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A concrete estimate for the weak Poincaré inequality on loop space

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Abstract The aim of the paper is to study the pinned Wiener measure on the loop space over a simply connected compact Riemannian manifold together with a Hilbert space structure and the Ornstein–Uhlenbeck operator d^*d. We give a concrete estimate for the weak Poincaré inequality, assuming positivity of the Ricci curvature of the underlying manifold. The order of the rate function is $s^{-\alpha}$ for any $\alpha > 0$.

Keywords Brownian bridge measure · Loop space · Orstein–Uhlenbeck operator · Weak Poincaré inequality

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1 Introduction

A.

Let M be a smooth connected compact complete Riemannian manifold (through this paper, we all assume M to be a smooth connected complete Riemannian manifold, so

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we do not state these conditions in the remaining part). For $a, b \in M$, we consider the pinned path space $\Omega_{a,b}$ over M,

$$\Omega_{a,b} = \{\omega \in C([0, 1], M); \omega(0) = a, \omega(1) = b\},$$

which is a smooth Finsler manifold with compatible distance function

$$d_\infty(\omega, \gamma) := \sup_{s\in[0,1]} d(\omega(s), \gamma(s)).$$

When $a = b$, we have the loop space over M, based at a. Let $\mathcal{F}C_b^\infty(\Omega_{a,b})$ be the collection of smooth cylinder function on $\Omega_{a,b}$. Each $F \in \mathcal{F}C_b^\infty(\Omega_{a,b})$ is determined by a smooth function f on M^n and a partition $0 < s_1 < \cdots < s_n < 1$ of $[0, 1]$:

$$F(\omega) = (\mathrm{e}v_{s_1,\ldots,s_n})_* f = f(\omega(s_1), \ldots, \omega(s_n)). \tag{1.1}$$

For each $T > 0$, endow $\Omega_{a,b}$ with the pinned Wiener measure $\mathbf{P}^T_{a,b}$ (also called the Brownian bridge measure), which is derived by pushing forward the standard Brownian bridge measure on the space of the pinned curves of $C([0, T]; M)$ with starting point a and ending point b, to $C([0, 1]; M)$ through the rescaling map $\omega(t) \mapsto \omega(\frac{t}{T})$. The measure $\mathbf{P}^T_{a,b}$ can be equally defined through its integration over smooth cylindrical functions of type (1.1):

$$\int F(\omega)\mathbf{P}^T_{a,b}(d\omega) = \frac{1}{p_T(a, b)} \int_{M^n} f(x_1, x_2, \ldots x_n)$$
$$\cdot p_{s_1 T}(a, x_1) p_{(s_2-s_1)T}(x_1, x_2) \ldots p_{(1-s_n)T}(x_n, b) \prod_{i=1}^n \mathrm{d}x_i.$$

where $p_t(x, y)$ is the heat kernel on M. Write $\mathbf{P}_{a,b}$ for $\mathbf{P}^1_{a,b}$ for simplicity, and the corresponding expectation is denoted by $\mathbf{E}_{a,b}$.

B.

Let $\omega(s)$ be the canonical process on $\Omega_{a,b}$, $\mathcal{F}_s$ be the natural filtration and $\mathcal{F} = \mathcal{F}_1$. Then $\omega(s)$ is a semi-martingale with $(\Omega_{a,b}, \mathcal{F}, \mathcal{F}_s, \mathbf{P}^T_{a,b})$, see [9]. Denote by $/\!/_{s,t}(\omega)$: $T_{\omega(s)}M \to T_{\omega(t)}M$ the stochastic parallel translation along the continuous path $\omega(\cdot)$, which is $\mathbf{P}^T_{a,b}$ a.s. defined. Write $/\!/_s = /\!/_{0,s}$. Let $\mathbf{H}$ be the space of finite energy curves in $\mathbb{R}^n$,

$$\mathbf{H} = \left\{ h : [0, 1] \to \mathbb{R}^n \;\middle|\; \int_0^1 |\dot{h}(s)|^2 ds < \infty,\ h \text{ is absolutely continuous} \right\}.$$

We identify $T_a M$ with $\mathbb{R}^n$ and define the Bismut's tangent space $\mathbf{H}^0_\omega$ in $\Omega_{a,b}$:

$$\mathbf{H}^0_\omega = \left\{ /\!/_{\cdot}(\omega) h_{\cdot} \,\middle|\, h \in H, h_0 = 0, h_1 = 0 \right\},$$

which is a Hilbert space with the inner product:

$$(X, Y)_{\mathbf{H}^0_\omega} = \int_0^1 \left\langle \frac{d}{dt}(/\!/_{t,0}(\omega) X_t)), \frac{d}{dt}(/\!/_{t,0}(\omega) Y_t)) \right\rangle_{T_a M} \mathrm{d}t$$

and corresponding norm $| - |_{\mathbf{H}^0_\omega}$.

Consider the differential operator d which sends a differentiable function on $\Omega_{a,b}$ (viewed as a Finsler manifold) to a differential 1-form. For F a smooth cylindrical function, dF can be considered as a bounded linear map on Bismut tangent space. By the Riesz representation theorem, there is the H^1 gradient $\mathbf{D}_0 F(\omega) \in \mathbf{H}^0_\omega$, given by

$$(\mathbf{D}_0 F(\omega), X_h)_{\mathbf{H}^0_\omega} = dF(X_h),$$

for all vectors X_h of the form $X_h(s) = /\!/_s(\omega) h_s$ in $\mathbf{H}^0_\omega$. In particular, for cylindrical function $F = (ev_{s_1,\ldots,s_n})_* f$ of the form (1.1),

$$\mathbf{D}_0 F(\omega)(t) = \sum_{i=1}^n /\!/_{s_i,t}(\omega) \nabla_i f(\omega(s_1), \omega(s_2), \ldots \omega(s_n)) \cdot G_0(s_i, t)$$

where $\nabla_i f$ is the gradient of f for the ith variable and $G_0(s,t) = s \wedge t - s \cdot t$, $0 \leqslant s, t \leqslant 1$, is the Green's function of the operator $-\frac{d^2}{ds^2}$ with Dirichlet boundary conditions on (0, 1). Also we have,

$$|\mathbf{D}_0 F|^2_{\mathbf{H}^0_\omega} = \sum_{i,j=1}^n G_0(s_i, s_j) \cdot \langle /\!/_{s_i,s_j}(\omega) \nabla_i f(\omega(s_1), \ldots \omega(s_n)), \nabla_j f(\omega(s_1), \ldots \omega(s_n)) \rangle_{\omega(s_j)} \tag{1.2}$$

For each $T > 0$, the quadratic form defined on smooth cylinder function by

$$\widetilde{\mathscr{E}}^T_{a,b}(F, F) := \int_{\Omega_{a,b}} |\mathbf{D}_0 F|^2_{\mathbf{H}^0_\omega} \mathbf{P}^T_{a,b}(d\omega),$$

can be extended to a Dirichlet form $\mathscr{E}^T_{a,b}$, which is due to an integration by parts formula, see [9]. The domain of the Dirichlet form $\mathscr{D}(\mathscr{E}^T_{a,b})$ is the the same as the closure of the gradient operator $\mathbf{D}_0$. Follow the custom, we call this Dirichlet form the O–U Dirichlet form. And we denote $\mathscr{E}_{a,b}$ for $\mathscr{E}^1_{a,b}$ for simplicity.

If μ is a probability measure, we denote by $\mathbf{E}[F;\mu]$ the average of a function $F \in L^2(\mu)$ with respect to this measure and $\mathbf{Var}(F;\mu) = \mathbf{E}(F^2;\mu) - [\mathbf{E}(F;\mu)]^2$ the corresponding variance. The main theorem of the paper is:

Theorem 5.1 *Let M be a simply connected compact manifold with strict positive Ricci curvature. For any small $\alpha > 0$, there exists a constant $s_0 > 0$ such that the following weak Poincaré inequality holds, i.e.*

$$\mathbf{Var}(F;\mathbf{P}_{a,a}) \leqslant \frac{1}{s^\alpha}\mathscr{E}_{a,a}(F,F) + s||F||_\infty^2, \quad s \in (0, s_0), \quad F \in \mathscr{D}(\mathscr{E}_{a,a}).$$

And the constant s_0 does not depend on the starting point $a \in M$.

C. Historical remark

The Ornstein–Uhlenbeck operator and the Ornstein–Uhlenbeck process play an important role in the development of the L^2 theory on path and loop spaces, cf. [13]. The study of the functional inequalities for O–U Dirichlet form with respect to the Wiener measure (on path space) and to the pinned Wiener measure (on loop space) goes back a long way. For the Wiener measure on path space over a compact manifold, it turns out that there is no fundamental topological or geometrical obstruction to the validity of the Poincaré inequality. See e.g. the work of Fang [15] for the existence of a Poincaré inequality for O–U Dirichlet form and that of Aida and Elworthy [6], Hsu [19], Capitaine et al. [7] for the existence of a logarithmic Sobolev inequality. There is also the approach of gradient stochastic different equations, e.g. Elworthy and Li [14], Elworthy et al. [13].

But for the loop space over a compact manifold M, the problem seems much more complicated. We limit ourselves here to the case of functional inequalities with respect to the pinned Wiener measure. Gross [17] pointed out that Logarithmic Sobolev inequality does not hold when $M = S^1$ and he proved instead a Logarithmic Sobolev inequality plus a potential term when M was a compact Lie group. In general the geometry and the topology of the manifold will play a significant role. In particular, a Poincaré inequality does not hold for the O–U Dirichlet form if the underlying manifold is not simply connected, since the indicator function of each connected component of the loop space is on the domain of the O–U Dirichlet form, see Aida [3]. Furthermore, in [10], Eberle constructed a simply connected compact Riemannian manifold on the loop space over which the Poincaré inequality for O–U Dirichlet form did not hold. As transpired in his proof, the validity of the Poincaré inequality may depend on the starting point of the based loop space. A Clark–Ocone formula with a potential was deduced by Gong and Ma [16], which led to their discovery of a Logarithmic Sobolev inequality with a potential on loop space over general compact manifold. See also Aida [1]. In their results, the simply connected condition is not needed for the underlying manifold. Aida [4], on the other hand, deduced a Clark–Ocone formula which led to a Logarithmic Sobolev inequality for a modified Dirichlet form, under suitable conditions on the small time asymptotics of the Hessian of the logarithm of the heat kernel of the underlying manifold. Built on that, a Poincaré inequality is shown to hold for the O–U Dirichlet form on the loop space over hyperbolic space, see Chen et al. [8].

