *If M is not compact then*

$$\sup_{x\in M} \lambda_0(B(x, r)) \ge \lambda_{ess}$$

for any $r > 0$ and any operator L of the form $\Delta + \nabla V$, where V is a C^2 function on M. Consequently if there is a C^2 function $V : M \to R$ and a positive number r such that

$$\lambda_0(r) := \sup_{x\in M} \lambda_0(B(x, r)) < \lambda_{ess}, \tag{2.1}$$

then M is compact.

Proof. Suppose that M is noncompact. Set $a = \frac{1}{2}(\lambda_{ess} - \lambda_0(r))$. By Donnely-Li's decomposition principle [8], $\sigma_{ess}(-L|_{D^c}) = \sigma_{ess}(-L)$ for compact sets D. Thus, $\lambda_0(D^c) \to \lambda_{ess}$ as D approaches M. If $a := \frac{1}{2}(\lambda_{ess} - \lambda_0(r)) > 0$, then there is a compact domain D such that

$$\lambda_0(D^c) \geq \lambda_{ess} - a = \frac{1}{2}\lambda_{ess} + \frac{1}{2}\lambda_0(r).$$

Now for any r we can find x such that $B(x,r) \cap D = \phi$. Thus by the domain monotonicity of the first Dirichlet eigenvalue

$$\lambda_0(r) \geq \lambda_0(B(x,r)) \geq \lambda_0(D^c) \geq \frac{1}{2}\lambda_{ess} + \frac{1}{2}\lambda_0(r),$$

which implies $a \leq 0$. □

In the following we shall use (1.4) and upper bounds of L acting on distance functions to obtain (2.1). Let ρ_x be the Riemannian distance function from x, and cut(x) the cut locus of x.

Let us first recall a comparison lemma:

Lemma 2.2. *Let γ be a positive continuous function on $(0,\infty)$ such that $L\rho_x(y) \leq \gamma(\rho_x(y))$ for any x and $y \notin \{x\} \cup cut(x)$. Define a measure ν on $[0,\infty)$ with*

$$\nu(dr) = \mathrm{e}^{\int_1^r \gamma(s)\, ds}\, dr.$$

Let Λ^γ be the principal eigenvalue of $L^\gamma := \frac{d^2}{dr^2} + \gamma \frac{d}{dr}$:

$$\Lambda^\gamma := \inf\left\{ \int_0^\infty |h'|^2(r)\, \nu(dr) :\ h \in C_0^\infty([0,\infty)),\ \nu(h^2) = 1 \right\}.$$

Then

$$\limsup_{s\uparrow\infty} \sup_{x\in M} \lambda_0(B(x,s)) \leq \Lambda^\gamma.$$

Proof. Let $\Lambda_{0,s}^\gamma$ be the first eigenvalue of $L^\gamma = \frac{d^2}{dr^2} + \gamma(r)\frac{d}{dr}$ on $[0,s]$ with Neumann boundary at 0 and Dirichlet boundary at s, and $h_s$the corresponding (positive) eigenfunction. Then h_s is decreasing since it has no critical point on $(0,s]$ as shown in Chen-Wang [7] (Proposition 6.4).

Now for any x, $h_s \circ \rho_x$ is defined on $B(x,s)$ and

$$\begin{aligned}(\Delta + V)(h_s \circ \rho_x) &= \Delta(h_s \circ \rho_x) + h_s'(\rho_x)\langle \nabla V, \nabla \rho_x\rangle \\ &= h_s''(\rho_x) + h_s'(\rho_x)\, L\rho_x \\ &\geq h_s''(\rho_x) + h_s'(\rho_x)\gamma(\rho_x) \\ &= -\Lambda_{0,s}^\gamma h_s \circ \rho_x\end{aligned}$$

outside of the cut locus of x. Since the cut locus of x has measure 0,

$$(\Delta + \nabla V)(h_s \circ \rho_x) \geq -\Lambda_{0,s}^\gamma h_s \circ \rho_x$$

on $B(x,s)$ in the sense of distribution (see e.g. Appendix in Yau [21]). Therefore $\lambda_0(B(x,s)) \le \Lambda^\gamma_{0,s}$ and

$$\lim_{s\uparrow\infty} \sup_{x\in M} \lambda_0(B(x,s)) \le \lim_{s\uparrow\infty} \Lambda^\gamma_{0,s} = \Lambda^\gamma. \qquad \square$$