Another development in the positive direction comes from Eberle [11], where it was shown that a local Poincaré ineqaulity hold for the O–U Dirichlet form on loop space over compact manifold. A parallel result was given by Aida [2]: when M was simply connected, the O–U Dirichlet form had the weak spectral gap property. By the weak spectral gap property for a Dirichlet form $\mathscr{E}$ in $L^2(\mathbf{P})$ it is meant that $F_n \to 0$ in probability for any sequence of functions $\{F_n\}_{n=1}^{\infty} \subset \mathscr{D}(\mathscr{E})$ satisfying the following conditions,

$$\sup_n ||F_n||_{L^2} \leqslant 1, \quad E(F_n) = 0, \quad \lim_{n\to\infty} \mathscr{E}(F_n, F_n) = 0,$$

see also Kusuoka [20]. Although we do not know the relation between Eberle's local Poincaré ineqaulity and Aida's weak spectral gap property, it was noted in Röckner and Wang in [22], the weak spectral gap property was equivalent to the following weak Poincaré inequality:

$$\mathbf{Var}_{\mu}(f) \leqslant \beta(s)\mathscr{E}(f, f) + s||f||_{\infty}^2, \quad s \in (0, s_0) \quad f \in \mathscr{D}(\mathscr{E}) \cap L^{\infty}(\mu)$$

Here $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing function and $s_0 > 0$ is a constant. And in [5], Aida used such weak Poincaré inequality to give an estimate on the spectral gap of a Schrödinger operator on the loop space. We refer the reader to Wang [24] for analysis, development and historical references on such inequalities. Our contribution here is the concrete estimate of $\beta(s)$ in the inequality above. Here we need to find suitable exhausting local sets replacing the role played by geodesic balls in the proof of weak Poincaré inequality on finite dimensional manifolds (see [24]). We use a slightly different collection of local sets from that used by Eberle in [11] for technical reasons. The main difficulty here is to get a suitable estimate of the constants for the local Poincaré inequalities on such local sets as in the proof for finite dimensional case. In fact, we do not derive a local Poincaré inequality, some additional term of the L^{∞} norm will appear in the estimate, but finally we can control such terms to get a global weak Poincaré inequality.

The paper is organised as follows. In Sect. 2, we introduce notation and state some results, especially that of Eberle [11,12] on which our proof is based on. In Sect. 3, we give some variance estimate for small time by using the estimates and ideas in [11]. In Sect. 4, A weak Poincaré type inequality for the distribution of the Brownian bridge evaluated at N equal time intervals is given. We use a combination of small time asymptotics and Poincaré inequality for the Wiener measure to control the growth of the constants with N. In particular, some of the methods in this section are inspired by [12] and [18]. In Sect. 5, the main theorem is proved by reducing the variance of a function on the loop space to the variance of a function on a product manifold which is localized to subsets which are chains of small geodesic balls, and the variance of functions on some sub-path with respect to pinned Wiener measure with small time parameter.

2 Notations and known results

Let $\{B_s\}$ be the T_aM valued stochastic anti-development of the canonical process $\omega(s)$, which is a semi-martingale with $(\Omega_{a,b}, \mathcal{F}, \mathcal{F}_s, \mathbf{P}^T_{a,b})$. It is however not a Brownian motion, see [9]. Denote by $L(\mathbb{R}^n; T_aM)$ the set of all linear maps from $\mathbb{R}^n$ to T_aM.

Lemma 2.1 [11] *Let $\{A_s(\omega), \omega \in \Omega_{a,b}, 0 \leqslant s \leqslant 1\}$ be a $L(\mathbb{R}^n; T_aM)$ valued adapted process such that $s \to A_s(\omega)$ is C^1 for every ω and*

$$\sup_{\omega\in\Omega_{a,b}} \sup_{s\in[0,1]} |A'_s(\omega)| < \infty.$$

Suppose that $H_s(\omega) = A_s(\omega)h_s$ for some $h \in \mathbf{H}$ with $h_1 = 0$ and $X_\cdot(\omega) = /\!/_\cdot(\omega)A_\cdot(\omega)h_\cdot$. Define

$$\delta^T_u X := \int_0^u \left[T^{-1}H'_s + \frac{1}{2}/\!/_s^{-1}\mathrm{Ric}^\#_{\omega(s)}(/\!/_s H_s)\right] dB_s, \quad 0 \leqslant u < 1$$

Then

$$\delta^T X := \lim_{u\to 1} \delta^T_u X$$

exists in $L^1(\Omega_{a,b}; \mathbf{P}^T_{a,b})$ and the limit is in $L^2(\Omega_{a,b}; \mathbf{P}^T_{a,b})$.

If $a, b \in M$ are not in the cut locus of each other, we take $A_\cdot$ such that $/\!/_\cdot A_\cdot$ is the damped stochastic parallel transport and take $H_\cdot$ to be parallel push back of the Jacobi fields along the unique geodesic connecting a and b with initial vector v. By a result of Malliavin–Stroock, the variance of $\delta^T X$ defined in above lemma with respect to $P^T_{a,b}$ are uniformly bounded for a, b, v, T in compact sets. In fact, we have the following lemma,

Lemma 2.2 [11,21] *Let $b \in M \setminus Cut(a)$, $v \in T_aM$, and $T > 0$. There is a vector $X^{T,a,b,v}_s = /\!/_s H_s$, with initial value $X^{T,a,b,v}_0 = v$ and H as in Lemma 2.1, such that for every $T_0 > 0$ and $r \in (0, inj_M)$,*

$$\sup_{T\in(0,T_0]} \rho(T, r) < \infty,$$

where

$$\rho(T, r) := \sup_{a,b\in M, d(a,b)\leqslant r,\ v\in T_aM, |v|=1} \left\{\mathbf{Var}\left(\delta^T X^{T,a,b,v}; \mathbf{P}^T_{a,b}\right)\right\}. \tag{2.3}$$

The next lemma deals with the derivative with the starting point of the expectation under pinned Wiener measure,

Lemma 2.3 [11] *Let* $v \in T_a M$*. For each* $X_s(\omega) = /\!/_s(\omega) H_s(\omega)$ *with* $H_s(\omega)$ *as in Lemma* 2.1*, and that* $X_0 = v$*,* $\mathbf{P}^T_{a,b}$ *a.s,*

$$d_a \left(\mathbf{E}^T_{\cdot,b}[F]\right)[v] = \mathbf{E}^T_{a,b}[dF(X)] - \mathbf{Cov}\left(\delta^T X, F; \mathbf{P}^T_{a,b}\right) \tag{2.4}$$

for each smooth cylinder function $F \in \mathcal{F}C_b^\infty(\Omega_{a,b})$*.*

X_s is a vector field on pinned path space if and only if $v = 0$. By the proof of Theorem 3.2 in [11], $\mathbf{E}^T_{a,b}[\delta^T X] = 0$ when $v = 0$, and in this case it is the integration by parts formula on pinned path space:

$$\mathbf{E}^T_{a,b}[dF(X)] = \mathbf{E}^T_{a,b}\left[F\delta^T X\right]$$

For two paths ω_1, ω_2 with $\omega_1(1) = \omega_2(0)$, define $\omega_1 \vee \omega_2$ as following:

$$\omega_1 \vee \omega_2(s) = \begin{cases} \omega_1(2s) & \text{if } s \in [0, 1/2], \\ \omega_2(2s-1) & \text{if } s \in [1/2, 1]. \end{cases}$$

For each ω in $\Omega_{a,b}$, we can find one and only one pair of $\widetilde{\omega_1}$, $\widetilde{\omega_2}$ to satisfy that $\omega = \widetilde{\omega_1} \vee \widetilde{\omega_2}$. For each fixed $T > 0$, $\omega \in \Omega_{a,b}$ with $a, b \notin \mathrm{Cut}(\omega(1/2))$ and $\omega = \widetilde{\omega_1} \vee \widetilde{\omega_2}$, $v \in T_{\omega(1/2)}M$, let

$$\widehat{X}^{T,v}_s(\omega) = \begin{cases} X^{T/2,\omega(1/2),\omega(0),v}_{1-2s}(\widetilde{\omega_1}^{-1}) & \text{if } s \in [0, 1/2], \\ X^{T/2,\omega(1/2),\omega(1),v}_{2s-1}(\widetilde{\omega_2}) & \text{if } s \in [1/2, 1] \end{cases}$$

where $X^{T,a,b,v}_s$ is as in Lemma 2.2 and $\widetilde{\omega_1}^{-1}(s) := \widetilde{\omega_1}(1-s)$, $0 \leqslant s \leqslant 1$, is the time reverse of the path $\widetilde{\omega_1}$.

For $F \in \mathcal{F}C_b^\infty(\Omega_{a,b})$ and $\omega \in \Omega_{a,b}$, let

$$\mathbf{\Gamma}^T(F)(\omega) = \begin{cases} \sup_{\{v \in T_{\omega(1/2)}M, |v|=1\}} (dF(\widehat{X}^{T,v}_s))^2(\omega), & \text{if } a, b \notin \mathrm{Cut}(\omega(1/2)) \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

For each smooth cylinder function $F \in \mathcal{F}C_b^\infty(\Omega_{a,b})$, there exists a unique function $\widetilde{F}(\widetilde{\omega_1}, \widetilde{\omega_2})$, defined on $\bigcup_{z \in M} \Omega_{a,z} \times \Omega_{z,b}$, such that $\widetilde{F}(\widetilde{\omega_1}, \widetilde{\omega_2}) = F(\widetilde{\omega_1} \vee \widetilde{\omega_2}) = F(\omega)$ for each ω in $\Omega_{a,b}$ with $\omega = \widetilde{\omega_1} \vee \widetilde{\omega_2}$.

Lemma 2.4 [11] *For* $a, b \in M$*,* $T > 0$*, and* $r > 0$*, denote* $U^r_{a,b} = B_r(a) \cap B_r(b)$ *(*$B_r(a)$ *means the ball with center* a *and radius* r*) and*

$$\mu^T_{a,b}(dx) = \frac{p_{T/2}(a,x)p_{T/2}(x,b)}{p_T(a,b)} dx. \tag{2.6}$$

- *There exists a positive number R_1, such that when $r \in (0, R_1)$,*

$$\begin{aligned}\mathbf{Var}(F; \mathbf{P}^T_{a,b}) \leqslant & 2q(T,r)\mathbf{E}^T_{a,b}[\mathbf{\Gamma}^T(F)] \\ & + (1+4q(T,r)\rho(T/2,r)) \int\limits_{U^r_{a,b}} \left\{ \mathbf{E}^{T/2,1}_{a,x}\left[\mathbf{Var}_2\left(\widetilde{F}(\widetilde{\omega_1}, \widetilde{\omega_2}); \mathbf{P}^{T/2}_{x,b}\right)\right] \right. \\ & \left. + \mathbf{E}^{T/2,2}_{x,b}\left[\mathbf{Var}_1(\widetilde{F}(\widetilde{\omega_1}, \widetilde{\omega_2}); \mathbf{P}^{T/2}_{a,x})\right]\right\} \mu^T_{a,b}(dx) \end{aligned} \tag{2.7}$$

holds for every smooth cylinder function $F : \Omega_{a,b} \mapsto \mathbb{R}$ such that $F(\omega) = 0$ if $\omega(1/2)$ is not in $U^r_{a,b}$. Here $\mathbf{\Gamma}^T(F)$, $\rho(T/2, r)$ are defined by (2.5) and (2.3) respectively. $\mathbf{E}^{T/2,i}$, $\mathbf{Var}_i$ indicates that the corresponding expectation or variance is taken with respect to the ith-subpath $\widetilde{\omega}_i$, $i = 1, 2$,

- *The constant $q(T, r)$ in above inequality does not depend on $a, b \in M$ and satisfies*

$$\overline{\lim_{T\downarrow 0}}\, T^{-1}q(T,r) \leqslant \frac{1+Kr^2}{4} \quad \forall r \in (0, R_1). \tag{2.8}$$

for some $K > 0$.

3 Estimates on the variance with small time parameter

The following lemma gives a short time asymptotics of the variance. It is crucial for the proof of the main result in this section. For simplicity, in the remaining part of the paper, the constants, e.g. C, T, and N, will change according to different situation but we will clarify which parameter such C depends on. At first, we state a lemma deriving from Lemma 2.4 by a cut-off procedure,

Lemma 3.1 *There exists a number $R_1 > 0$ such that for all $a, b \in M$ with $d(a, b) < r < R_1$, if $T < T_1(\eta, r)$ for a positive number $T_1(\eta, r)$, the following inequality holds for any number $0 < \eta < 1$ and smooth cylindrical function F on $(\Omega_{a,b}, P^T_{a,b})$,*

$$\begin{aligned} & \mathbf{Var}(F; \mathbf{P}^T_{a,b}) \\ & \leqslant 4q(T,r)\mathbf{E}^T_{a,b}\left[\mathbf{\Gamma}^T(F)\mathbf{1}_{\{\omega(1/2)\in U^r_{a,b}\}}\right] \\ & \quad + (1+4q(T,r)\rho(T/2,r)) \int\limits_{U^r_{a,b}} \left\{ \mathbf{E}^{T/2,1}_{a,x}\left[\mathbf{Var}_2\left(\widetilde{F}(\widetilde{\omega_1}, \widetilde{\omega_2}); \mathbf{P}^{T/2}_{x,b}\right)\right] \right. \\ & \quad \left. + \mathbf{E}^{T/2,2}_{x,b}\left[\mathbf{Var}_1\left(\widetilde{F}(\widetilde{\omega_1}, \widetilde{\omega_2}); \mathbf{P}^{T/2}_{a,x}\right)\right]\right\} \mu^T_{a,b}(dx) \\ & \quad + \left(6 + \frac{128q(T,r)}{\eta^2 r^2}\right) e^{-\frac{(1-4\eta)r^2}{2T}} ||F||^2_\infty. \end{aligned} \tag{3.9}$$

Here the measure $\mu^T_{a,b}(dx)$ is the distribution of the mid-point of the Brownian Bridge, given by (2.6), $U^r_{a,b} := B_r(a) \cap B_r(b)$, $q(T, r)$ is some constant satisfying (2.8), and

$\mathbf{E}^{T/2,i}$, $\mathbf{Var}_i$ *indicates that the expectation or variance is taken with respect to the subpath* $\widetilde{\omega}_i$.

Remark The constants R_1 and $q(T,r)$ are the same as that in Lemma 2.4, and R_1 is smaller than the injectivity radius of M.

Proof Step 1 For a positive r smaller than the injectivity radius of M, $a, b \in M$ with $d(a,b) < r$, define a function $\Psi_{a,b} : \Omega_{a,b} \to \mathbb{R}$ by

$$\Psi_{a,b}(\omega) := \varphi(d(a,\omega(1/2))) \cdot \varphi(d(b,\omega(1/2))). \tag{3.10}$$

Here the smooth function $\varphi : \mathbb{R}^+ \to \mathbb{R}$ satisfies the following conditions,

$$\varphi(s) = \begin{cases} 1, & \text{if } s \leqslant (1-\eta)r, \\ 0, & \text{if } s \geqslant r \end{cases} \quad \text{and} \quad |\varphi'| \leqslant \frac{2}{\eta r}. \tag{3.11}$$

Then the function $\Psi_{a,b}$ is on $\mathscr{D}(\mathscr{E}^T_{a,b})$, the domain of the O–U Dirichlet form, and $|\mathbf{D}_0\Psi_{a,b}(\omega)|_{\mathbf{H}^0_\omega} \leqslant \frac{4}{\eta r}$. Furthermore we show below that for all small $\eta > 0$ there is constant $T_1(\eta, r)$ such that if $T < T_1(\eta, r)$,

$$\mathbf{P}^T_{a,b}(\Psi_{a,b} \neq 1) \leqslant 2\mathrm{e}^{-\frac{(1-4\eta)r^2}{2T}}. \tag{3.12}$$

We begin with estimating the probability

$$\mathbf{P}^T_{a,b}\left(d(a,\omega(1/2)) > (1-\eta)r\right) = \mu^T_{a,b}\left((B_{(1-\eta)r}(a))^c\right).$$

By Varadhan's estimate [23],

$$\lim_{T\downarrow 0} T \log p_T(a,b) = -\frac{d^2(a,b)}{2} \quad \text{uniformly on } M \times M.$$

Hence for any $0 < \eta < 1$, there exists a constant $T_1(\eta, r) > 0$, such that for every $0 < T < T_1(\eta, r)$,

$$-\frac{d^2(a,b)}{2T} - \frac{\eta^2 r^2}{4T} \leqslant \log p_T(a,b) \leqslant -\frac{d^2(a,b)}{2T} + \frac{\eta^2 r^2}{4T}. \tag{3.13}$$

In the calculations that follows we assume that $0<T<T_1(\eta,r)$. Note that $d(a,b)<r$,

$$\begin{aligned}
&\mathbf{P}^T_{a,b}\left(d(a,\omega(1/2))>(1-\eta)r\right)\\
&=\frac{1}{p_T(a,b)}\int\limits_{\{d(a,x)>(1-\eta)r\}} p_{T/2}(a,x)p_{T/2}(x,b)dx\\
&\leqslant \mathrm{e}^{\frac{d^2(a,b)}{2T}+\frac{\eta^2r^2}{4T}}\int\limits_{\{d(a,x)>(1-\eta)r\}} \mathrm{e}^{-\frac{d^2(a,x)}{T}+\frac{\eta^2r^2}{2T}}p_{T/2}(x,b)dx\\
&\leqslant \mathrm{e}^{\frac{r^2}{2T}+\frac{\eta^2r^2}{4T}}\mathrm{e}^{-\frac{(1-\eta)^2r^2}{T}+\frac{\eta^2r^2}{2T}}\leqslant \mathrm{e}^{-\frac{(1-4\eta)r^2}{2T}}.
\end{aligned}$$

Similarly,

$$\mathbf{P}^T_{a,b}\left(d(b,\omega(1/2))>(1-\eta)r\right)\leqslant \mathrm{e}^{-\frac{(1-4\eta)r^2}{2T}}.$$

Hence

$$\begin{aligned}
\mathbf{P}^T_{a,b}(\Psi_{a,b}\neq 1)&\leqslant \mathbf{P}^T_{a,b}\left(d(a,\omega(1/2))>(1-\eta)r\right)+\mathbf{P}^T_{a,b}\left(d(b,\omega(1/2))>(1-\eta)\right)\\
&\leqslant 2\mathrm{e}^{-\frac{(1-4\eta)r^2}{2T}}.
\end{aligned}$$