Theorem 2.3. *Suppose* $\int_M \mathrm{e}^{V(x)} \mathrm{d}\, x < \infty$ *and* $L\rho_x \le \gamma \circ \rho$. *Let* Λ^γ *be the principal eigenvalue of* $\frac{d^2}{dr^2} + \gamma \frac{d}{dr}$ *on* $[0,\infty)$. *Then the inequality* (1.4) *does not hold for any* $r_0 < \frac{1}{\Lambda^\gamma}$ *unless the manifold is compact.*

Proof. Suppose that the inequality (1.4) holds for some $r_0 < \frac{1}{\Lambda^\gamma}$ then by [19]

$$\lambda_{ess}(-L) \ge \frac{1}{r_0} > \Lambda^\gamma.$$

By the eigenvalue comparison lemma $L(\rho) \le \gamma(\rho)$ implies that there exists $s > 0$ such that

$$\sup_{x\in M} \lambda_0(B(x,s)) \le \Lambda^\gamma_s < \lambda_{ess}(-L).$$

Theorem 2.1 now applies to imply the compactness of the manifold. $\square$

It is known that Ricci curvature bounded from below implies that $\Delta\rho_x \le c(1+\rho_x^{-1})$ for some constant c and any $x \in M$. In general this is true for $L = \Delta + \nabla V$ if the following curvature dimension condition holds:

$$\Gamma_2(f,f) := \frac{1}{2} L|\nabla f|^2 - \langle \nabla f, \nabla Lf\rangle \ge -K|\nabla f|^2 + \frac{1}{n}(Lf)^2, \quad f \in C^\infty(M), \quad (2.2)$$

where $K \ge 0$, $n > 1$ are constants. This inequality is equivalent to that the Ricci curvature being bounded from below by $-K$ in the case that $L = \Delta$ and $n = d$, the dimension of the manifold. It was shown in Qian [17] that (2.2) implies that $L\rho_x \le \gamma(\rho_x)$ outside of $\{x\} \cup \mathrm{cut}(x)$ where

$$\gamma(r) = \sqrt{K(n-1)}\coth[r\sqrt{K/(n-1)}]. \quad (2.3)$$

This consideration leads to the following corollary:

Corollary 2.4. *Assume* (2.2) *and* $\int_M \mathrm{e}^{V(x)} \mathrm{d}\, x < \infty$. *Then* (1.4) *cannot hold for any* $r_0 < 4/K(n-1)$ *unless the manifold is compact.*

Proof. Cheeger's inequality implies that the principal eigenvalue of $\frac{d^2}{dr^2} + \gamma\frac{d}{dr}$ is less than or equal to $K(n-1)/4$. See Chavel [6]. Theorem 2.3 now applies. $\square$

Let P_t be the semigroup associated to the heat equation $\frac{\partial}{\partial t} = L$. We relate the spectrum of $-L$ to the integral kernel $p_t(x,x)$, with respect to the measure μ, of the semigroup P_t.

Proposition 2.5. *Assume* $\int_M \mathrm{e}^{V(x)} \mathrm{d}\, x < \infty$. *If* $\int_M p_t(x,x)\mu(dx) < \infty$ *for some* $t > 0$, *then* $\lambda_{ess}(-\Delta - \nabla V) = \infty$, *or equivalently,* (1.4) *holds for some function* β

with $r_0 = 0$. Consequently $\int_M p_t(x,x)\mu(dx) < \infty$ for some $t > 0$ and the curvature dimension inequality (2.2) *together imply that the manifold is compact.*

Proof. The relation of the super Poincaré inequality (1.4) and the essential spectrum of $(-L)$ is given by Theorem 2.1 in [19]. We shall show $\lambda_{ess}(-L) = \infty$. For any f with $\mu(f^2) \le 1$, one has $(P_t f(x))^2 \le p_{2t}(x,x), t > 0, x \in M$. Therefore, if $\int p_t(x,x)\mu(\mathrm{d}x) < \infty$ then $P_{t/2}$ is $L^2(\mu)$-uniformly integrable and hence P_t is compact in $L^2(\mu)$, see e.g. Theorem 2.3 in [20]. Thus, the proof is complete since $\sigma_{ess}(L) = \emptyset$ if P_t is compact. □

Corollary 2.6. *Assume* (2.2) *and $\int_M \mathrm{e}^{V(x)}\mathrm{d}\,x < \infty$. Let $\rho := \rho_{x_0}$ for a fixed $x_0 \in M$. Then M is compact provided one of the following holds:*
(1) *$K = 0$ and $\mu(\rho^n) < \infty$.*
(2) *$K > 0$ and $\mu\Big(\rho^{n/2}\exp\Big[\frac{1}{2}\sqrt{nK}\big(\sqrt{2}+1\big)\rho\Big]\Big) < \infty$.*