Step 2 Let R_1 be the constant in Lemma 2.4. Assume that $r<R_1$ and we first observe that

$$\begin{aligned}
\mathbf{Var}\left(F;\mathbf{P}^T_{a,b}\right)&=\mathbf{E}^T_{a,b}F^2-\left(\mathbf{E}^T_{a,b}F\right)^2\\
&\leqslant \mathbf{E}^T_{a,b}\left(F\Psi_{a,b}\right)^2+||F||^2_\infty\mathbf{P}^T_{a,b}\left(\Psi_{a,b}\neq 1\right)\\
&\quad-\left(\mathbf{E}^T_{a,b}[F-F\Psi_{a,b}+F\Psi_{a,b}]\right)^2\\
&\leqslant \mathbf{Var}\left(F\Psi_{a,b};\mathbf{P}^T_{a,b}\right)+3||F||^2_\infty\mathbf{P}^T_{a,b}(\Psi_{a,b}\neq 1)\\
&\leqslant \mathbf{Var}\left(F\Psi_{a,b};\mathbf{P}^T_{a,b}\right)+6\mathrm{e}^{-\frac{(1-4\eta)r^2}{2T}}||F||^2_\infty
\end{aligned}\tag{3.14}$$

Since $\Psi_{a,b}(\omega)=0$ when $\omega(1/2)\notin U^r_{a,b}$, Lemma 2.4 applies to $F\Psi_{a,b}$ and we have,

$$\begin{aligned}
\mathbf{Var}(F\Psi_{a,b};\mathbf{P}^T_{a,b})\leqslant\ & 2q(T,r)\mathbf{E}^T_{a,b}[\mathbf{\Gamma}^T(F\Psi_{a,b})]\\
&+(1+4q(T,r)\rho(T/2,r))\\
&\times\int\limits_{U^r_{a,b}}\Big\{\mathbf{E}^{T/2,1}_{a,x}[\mathbf{Var}_2(\widetilde{F\Psi_{a,b}}(\widetilde{\omega_1},\widetilde{\omega_2});\mathbf{P}^{T/2}_{x,b})]\\
&+\mathbf{E}^{T/2,2}_{x,b}[\mathbf{Var}_1(\widetilde{F\Psi_{a,b}}(\widetilde{\omega_1},\widetilde{\omega_2});\mathbf{P}^{T/2}_{a,x})]\Big\}\,\mu^T_{a,b}(dx),
\end{aligned}$$

We next deal with the terms $\mathbf{Var}_i(\widetilde{F\Psi_{a,b}}(\widetilde{\omega_1},\widetilde{\omega_2});\mathbf{P}_{x,b}^{T/2})$. Since $\Psi_{a,b}(\omega)=\varphi(d(a,\omega(1/2)))\cdot\varphi(d(b,\omega(1/2)))$ is determined by $\omega(1/2)$, for $i=1,2$.

$$\mathbf{Var}_i\left(\widetilde{F\Psi_{a,b}}(\widetilde{\omega_1},\widetilde{\omega_2});\mathbf{P}_{x,b}^{T/2}\right)=\varphi(d(a,x))^2\cdot\varphi(d(x,b))^2\cdot\mathbf{Var}_i\left(\widetilde{F}(\widetilde{\omega_1},\widetilde{\omega_2});\mathbf{P}_{x,b}^{T/2}\right)$$
$$\leqslant\mathbf{Var}_i\left(\widetilde{F}(\widetilde{\omega_1},\widetilde{\omega_2});\mathbf{P}_{x,b}^{T/2}\right)I_{\{x\in U_{a,b}^r\}}.$$

Consequently

$$\begin{aligned}\mathbf{Var}(F\Psi_{a,b};\mathbf{P}_{a,b}^T)\leqslant{}&2q(T,r)\mathbf{E}_{a,b}^T[\mathbf{\Gamma}^T(F\Psi_{a,b})]\\&+(1+4q(T,r)\rho(T/2,r))\\&\times\int_{U_{a,b}^r}\Big\{\mathbf{E}_{a,x}^{T/2,1}[\mathbf{Var}_2(\widetilde{F}(\widetilde{\omega_1},\widetilde{\omega_2});\mathbf{P}_{x,b}^{T/2})I_{\{x\in U_{a,b}^r\}}]\\&+\mathbf{E}_{x,b}^{T/2,2}[\mathbf{Var}_1(\widetilde{F}(\widetilde{\omega_1},\widetilde{\omega_2});\mathbf{P}_{x,b}^{T/2})I_{\{x\in U_{a,b}^r\}}]\Big\}\,\mu_{a,b}^T(dx),\end{aligned}\tag{3.15}$$

Let $S_aM:=\{v\in T_aM,\ |v|=1\}$. For each $\omega\in\Omega_{a,b}$, by the definition of $\mathbf{\Gamma}^T$ in (2.5),

$$\begin{aligned}&\mathbf{\Gamma}^T(F\Psi_{a,b})(\omega)\\&=\sup\left\{[d(F\Psi_{a,b})(\widehat{X}^{T,v})]^2(\omega);\,v\in S_{\omega(1/2)}M\right\}\\&\leqslant2\sup\left\{[dF(\widehat{X}^{T,v})]^2\Psi_{a,b}^2;\,v\in S_{\omega(1/2)}M\right\}\\&\quad+2\sup\left\{F^2[d\Psi_{a,b}(\widehat{X}^{T,v})]^2(\omega);\,v\in S_{\omega(1/2)}M\right\}\\&\leqslant2\mathbf{\Gamma}^T(F)(\omega)I_{\{\omega(1/2)\in U_{a,b}^r\}}\\&\quad+2||F||_\infty^2\sup\left\{\langle v,\nabla_x[\varphi(d(a,x))\cdot\varphi(d(x,b))]\rangle^2;\,v\in S_xM\right\}\\&\leqslant2\mathbf{\Gamma}^T(F)(\omega)I_{\{\omega(1/2)\in U_{a,b}^r\}}+\frac{32}{\eta^2r^2}||F||_\infty^2I_{\{\Psi_{a,b}(\omega)\neq1\}}.\end{aligned}\tag{3.16}$$

The required inequality (3.9) follows from (3.14), (3.15) and (3.16). □

Proposition 3.2 *There is a constant R_0 such that for each $0<\eta<1/4$ the following inequality holds on $(\Omega_{a,b},\mathbf{P}_{a,b}^T)$ provided that $d(a,b)<r<R_0$ and $0<T<T_0(\eta,r)$ for some $T_0(\eta,r)>0$:*

$$\begin{aligned}\mathbf{Var}(F;\mathbf{P}_{a,b}^T)\leqslant{}&TC(r)\mathbf{E}_{a,b}^T\left(|\mathbf{D}_0F|_{\mathbf{H}_\omega^0}^2\right)\\&+C(\eta,r)e^{-\frac{(1-4\eta)r^2}{2T}}||F||_\infty^2,\quad F\in\mathcal{F}C_b^\infty(\Omega_{a,b})\end{aligned}$$

Here $C(\eta,r)$, $C(r)$ are independent of T.

Remark Note that r must be less than the injective radius here, so we can not make the constant in front of $||F||_\infty^2$ tend to 0, and the above estimate is not a weak Poincaré inequality.

Proof By approximation procedure, it suffices to show the inequality holds for all smooth cylindrical functions with the following form,

$$F(\omega)=f\left(\omega\left(\frac{1}{2^m}\right),\omega\left(\frac{2}{2^m}\right),\ldots,\omega\left(\frac{2^m-1}{2^m}\right)\right),\quad f\in C^\infty(M^{2^m-1}),\quad m\in\mathbb{N}^+. \tag{3.17}$$

For any $\omega\in\Omega_{a,b}$, let

$$\omega_i(s)=\omega\left(\frac{i-1+s}{2^k}\right),\quad s\in[0,1],\quad k\in\mathbb{N}^+,\quad 1\leqslant i\leqslant 2^k$$

For simplicity, we did not reflect the index k in the definition of the new path ω_i. For each smooth cylinder function F and positive integer k we define a unique function $F^{[k]}$ of 2^k sub-paths. It is defined on $\bigcup_{z=(x_1,\ldots x_{2^k-1})\in M^{2^k-1}}\prod_{i=1}^{2^k}\Omega_{x_{i-1},x_i}$ (here $x_0=a$ and $x_{2^k}=b$), such that for each $\omega\in\Omega_{a,b}$

$$F^{[k]}(\omega_1,\ldots,\omega_{2^k})=F(\omega).$$

In fact, $\int F^{[k]}(\omega_1,\ldots\omega_{2^k})\prod_{i=1}^{2^k}\mathbf{P}^{T/2^k}_{x_{i-1},x_i}(d\omega_i)$ is a smooth version of the conditional expectation $\mathbf{E}^T_{a,b}[F|\omega(1/2^k)=x_1,\ldots\omega(1-1/2^k)=x_{2^k-1}]$ and $F^{[1]}$ is the same as $\widetilde{F}$ in Lemmas 2.4 and 3.1.

For $N\geqslant 1$ and $T>0$ we define the probability measure $\mu^{N,T}_{a,b}$ in M^{N-1} as,

$$\mu^{N,T}_{a,b}(\mathrm{d}x):=\frac{p_{\frac{T}{N}}(b,x_{N-1})p_{\frac{T}{N}}(x_{N-1},x_{N-2})\ldots p_{\frac{T}{N}}(x_1,a)}{p_T(a,b)}\mathrm{d}x_{N-1}\ldots\mathrm{d}x_1. \tag{3.18}$$

Fix a number $0<r<R_1$ for R_1 as in Lemma 3.1, $0<\eta<1/4$ and a positive number $T<T_1(\eta,r)$. For the variance terms for $\widetilde{F}$ as a function of any of the two subpaths $\widetilde{\omega_1},\widetilde{\omega_2}$ on the right side of inequality (3.9), we can apply (3.9) again, on each sub-path while keeping the other fixed, to obtain an estimate on the variance of F in terms of the variances and the operation $\mathbf{\Gamma}^{T/2}$ for $F^{[2]}$ as a function of any of the four subpaths (note that $x\in U^r_{a,b}$, so we can use Lemma 3.1 here). Repeat this procedure by mid-dividing the path and applying (3.9). The variance terms will finally vanish after a repetition of m times for the smooth cylinder function of type (3.17),

and we have,

$$
\begin{aligned}
&\mathbf{Var}(F;\mathbf{P}^T_{a,b})\\
&\leqslant 4\sum_{k=0}^{m-1} G(k,T,r)q(T/2^k,r)\\
&\times\left(\sum_{j=1}^{2^k}\int\limits_{U_j}\left\{\mathbf{E}^{T/2^k,j}_{x_{j-1},x_j}\left[\mathbf{\Gamma}^{T/2^k,j}(F^{[k]})(\omega_1,\ldots\omega_{2^k})I_{\{\omega_j(1/2)\in U^r_{x_{j-1},x_j}\}}\right]\right.\right.\\
&\left.\left.\times\prod_{i\neq j}\mathbf{P}^{T/2^k}_{x_{i-1},x_i}(d\omega_i)\right\}\mu^{2^k,T}_{a,b}(dx)\right)\\
&+\sum_{k=0}^{m-1} G(k,T,r)\left(6+\frac{128q(T/2^k,r)}{\eta^2 r^2}\right)2^k \mathrm{e}^{-\frac{2^k(1-4\eta)r^2}{2T}}||F||^2_\infty \qquad (3.19)
\end{aligned}
$$

where $G(0,T,r)=1$, $G(k,T,r)=\prod_{i=1}^k(1+4q(T/2^{i-1},r)\rho(T/2^i,r))$ for each $k>0$, $U_j=\{x=(x_1,\ldots,x_{2^k-1})\in M^{2^k-1}: d(x_{j-1},x_j)<r\}$ for $j=1,2\ldots,2^k$ ($x_0=a$ and $x_{2^k}=b$). We denote by $\mathbf{E}^{T/2^k,j}_{x_{j-1},x_j}$ and $\mathbf{\Gamma}^{T/2^k,j}(F^{[k]})$ the corresponding expectation and the operation $\mathbf{\Gamma}^{T/2^k}$ (defined in (2.5)) with respect to the jth sub-path for function $F^{[k]}$.

By (5.8) in the proof of lemma 5.1 in [11] if T is small enough,

$$
\sup_{k\in\mathbb{N}} G(k,T,r)<C(r). \qquad (3.20)
$$

By this and Lemma 3.3 below we can find a positive number $R_0<R_1$, such that for each $0<r<R_0$, there is a $T_2(r)>0$, when $T<T_2(r)$ the following holds for all positive integer m:

$$
\begin{aligned}
&\sum_{k=0}^{m-1} G(k,T,r)q(T/2^k,r)\\
&\cdot\left(\sum_{j=1}^{2^k}\int\limits_{U_j}\left\{\mathbf{E}^{T/2^k,j}_{x_{j-1},x_j}\left[\mathbf{\Gamma}^{T/2^k,j}(F^{[k]})(\omega_1,\ldots\omega_{2^k})\mathbf{1}_{\{\omega_j(1/2)\in U^r_{x_{j-1},x_j}\}}\right]\right.\right.\\
&\left.\left.\times\prod_{i\neq j}\mathbf{P}^{T/2^k}_{x_{i-1},x_i}(d\omega_i)\right\}\mu^{2^k,T}_{a,b}(dx)\right)\\
&\leqslant TC(r)\mathbf{E}^T_{a,b}|\mathbf{D}_0F|^2_{\mathbf{H}^0_\omega} \qquad (3.21)
\end{aligned}
$$

Note that by Lemma 2.4 there is $T_0(\eta,r)<\min(T_2(r),T_1(\eta,r))$ such that if $T<T_0(\eta,r)$, then $|q(T,r)|\leqslant C(r)$ for some constans $C(r)$ and (3.20) holds. Using this

bound we can see that for $0 < \eta < 1/4$, $T < T_0(\eta, r)$,

$$\sup_{k\in\mathbb{N}} G(k, T, r)\left(6 + \frac{128q(T/2^k, r)}{\eta^2 r^2}\right) \leqslant C(r, \eta)$$

and

$$\sum_{k=0}^{m-1} G(k, T, r)\left(6 + \frac{128q(T/2^k, r)}{\eta^2 r^2}\right) 2^k e^{-\frac{2^k(1-4\eta)r^2}{2T}} \leqslant C(r, \eta)\sum_{k=0}^{\infty} 2^k e^{-\frac{2^k(1-4\eta)r^2}{2T}}$$
$$\leqslant C(r, \eta)e^{-\frac{(1-4\eta)r^2}{2T}} \quad (3.22)$$

We conclude the proof from (3.19)–(3.22). □

Finally we complete the proof of the Proposition by proving the lemma below.

Lemma 3.3 *Let $U_j = \{x = (x_1, \ldots, x_{2^k-1}) \in M^{2^k-1} : d(x_{j-1}, x_j) < r\}$ for $j = 1, 2 \ldots 2^k$ ($x_0 = a$ and $x_{2^k} = b$). We can find a $R_2 > 0$, for each $0 < r < R_2$, there is a number $T(r) > 0$, when $T < T(r)$, we have*

$$\sum_{k=0}^{m-1} q(T/2^k, r)\cdot\left(\sum_{j=1}^{2^k}\int_{U_j}\left\{\mathbf{E}^{T/2^k, j}_{x_{j-1},x_j}\left[\mathbf{\Gamma}^{T/2^k, j}(F^{[k]})(\omega_1, \ldots \omega_{2^k})\mathbf{1}_{\{\omega_j(1/2)\in U^r_{x_{j-1},x_j}\}}\right]\right.\right.$$
$$\left.\left.\times\prod_{i\neq j}\mathbf{P}^{T/2^k}_{x_{i-1},x_i}(d\omega_i)\right\}\mu^{2^k,T}_{a,b}(dx)\right)$$
$$\leqslant TC(r)\mathbf{E}^T_{a,b}|\mathbf{D}_0 F|^2_{\mathbf{H}^0_\omega}$$

Here $C(r)$ is independent of T.

Proof Following the notation from [11], let $\{h_{k,j};\ k \geqslant 0, 1 \leqslant j \leqslant 2^k\}$ be the orthonormal basis of $H^{1,2}_0([0, 1]; \mathbb{R})$ consisting of Schauder functions, i.e. $h_{0,1}(s) = s \wedge (1 - s)$,

$$\begin{cases} h_{k,j}(s) = 2^{-k/2}h_{0,1}(2^k s - (j-1)) & \text{if } s \in [(j-1)2^{-k}, j2^{-k}], \\ h_{k,j}(s) = 0 & \text{otherwise} \end{cases}$$

for $k \geqslant 1$ and $1 \leqslant j \leqslant 2^k$. Let $d = dim(M)$. We choose $\{e_i, 1 \leqslant i \leqslant d\}$, a family of measurable vector fields on M with $\{e_i(z); 1 \leqslant i \leqslant d\}$ an orthonomal basis on $T_z M$ for every $z \in M$. These give rise to an orthonormal basis of $\mathbf{H}^0_\omega$:

$$Z^{k,j,i}_s(\omega) = h_{k,j}(s)/\!/_{1/2,s}(\omega)e_i(\omega(1/2)),\ s \in [0, 1],\ k \geqslant 0,\ 1 \leqslant j \leqslant 2^k,\ 1 \leqslant i \leqslant d.$$

For each $F \in \mathcal{F}C_b^\infty(\Omega_{a,b})$, let

$$\Lambda_{k,j}(F)(\omega) = \sum_{i=1}^{d} \left[dF(Z^{k,j,i})\right]^2 = \sum_{i=1}^{d} \left(\mathbf{D}_0 F, Z^{k,j,i}\right)^2_{\mathbf{H}^0_\omega}.$$

Then we have

$$|\mathbf{D}_0 F(\omega)|^2_{\mathbf{H}^0_\omega} = \sum_{k=0}^{\infty} \sum_{j=1}^{2^k} \Lambda_{k,j}(F)(\omega), \quad \forall \omega \in \Omega_{a,b}.$$

By Lemma 4.3 in [11], there exist constants $R_2 > 0$, such that for each $r \in (0, R_2)$, there is a $\widetilde{T}(r) > 0$ such that when $T < \widetilde{T}(r)$

$$\mathbf{\Gamma}^T(F)(\omega)\mathbf{1}_{\{\omega(1/2)\in U^r_{a,b}\}} \leqslant C(r)\Lambda_{0,1}(F)(\omega) + \sum_{l=0}^{\infty} C(r)(T + 2^{-l}) \sum_{n=1}^{2^l} \Lambda_{l,n}(F)(\omega),$$

for each smooth cylinder function F and $\omega \in \Omega_{a,b}$,

Thus, we obtain

$$\begin{aligned}
&\mathbf{\Gamma}^{T/2^k,j}(F^{[k]})(\omega_1,\ldots\omega_{2^k})\mathbf{1}_{\{\omega_j(1/2)\in U^r_{x_{j-1},x_j}\}} \\
&\leqslant C(r)\Lambda^{k,j}_{0,1}(F^{[k]})(\omega_1,\ldots\omega_{2^k}) + \sum_{l=0}^{\infty} C(r)(T/2^k + 2^{-l}) \\
&\times \sum_{n=1}^{2^l} \Lambda^{k,j}_{l,n}(F^{[k]})(\omega_1,\ldots\omega_{2^k}).
\end{aligned} \tag{3.23}$$

Here $\Lambda^{k,j}_{l,n}(F^{[k]})$ means the corresponding operation $\Lambda_{l,n}$ is taken with respect to the jth subpath for function $F^{[k]}$. Note that

$$\begin{aligned}
\Lambda^{k,j}_{l,n}(F^{[k]})(\omega_1,\ldots\omega_{2^k}) &= \sum_{i=1}^{d} \left[dF^{[k]}(Z^{l,n,i})\right]^2 \left(\omega_1,\ldots,\omega_{j-1},\bullet,\omega_{j+1},\ldots\omega_{2^k}\right) \\
&= \sum_{i=1}^{d} 2^k \left[dF(Z^{l+k,(j-1)2^l+n,i})\right]^2 (\omega) \\
&= 2^k \Lambda_{l+k,(j-1)2^l+n}(F)(\omega).
\end{aligned} \tag{3.24}$$

The second equality above is due to the defintion of $Z^{l,n,i}$ and some time rescaling procedure. By (3.23) and (3.24) we obtain,

$$\sum_{k=0}^{m-1} q(T/2^k, r) \sum_{j=1}^{2^k} \mathbf{\Gamma}^{T/2^k,j}(F^{[k]})(\omega_1, \dots \omega_{2^k}) \mathbf{1}_{\{\omega_j(1/2) \in U^r_{x_{j-1}x_j}\}}$$
$$\leqslant \sum_{k=0}^{m-1} \left\{ C(r)q(T/2^k, r)2^k + \sum_{l=0}^{k} C(r)(2^{k-2l} + T)q(T/2^{k-l}, r) \right\} \sum_{j=1}^{2^k} \Lambda_{k,j}(F)(\omega). \tag{3.25}$$

Let $g(k, T, r) := C(r)q(T/2^k, r)2^k + \sum_{l=0}^{k} C(r)(2^{k-2l} + T)q(T/2^{k-l}, r)$, by the estimate (2.8) for $q(T, r)$ we can find a $T(r) < \widetilde{T}(r)$, such that for each $T < T(r)$, $\sup_{k\in\mathbb{N}} g(k, T, r) \leqslant TC(r)$, where $C(r)$ is a constant independent of T and k.