Proof. By Proposition 2.5, in both cases we only need to prove that $\int_M p_t(x,x)\mu(\mathrm{d}x) < \infty$ holds for some $t > 0$. First observe, by Corollary 2 in [2] (see [3] for more details),

$$\begin{aligned} p_t(x,x)\exp\bigg[-\frac{\big(\rho_x(y)+\sqrt{nK}\,s\big)^2}{4s} - \frac{\sqrt{nK}}{2}\min\Big\{\big(\sqrt{2}-1\big)\rho_x(y), \frac{\sqrt{nK}}{2}s\Big\}\bigg] \\ \le \Big(\frac{t+s}{t}\Big)^{n/2} p_{t+s}(x,y), \quad t,s > 0, x,y \in M. \end{aligned} \tag{2.4}$$

For part (1), take $s = \rho(x)^2 + 1$ in (2.4) and integrate both sides over y with respect to μ to obtain

$$cp_t(x,x)\mu(B(x_0,1)) \le \Big(\frac{t+\rho(x)^2+1}{t}\Big)^{n/2}$$

for some $c > 0$. Thus

$$\int_M p_t(x,x)\mu(\mathrm{d}x) \le c_1(1+t^{-n/2}) < \infty$$

for some $c_1 > 0$ and all $t > 0$.

For part (2) take $s = (\rho(x)+1)/\sqrt{nK}$ in (2.4) to see

$$p_t(x,x) \le c(t)(\rho(x)+1)^{n/2}\exp\bigg[\frac{\sqrt{nK}}{2}\Big(\sqrt{2}+1\Big)\rho(x)\bigg]$$

for some $c(t) > 0$. Hence $\int_M p_t(x,x)\mu(\mathrm{d}x) < \infty$ for all $t > 0$. □

So far we conclude that (1.4) together with the curvature-dimension condition (2.2) implies the compactness of M. Below we show that (1.4) alone, with a good enough function β, also implies the compactness of the manifold.

Proposition 2.7. *Assume* $\int_M e^{V(x)}dx < \infty$. *If* (1.4) *holds for* $r_0 = 0$ *some* β *satisfying*

$$C(\delta) := \int_1^\infty \frac{1}{r^2}\log\beta\Big(\frac{1}{\delta r^2}\Big)dr < \infty \tag{2.5}$$

for some $\delta > 1$, *then* M *is compact with diameter*

$$D \leq \inf_{\delta>1}\Big\{\log\frac{\delta\mu(e^\rho)}{\delta-1} + C(\delta)\Big\}.$$

Conversely, if M *is compact then* (1.4) *holds for* $r_0 = 0$ *and* $\beta(r) = c(1 + r^{-d/2})$ *for some* $c > 0$, *hence* (2.5) *holds for all* $\delta > 1$.

Proof. The first assertion follows from Theorem 6.1 in [19], while the second assertion follows from Corollary 3.3 in [19] by the Sobolev inequality on compact manifolds. □

3. A Measure-Curvature Argument

In this section we shall assume that the essential spectrum of $-L$ is empty, i.e. $\lambda_{ess} = \infty$. Recall that according to [19] this is equivalent to the super Poincaré inequality

$$\mu(f^2) \leq r\mu(|\nabla f|^2) + \beta(r)\mu(|f|)^2, \quad r > 0, f \in C_0^\infty(M) \tag{3.1}$$

a decreasing function $\beta : (0,\infty) \to (0,\infty)$. Consider the following generalized curvature dimension inequality:

$$\Gamma_2(f,f) \geq -(n-1)(k\circ\rho)|\nabla f|^2 + \frac{1}{n}(Lf)^2, \quad f \in C_0^\infty(M), \tag{3.2}$$

where $\rho := \rho_{x_0}$ for a fixed point x_0, $n > 1$ and k is an increasing function from $(0,\infty)$ to $(0,\infty)$. When $L = \Delta$ and $n = d$ is the dimension of the manifold, (3.2) is equivalent to $\mathrm{Ric}_x \geq -(d-1)k\circ\rho(x), x \in M$. We allow k to be unbounded. Now (3.1) implies decay of $\mu(\rho > r)$ while (3.2) provides a lower bound of $\mu(\rho > r)$. The two together with appropriate choices of β and k should force the manifold to be compact.