So by (3.25), for $T < T(r)$, we have,

$$\sum_{k=0}^{m-1} q(T/2^k, r) \cdot \left(\sum_{j=1}^{2^k} \int_{U_j} \left\{ \mathbf{E}^{T/2^k,j}_{x_{j-1},x_j} \left[\mathbf{\Gamma}^{T/2^k,j}(F^{[k]})(\omega_1, \dots \omega_{2^k}) \mathbf{1}_{\{\omega_j(1/2) \in U^r_{x_{j-1},x_j}\}} \right] \right.\right.$$
$$\left.\left. \times \prod_{i\neq j} \mathbf{P}^{T/2^k}_{x_{i-1},x_i}(d\omega_i) \right\} \mu^{2^k,T}_{a,b}(dx) \right)$$
$$\leqslant \sum_{k=0}^{m-1} g(k, T, r) \sum_{j=1}^{2^k} \mathbf{E}^T_{a,b}[\Lambda_{k,j}(F)]$$
$$\leqslant TC(r)\mathbf{E}^T_{a,b}|\mathbf{D}_0 F|^2_{\mathbf{H}^0_\omega}$$

□

4 An estimate over discretized loop space

For each $r \in \mathbb{R}^+$ and integer $N \geqslant 1$, define the subset $U^{r,N}_{a,b}$ of M^{N-1} as,

$$U^{r,N}_{a,b} := \left\{ (x_1, \dots x_{N-1}) \in M^{N-1};\ d(x_{i-1}, x_i) < r,\ 1 \leqslant i \leqslant N,\ x_0 = a,\ x_N = b \right\}. \tag{4.26}$$

And recall that $\mu^{N,T}_{a,b}$ is the probability measure on M^{N-1} defined in (3.18), which is also the joint distribution of $(\omega(i/N),\ i = 1, 2, \dots N-1)$ under $\mathbf{P}^T_{a,b}$. We have the following estimate of the variance with respect to $\mu^{N,1}_{a,b}$.

Proposition 4.1 *Let M be a compact simply connected manifold with strict positive Ricci curvature. For any $0 < \eta < 1/8$, $0 < r < R_0$, there exists an integer*

$N_1(\eta, r) > 0$, *such that for* $l > N_1(\eta, r)$ *there exists an integer* $N_2(\eta, l, r)$ *with the property that if* $N > N_2(\eta, l, r)$

$$\begin{aligned}&\mathbf{Var}(f; \mu_{a,a}^{N,1})\\ &\leqslant C(l,r)N^{C(l,r)}\mathrm{e}^{2N\eta r^2}\sum_{i=1}^{N-1}\int\limits_{M^{N-1}}|\nabla_i f|^2 \mathrm{d}\mu_{a,a}^{N,1}\\ &+C(l,\eta,r)N^{C(l,r)}\mathrm{e}^{-\frac{N(1-8\eta)r^2}{2}}||f||_\infty^2\end{aligned}$$

for all $f \in C^\infty(M^{N-1})$ *with* $\mathrm{supp}(f) \subset \overline{U}_{a,a}^{r,N}$.

Proof Step 1 For any integers $N > l > 1$ and a function $f \in C^\infty(M^{N-1})$. Define a smooth cylinder function $\widehat{F} : \Omega_{a,a} \mapsto \mathbb{R}$ as,

$$\widehat{F}(\omega) := f(\omega(1/N), \dots, \omega(1-1/N)). \tag{4.27}$$

For $x = (x_1, \dots, x_{N-l}) \in M^{N-l}$ we define a function $f_l : M^{N-l} \to \mathbb{R}$ as following,

$$f_l(x_1, \dots, x_{N-l}) = \int\limits_{M^{l-1}} f(x_1,\dots,x_{N-l}, y_1,\dots,y_{l-1})\ \mu_{x_{N-l},a}^{l,\frac{l}{N}}(dy_1\dots dy_{l-1}) \tag{4.28}$$

In fact

$$f_l(x_1, \dots x_{N-l}) = \mathbf{E}_{a,a}\left[\widehat{F}(\omega) \mid \omega(1/N) = x_1, \dots, \omega\left(\frac{N-l}{N}\right) = x_{N-l}\right].$$

For such x define a function on $\Omega_{x_{N-l},a}$ by

$$\widetilde{F}_l(x_1, \dots x_{N-l}, \omega) := f(x_1, \dots x_{N-l}, \omega(1/l), \dots, \omega(1-1/l)), \quad \omega \in \Omega_{x_{N-l},a}. \tag{4.29}$$

Clearly

$$f_l(x_1, \dots x_{N-l}) = \mathbf{E}_{x_{N-l},a}^{\frac{l}{N}}\left[\widetilde{F}_l(x_1, \dots x_{N-l}, \bullet)\right].$$

Let $\wp_l = \sigma\{\omega(i/N),\ 1 \leqslant i \leqslant N-l\}$, an σ-algebra on $\Omega_{a,a}$ and $\mu_{a,a}^{N,l,T}$ be the probability measure on M^{N-l}:

$$\begin{aligned}&\mu_{a,a}^{N,l,T}\ (dx_1, \dots, dx_{N-l})\\ &\quad= \frac{p_{\frac{lT}{N}}(a, x_{N-l})p_{\frac{T}{N}}(x_{N-l}, x_{N-l-1})\cdots p_{\frac{T}{N}}(x_1, a)}{p_T(a,a)}dx_{N-l}\dots dx_1.\end{aligned} \tag{4.30}$$

We have

$$\begin{aligned}&\mathbf{Var}(f;\mu_{a,a}^{N,1})\\&=\mathbf{Var}(\widehat{F};\mathbf{P}_{a,a})\\&=\mathbf{E}_{a,a}\left[(\widehat{F}-\mathbf{E}_{a,a}[\widehat{F}|\wp_l])^2\right]+\mathbf{E}_{a,a}\left[(\mathbf{E}_{a,a}[\widehat{F}|\wp_l]-\mathbf{E}_{a,a}[\widehat{F}])^2\right]\\&=\int\limits_{M^{N-l}}\mathbf{Var}\left(\widetilde{F}_l;\mathbf{P}^{\frac{l}{N}}_{x_{N-l},a}\right)\mu_{a,a}^{N,l,1}(dx)+\mathbf{Var}\left(f_l,\mu_{a,a}^{N,l,1}\right)\end{aligned}\tag{4.31}$$

Step 2 Now we are going to estimate $\mathbf{Var}(f_l,\mu_{a,a}^{N,l,1})$. Let $\mathbf{P}_a^1$ be the distribution of a standard Brownnian motion on compact manifold M starting from a with time parameter 1, which is a probability measure on the path space Ω_a over M with starting point a and time 1. Let

$$\gamma_a^{N,l,1}(dx_1,\dots dx_{N-l}):=p_{\frac{1}{N}}(a,x_1),\dots p_{\frac{1}{N}}(x_{N-l-1},x_{N-l})\mathrm{d}x_1,\dots \mathrm{d}x_{N-l}$$

be a probability measure on M^{N-l}, which is the joint distribution of $(\omega(i/N),\omega\in\Omega_a,\ i=1,2,\dots N-l)$ under $\mathbf{P}_a^1$. By the Poincaré inequality for $\mathbf{P}_a^1$ on the path space over compact manifold [15] we get,

$$\mathbf{Var}(f_l;\gamma_a^{N,l,1})=\mathbf{Var}(\overline{F}_l;\mathbf{P}_a^1)\leqslant C\mathbf{E}_a^1|\mathbf{D}\overline{F}_l|^2_{\mathbf{H}_\omega}\leqslant CN\sum_{i=1}^{N-l}\int\limits_{M^{N-l}}|\nabla_i f_l|^2 d\gamma_a^{N,l,1}$$

where $\overline{F}_l(\omega):=f_l(\omega(1/N),\dots\omega(1-l/N))$ for $\omega\in\Omega_a$ and $\mathbf{D}$ is the gradient operator related to Bismut tangent norm $|.|_{\mathbf{H}_\omega}$ in path space over compact manifold M. We also use the relation

$$|\mathbf{D}\overline{F}_l(\omega)|^2_{\mathbf{H}_\omega}\leqslant(N-l)\sum_{i=1}^{N-l}|\nabla_i f_l(\omega(1/N),\dots\omega(1-l/N))|^2,\quad \omega\in\Omega_a,$$

in above inequality which can be checked by direct computation. Thus, we have

$$\begin{aligned}\mathbf{Var}(f_l;\mu_{a,a}^{N,l,1})&=\mathbf{Var}\left(f_l;\frac{p_{\frac{l}{N}}(a,x_{N-l})}{p_1(a,a)}\gamma_a^{N,l,1}\right)\\&\leqslant C\cdot\operatorname{osc}\left(p_{\frac{l}{N}}(a,\cdot)\right)\cdot N\cdot\sum_{i=1}^{N-l}\int\limits_{M^{N-l}}|\nabla_i f_l|^2\mathrm{d}\mu_{a,a}^{N,l,1}.\end{aligned}\tag{4.32}$$

Here for g a function on M, $\mathrm{osc}(g(\cdot)) := \frac{\sup_{x\in M} g(x)}{\inf_{x\in M} g(x)}$ is the oscillation of g. By Varadhan's estimate (3.13), if $\frac{l}{N} < T_1(\eta, r)$ then

$$\mathrm{osc}\left(p_{\frac{l}{N}}(a,\cdot)\right) \leqslant \mathrm{e}^{\frac{N}{l}(\eta^2 r^2 + \frac{D^2}{2})}$$

for D the diameter of the compact manifold M. So by this and (4.32) when $\frac{l}{N} < T_1(\eta, r)$, we have

$$\mathbf{Var}\left(f_l; \mu_{a,a}^{N,l,1}\right) \leqslant CN\mathrm{e}^{\frac{N}{l}(\eta^2 r^2 + \frac{D^2}{2})} \sum_{i=1}^{N-l} \int_{M^{N-l}} |\nabla_i f_l|^2 \mathrm{d}\mu_{a,a}^{N,l,1}. \tag{4.33}$$

Step 3 Now we are going to estimate $|\nabla_i f_l|$ in terms of f. It is easy to see that for $i < N - l$,

$$|\nabla_i f_l|^2(x_1, \ldots x_{N-l}) \leqslant \int_{M^{l-1}} |\nabla_i f|^2(x_1, \ldots x_{N-l}, y_1, \ldots y_{l-1}) \mu_{x_{N-l},a}^{l,\frac{l}{N}}(dy) \tag{4.34}$$

and for $i = N - l$,

$$\begin{aligned} &|\nabla_{N-l} f_l|^2(x_1, \ldots x_{N-l}) \\ &\quad = \sup_{|v|=1} \mathrm{d}_{N-l} f_l(v) \\ &\quad \leqslant 2 \int_{M^{l-1}} |\nabla_{N-l} f|^2(x_1, \ldots x_{N-l}, y_1, \ldots y_{l-1}) \mu_{x_{N-l},a}^{l,\frac{l}{N}}(dy) \\ &\qquad +2 \sup_{|v|=1} \left| \mathrm{d}_z \left(\mathbf{E}_{z,a}^{\frac{l}{N}}(\widetilde{F}_l) \right) |_{z=x_{N-l}}(v) \right|^2. \end{aligned} \tag{4.35}$$

Here $\widetilde{F}_l(\omega) := f(x_1, \ldots x_{N-l}, \omega(1/l), \ldots \omega(1 - 1/l))$ is as defined in (4.29). As $d(a, x_{N-l}) \leqslant r$ may not hold, we can not choose the vector constructed in Lemma 2.2 when we apply Lemma 2.3 to estimate the differentiation. To estimate the differentiation of the expectation with respect to the starting point we choose another vector field $X^{l,v}(s) := /\!\!/_s (1 - ls)^+ v,\ 0 \leqslant s \leqslant 1$. Recall that (B_s) is the stochastic anti-development, in the expression for $\delta^{\frac{l}{N}} X$ in Lemma 2.1, and

$$B_s = \beta_s + \int_0^s \left(/\!\!/_u^{-1} \nabla \log p_{\frac{(1-u)l}{N}}(\omega(u), a) \right) du, \quad 0 \leqslant s < 1$$

for some process (β_s) whose distribution is the Brownnian motion with time parameter $\frac{l}{N}$ under the probability measure $\mathbf{P}^{\frac{l}{N}}_{x_{N-l},a}$ (see [9]). We get,

$$\begin{aligned}
\mathbf{Var}\left(\delta^{\frac{l}{N}} X^{l,v}; \mathbf{P}^{\frac{l}{N}}_{x_{N-l},a}\right) &\leqslant \mathbf{E}^{\frac{l}{N}}_{x_{N-l},a}\left(\delta^{\frac{l}{N}} X^{l,v}\right)^2 \\
&\leqslant \mathbf{E}^{\frac{l}{N}}_{x_{N-l},a}\Bigg[\int_0^{\frac{1}{l}}\left(-Nv + \frac{1}{2}\mathrm{Ric}_{\omega(s)}(/\!/_s(1-ls)v)\right) \\
&\quad \times \left(\mathrm{d}\beta_s + /\!/_s^{-1}\nabla \log p_{\frac{(1-s)l}{N}}(\omega(s), a)\mathrm{d}s\right)\Bigg]^2 \\
&\leqslant C(l)N^4, \quad v \in S_{x_{N-l}}M.
\end{aligned}$$

In the last step we used the estimate $|\nabla \log p_s(x,a)| \leqslant C[\frac{d(x,a)}{s} + \frac{1}{\sqrt{s}}](s > 0)$ for compact manifolds. Also note that $X^{l,v}(\frac{i}{l}) = 0$ for $1 \leqslant i \leqslant l$. By Lemma 2.3, we have,

$$\begin{aligned}
&\sup_{|v|=1}\left|\mathrm{d}_{N-l}\left(\mathbf{E}^{\frac{l}{N}}_{x_{N-l},a}(\widetilde{F}_l)\right)(v)\right|^2 \\
&\leqslant \sup_{|v|=1}\left\{\left|\mathbf{E}^{\frac{l}{N}}_{x_{N-l},a}[d\widetilde{F}_l(X^{l,v})]\right| + \left[\mathbf{Var}\left(\delta^{\frac{l}{N}} X^{l,v}; \mathbf{P}^{\frac{l}{N}}_{x_{N-l},a}\right)\right]^{1/2}\left[\mathbf{Var}\left(\widetilde{F}_l; \mathbf{P}^{\frac{l}{N}}_{x_{N-l},a}\right)\right]^{1/2}\right\}^2 \\
&\leqslant C(l)N^4\mathbf{Var}\left(\widetilde{F}_l; \mathbf{P}^{\frac{l}{N}}_{x_{N-l},a}\right).
\end{aligned} \tag{4.36}$$

Using (4.34)–(4.36) we derive the following estimate

$$\sum_{i=1}^{N-l}|\nabla_i f_l|^2(-) \leq \sum_{i=1}^{N-l}\int_{M^{l-1}}|\nabla_i f|^2(-,y)\mu^{l,\frac{l}{N}}_{x_{N-l},a}(dy) + C(l)N^4\mathbf{Var}\left(\widetilde{F}_l; \mathbf{P}^{\frac{l}{N}}_{x_{N-l},a}\right).$$

then from that Combining this with (4.31) and (4.33), we obtain the following,

$$\begin{aligned}
&\mathbf{Var}(f; \mu^{N,1}_{a,a}) \\
&\leqslant C(l)N\exp\left(\frac{N}{l}\left(\eta^2r^2 + \frac{D^2}{2}\right)\right)\sum_{i=1}^{N-l}\int_{M^{N-1}}|\nabla_i f|^2\mu^{N,1}_{a,a}(dx) \\
&\quad + \left[1 + C(l)N^5\exp\left(\frac{N}{l}\left(\eta^2r^2 + \frac{D^2}{2}\right)\right)\right]\int_{M^{N-l}}\mathbf{Var}\left(\widetilde{F}_l; \mathbf{P}^{\frac{l}{N}}_{x_{N-l},a}\right)\mu^{N,l,1}_{a,a}(dx)
\end{aligned} \tag{4.37}$$

Step 4 We estimate the variance $\mathbf{Var}(\widetilde{F}_l; \mathbf{P}^{\frac{l}{N}}_{x_{N-l},a})$ by the variance with respect to the Brownian bridge measure. Note that

$$\mathbf{Var}\left(\widetilde{F}_l; \mathbf{P}^{\frac{l}{N}}_{x_{N-l},a}\right) = \mathbf{Var}\left(f(x_1, \dots x_{N-l}, \bullet, \dots, \bullet); \mu^{l,\frac{l}{N}}_{x_{N-l},a}\right). \tag{4.38}$$

Let $\overline{\mu}^{l,\frac{l}{N}}_{x_{N-l},a}$ be normalization of $\mu^{l,\frac{l}{N}}_{x_{N-l},a}$ on the subset $U^{r,l}_{x_{N-l},a}$ of M^{l-1}, i.e.

$$\overline{\mu}^{l,\frac{l}{N}}_{x_{N-l},a}(A) = \mu^{l,\frac{l}{N}}_{x_{N-l},a}(A)/\mu^{l,\frac{l}{N}}_{x_{N-l},a}(U^{r,l}_{x_{N-l},a}), \quad A \subseteq U^{r,l}_{x_{N-l},a}.$$

For each smooth function g with support in $\overline{U}^{r,l}_{x_{N-l},a}$, we have,

$$\mathbf{Var}\left(g; \mu^{l,\frac{l}{N}}_{x_{N-l},a}\right) \leqslant \mu^{l,\frac{l}{N}}_{x_{N-l},a}\left(U^{r,l}_{x_{N-l},a}\right)\mathbf{Var}\left(g; \overline{\mu}^{l,\frac{l}{N}}_{x_{N-l},a}\right) + \frac{\left(1-\mu^{l,\frac{l}{N}}_{x_{N-l},a}(U^{r,l}_{x_{N-l},a})\right)}{\mu^{l,\frac{l}{N}}_{x_{N-l},a}(U^{r,l}_{x_{N-l},a})}||g||^2_\infty. \tag{4.39}$$