Theorem 3.1. *Assume* $\int_M e^{V(x)}dx < \infty$. *The manifold* M *is compact if* (3.2) *holds and*

$$\limsup_{r\to\infty}\frac{-\log\mu(\rho > r)}{(n-1)r\sqrt{k(2r+3)}} > 1. \tag{3.3}$$

Proof. Assume that M is noncompact. For any $r > 0$ there exists $x_r \in M$ such that $\rho(x_r) = r + 1$. Apply (3.2) to see

$$\Gamma_2(f,f)(x) \geq -(n-1)k(2r+3)|\nabla f|^2(x) + \frac{1}{n}(Lf)^2(x), \quad f \in C^\infty(M), x \in B(x_r, r+2).$$

On the other hand, see Qian [17],

$$L\rho_{x_r} \le (n-1)\sqrt{k(2r+3)}\coth\Big[\sqrt{k(2r+3)}\,\rho_{x_r}\Big]$$

on $B(x_r, r+2)\backslash(\{x_r\}\cup\mathrm{cut}(x_r))$. This implies, by a standard argument as in Lemma 2.2 in [11], that

$$\mu(B(x_r, r+2)) \le \mu(B(x_r,1))(r+2)^n \exp\Big[(n-1)(r+1)\sqrt{k(2r+3)}\Big].$$

Consequently

$$\mu(\rho > r) \ge \mu(B(x_r,1)) \ge \mu(B(x_0,1))(r+2)^{-n}\exp\Big[-(n-1)(r+1)\sqrt{k(2r+3)}\Big]$$

contradicting with (3.3). □

Corollary 3.2. *Assume* (3.1), (3.2) *and* $\int_M e^{V(x)}\mathrm{d}\,x < \infty$. *Then M is compact if* (3.3) *holds with $\mu(\rho > r)$ replaced by $p_c(r)$ for any $c > 0$ defined below:*

$$p_c(r) := \inf_{\lambda,\delta>1} \exp\Big\{(c-r)\lambda + \lambda\int_1^\lambda \frac{1}{s^2}\log\Big[\frac{\delta}{\delta-1}\beta\Big(\frac{1}{\delta s^2}\Big)\Big]\mathrm{d}s\Big\}.$$

Proof. By Theorem 6.1 in [19] (3.1) implies that $\mu(e^\rho) < \infty$ and

$$\mu(\exp[\lambda\rho]) \le \mu(e^\rho)^\lambda \exp\Big[\lambda\int_1^\lambda \frac{1}{r^2}\log\Big[\frac{\delta}{\mathrm{d}d-1}\Big(\frac{1}{(\delta r^2)}\Big)\Big]\mathrm{d}r\Big\}.$$

Therefore, $\mu(\rho > r) \le p_c(r)$ for $c := \log\mu(e^\rho)$. The proof is complete by Theorem 3.1. □

Corollary 3.3. *Assume* $\int_M e^{V(x)}\mathrm{d}\,x < \infty$ *and the super Poincaré inequality* (3.1) *holds for the function $\beta(r) = c_1\exp[c_2 r^{-\alpha}]$, where $c_1, c_2, \alpha > 0$ are constants, and* (3.2) *holds. Then the manifold is compact in each of the following situations*:

(1) $\alpha < 1/2$.

(2) $\alpha = 1/2$ *and* $\limsup_{r\to\infty}\dfrac{r}{\log k(r)} > 2c_2$.

(3) $\alpha > 1/2$ *and* $\limsup_{r\to\infty}\dfrac{r^{2/(2\alpha-1)}}{k(r)} > (n-1)^2\Big(\dfrac{2\alpha}{2\alpha-1}\Big)^{4\alpha/(2\alpha-1)}(2c_2)^{2/(2\alpha-1)}$.

Proof. Part (1) is covered by Proposition 2.7. For part (2) and (3) we only need to verify that (3.3) holds for $p_c(r)$ defined in the previous corollary. Note that for any $\sigma_1 > \sigma_2 > 1$, there exists $c_3 > 0$ such that for all $\lambda \ge 1$,

$$\lambda\int_1^\lambda \frac{1}{r^2}\log\Big[\frac{\sigma_2}{\sigma_2-1}\beta\Big(\frac{1}{(\sigma_2 r^2)}\Big)\Big]\mathrm{d}r \le \begin{cases} c_3 + \frac{c_2\sigma_1^\alpha}{2\alpha-1}\lambda^{2\alpha}, & \text{if } \alpha > 1/2,\\ c_3 + c_2\sqrt{\sigma_1}\lambda\log\lambda, & \text{if } \alpha = 1/2.\end{cases}$$

Then, for any $c > 0$ and any $\sigma > 1$,

$$\log p_c(r) \le \begin{cases} -r^{2\alpha/(2\alpha-1)}\Big(\frac{2\alpha-1}{2\alpha}\Big)^{2\alpha/(2\alpha-1)}(c_2\sigma)^{-1/(2\alpha-1)}, & \text{if } \alpha > \frac{1}{2}, r \gg 1,\\ -\exp[r/(c_2\sigma)], & \text{if } \alpha = \frac{1}{2}, r \gg 1.\end{cases}$$

Thus each of (2) and (3) implies (3.3) for $p_c(r)$ in place of $\mu(\rho > r)$. The result now follows from Corollary 3.2. □

It is known from [19] that (3.1) is equivalent to an F-Sobolev inequality (see [19] for details). In particular we consider the following generalized log-Sobolev inequality

$$\mu(f^2[\log(f^2+1)]^\delta) \le C_1\mu(|\nabla f|^2) + C_2, \quad f \in C_0^\infty(M), \mu(f^2) = 1, \tag{3.4}$$

where $\delta, C_1, C_2 > 0$ are constants. This leads to the next corollary. When $\delta \neq 1$, we will reduce the inequality to (3.1) to apply Corollary 3.3. But when $\delta = 1$ we will use a Herbst's argument to obtain estimates of $\mu(\rho > r)$ directly from (3.4). Certainly in the latter case the first method also applies, but the resulting condition (3) is worse than (4) below.

Corollary 3.4. *Assume* (3.2), (3.4) *and* $\int_M e^{V(x)} \mathrm{d}\, x < \infty$. *$M$ is compact provided at least one of the following holds.*

(1) $\delta > 2$.

(2) $\delta = 2$ *and* $\limsup_{r\to\infty} \dfrac{r}{\log k(r)} > 2\sqrt{C_1}$.

(3) $\delta < 2$ *and* $\limsup_{r\to\infty} \dfrac{r^{2\delta/(2-\delta)}}{k(r)} > \dfrac{(n-1)^2 C_1^{2/(2-\delta)} 4^{(2+\delta)/(2-\delta)}}{(2-\delta)^{4/(2-\delta)}}$.

(4) $\delta = 1$ *and* $\limsup_{r\to\infty} \dfrac{r^2}{k(r)} > 4(n-1)^2 C_1^2$.

Proof. We shall apply Corollary 3.3 by converting (3.4) to (3.1). Letting $F(t) := [\log(t+1)]^\delta$, we have

$$F^{-1}(t) = \exp[t^{1/\delta}] - 1 \le \exp[t^{1/\delta}], \quad t > 0.$$

By (3.4) and the proof of Theorem 3.1 in [19], we obtain

$$(t - C_2)\mu(f^2) \le t\sqrt{\exp[t^{1/\delta}]\mu(f^2)} + C_1\mu(|\nabla f|^2)$$

for all $t > 0$ and all $f \in C_0^\infty(M)$ with $\mu(|f|) = 1$. This implies that

$$\mu(f^2) \le \frac{C_1}{(1-\varepsilon)t - C_2}\mu(|\nabla f|^2) + \frac{t\exp[t^{1/\delta}]}{4\varepsilon}.$$

Taking $t = (C_1 r^{-1} + C_2)/(1-\varepsilon)$, we obtain (3.1) for

$$\beta(r) = \frac{C_1 r^{-1} + C_2}{4\varepsilon(1-\varepsilon)} \exp\left[\left(\frac{C_1 r^{-1} + C_2}{1-\varepsilon}\right)^{1/\delta}\right], \quad r > 0,$$

for any $\varepsilon \in (0,1)$. The required result now follows, for $\delta \neq 1$ from Corollary 3.3. If $\delta = 1$ then (3.4) implies

$$\mu(f^2 \log f^2) \le C_1\mu(|\nabla f|^2) + C_2, \quad f \in C_0^\infty(M),\ \mu(f^2) = 1, \tag{3.5}$$

for some $C_1, C_2 > 0$. By an argument due to Herbst (cf. p. 148 in [14]), (3.5) implies $\mu(\rho > r) \le c\exp[-r^2/C_1]$ for some constant $c > 0$. Part (4) now follows from Theorem 3.1. □

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