By asymptotic property (3.13), when $\frac{l}{N} < T_1(\eta, r)$, it satisfies that,

$$\begin{aligned}
&1-\mu^{l,\frac{l}{N}}_{x_{N-l},a}(U^{r,l}_{x_{N-l},a})\\
&\quad= \mu^{l,\frac{l}{N}}_{x_{N-1},a}(\{z:\ d(z_i, z_{i+1}) > r \text{ for some } 0 \leqslant i \leqslant l-1\})\\
&\quad\leqslant \sum_{i=0}^{l-1} \frac{\int_{\mathrm{d}(z_i,z_{i+1})>r} p_{\frac{1}{N}}(x_{N-l}, z_1)\dots p_{\frac{1}{N}}(z_{l-1}, a)\mathrm{d}z_1, \dots \mathrm{d}z_{l-1}}{p_{\frac{l}{N}}(x_{N-l}, a)}\\
&\quad\leqslant l \cdot \frac{\exp(-\frac{(1-4\eta)Nr^2}{2})}{\exp(-\frac{N}{2l}(\eta^2r^2 + D^2))}.
\end{aligned} \tag{4.40}$$

If we choose l sufficient large, e.g. $\frac{\eta^2r^2+D^2}{l} < 2(1-4\eta)r^2$, there is an integer $\widetilde{N}(\eta, l, r)$, such that whenever $N > \widetilde{N}(\eta, l, r)$

$$\mu^{l,\frac{l}{N}}_{x_{N-l},a}(U^{r,l}_{x_{N-l},a}) > \frac{1}{2} \tag{4.41}$$

Since the support of f is a subset of $\overline{U}^{r,N}_{a,a}$, then for each fixed $x_1, \dots x_{N-l}$,

$$\mathrm{supp}(f(x_1, \dots, x_{N-l}, \bullet, \dots, \bullet)) \subset \overline{U}^{r,l}_{x_{N-l},a}.$$

Hence by (4.39)–(4.41), for each integer l sufficiently big, there exists an integer $\widetilde{N}(\eta, l, r)$, for each $N > \widetilde{N}(\eta, l, r)$, we have,

$$
\begin{aligned}
&\mathbf{Var}\left(f(x_1, \dots x_{N-l}, \bullet, \dots, \bullet); \mu_{x_{N-l},a}^{l,\frac{l}{N}}\right)\\
&\leqslant \frac{1}{\lambda(U_{x_{N-l},a}^{r,l}; \overline{\mu}_{x_{N-l},a}^{l,\frac{l}{N}})} \times \sum_{i=1}^{l-1} \int |\nabla_{N-i} f|^2(x_1, \dots x_{N-l}, z_1, \dots z_{l-1}) \mu_{x_{N-l},a}^{l,\frac{l}{N}}(dz)\\
&\quad + 2l \cdot \frac{\exp(-\frac{(1-4\eta)Nr^2}{2})}{\exp(-\frac{N}{2l}(\eta^2 r^2 + D^2))} ||f||_\infty^2,
\end{aligned}
\tag{4.42}
$$

where

$$
\lambda\left(U_{x,a}^{r,l}; \overline{\mu}_{x,a}^{l,\frac{l}{N}}\right) := \inf_{g \in C_0^\infty(U_{x,a}^{r,l})} \frac{\int_{U_{x,a}^{r,l}} |\nabla g|^2 \mathrm{d}\overline{\mu}_{x,a}^{l,\frac{l}{N}}}{\mathbf{Var}\left(g; \overline{\mu}_{x,a}^{l,\frac{l}{N}}\right)}. \tag{4.43}
$$

Therefore, by (4.37) and (4.42), we have for each l big enough, $N > \widetilde{N}(\eta, l, r)$ and $\frac{l}{N} < T_1(\eta, r)$,

$$
\begin{aligned}
&\mathbf{Var}\left(f; \mu_{a,a}^{N,1}\right)\\
&\leqslant C(l) N \exp\left(\frac{N}{l}\left(\eta^2 r^2 + D^2/2\right)\right)\left[1 + \frac{N^4}{\inf_{x \in M} \lambda\left(U_{x,a}^{r,l}; \overline{\mu}_{x,a}^{l,\frac{l}{N}}\right)}\right]\\
&\quad \times \sum_{i=1}^{N-1} \int_{M^{N-1}} |\nabla_i f|^2 \mathrm{d}\mu_{a,a}^{N,1}\\
&\quad + \left[C(l) N^5 \exp\left(N\left(-\frac{(1-4\eta)r^2}{2} + \frac{3\eta^2 r^2 + 2D^2}{l}\right)\right)\right] ||f||_\infty^2.
\end{aligned}
\tag{4.44}
$$

Step 5 Finally, by (4.44) and the uniform estimate of $\lambda(U_{x,a}^{r,l}; \overline{\mu}_{x,a}^{l,\frac{l}{N}})$ in x derived in Lemma 4.2 below, for each integer l sufficiently big, there exists an integer $N_2(\eta, l, r) > 0$, such that if $N > N_2(\eta, l, r)$, then

$$
\begin{aligned}
&\mathbf{Var}\left(f; \mu_{a,a}^{N,1}\right)\\
&\leqslant \left[C(l,r) N^{C(l,r)} \exp\left(N\left(\frac{L(\varepsilon) + \eta^2 r^2 + D^2/2}{l} + 4D\varepsilon\right)\right)\right]\\
&\quad \times \sum_{i=1}^{N-1} \int_{M^{N-1}} |\nabla_i f|^2 \mathrm{d}\mu_{a,a}^{N,1}
\end{aligned}
$$

$$+\left[C(l,\eta,r)N^{C(l,r)}\exp\left(N\left(-\frac{(1-4\eta)r^2}{2}+\frac{3\eta^2r^2+2D^2}{l}\right)\right)\right]||f||_\infty^2. \tag{4.45}$$

Note that the constants C and L in the inequality above do not depend on N, and L does not depend on l and the starting point a. So for any fixed $\eta > 0$, $0 < r < R_0$, we first choose a $\varepsilon = \frac{\eta r^2}{4D}$ to make $4D\varepsilon = \eta r^2$, then take a l big enough such that $\frac{L(\varepsilon)+\eta^2r^2+D^2/2}{l} < \eta r^2$ and $\frac{3\eta^2r^2+2D^2}{l} < \eta r^2$ for the chosen $\varepsilon = \frac{\eta r^2}{4D}$ (i.e. $l > N_0(\eta, r)$ for some constant $N_0(\eta, r)$ which only depends on η and r). Hence by (4.45), there is a constants $N_1(\eta, r)$, such that for each integer $l > N_1(\eta, r)$, there exists an integer $N_2(\eta, l, r) > 0$, such that if $N > N_2(\eta, l, r)$ then we have, as required,

$$\begin{aligned}&\mathbf{Var}\left(f;\mu_{a,a}^{N,1}\right)\\&\leqslant C(l,r)N^{C(l,r)}e^{2N\eta r^2}\sum_{i=1}^{N-1}\int_{M^{N-1}}|\nabla_i f|^2 d\mu_{a,a}^{N,1}+C(l,\eta,r)N^{C(l,r)}e^{-\frac{N(1-8\eta)r^2}{2}}||f||_\infty^2.\end{aligned}$$

Lemma 4.2 *Let M be a compact simply connected manifold with strict positive Ricci curvature. For $a, x \in M$, $r < R_0$ and $N \in \mathbb{N}^+$, $\lambda(U_{x,a}^{r,l}; \overline{\mu}_{x,a}^{l,\frac{l}{N}})$ as defined in (4.43), there exists a constant $T(l, r)$, such that when $\frac{l}{N} < T(l, r)$, for each $\varepsilon > 0$ small enough,*

$$\inf_{x,a\in M}\lambda\left(U_{x,a}^{r,l};\overline{\mu}_{x,a}^{l,\frac{l}{N}}\right)\geqslant\frac{C(l,r)}{N^{C(l,r)}}\exp\left(-\left(\frac{L(\varepsilon)}{l}+4D\varepsilon\right)\cdot N\right).$$

where the constant $C(l, r)$ only depends on l, r and the constant $L(\varepsilon)$ only depends on ε, not on l.

Proof Step 1 Following [12] define a measure $\nu_{a,b}^{l,T}$ on M^{l-1} as,

$$\nu_{a,b}^{l,T}(dz)=\exp\left(-E_{a,b}^l(z)/T\right)dz,$$

where $z = (z_1, \ldots, z_{l-1}) \in M^{l-1}$, $dz = \Pi_{j=1}^{l-1} dz_j$ and

$$E_{a,b}^l(z_1,\ldots z_{l-1})=\frac{l}{2}\sum_{i=0}^{l-1}d(z_i,z_{i+1})^2,\quad z_0=a,\ z_l=b,$$

Let $\overline{\nu}_{a,b}^{l,T}(dz)$ be the restriction of $\nu_{a,b}^{l,T}(dz)$ on the subset $U_{a,b}^{r,l}$ of M^{l-1} normalized to have mass 1. From [12, lemma 3.2], for each fixed $l > 0$,

$$\overline{\lim_{T\downarrow 0}}\sup_{a,b\in M}\sup_{U_{a,b}^{r,l}}\operatorname{osc}\left(d\mu_{a,b}^{l,T}/d\nu_{a,b}^{l,T}\right)\leqslant C(l,r),$$

So, there is a $T(l,r) > 0$ such that for any $\frac{l}{N} < T(l,r)$,

$$\lambda\left(U^{r,l}_{x,a};\overline{\mu}^{l,\frac{l}{N}}_{x,a}\right) \geqslant \frac{1}{2C(l,r)}\lambda\left(U^{r,l}_{x,a};\overline{\nu}^{l,\frac{l}{N}}_{x,a}\right). \tag{4.46}$$

Step 2 As in [12], let $U^{r,l}_{a,b,\triangle} := \overline{U}^{r,l}_{a,b}/\sim$ be the one point compactification of $U^{r,l}_{a,b}$, which is obtained by identifying the boundary $\partial U^{r,l}_{a,b}$ as a single point $\triangle$. And let $\widetilde{C}([0,1];\overline{U}^{r,l}_{a,b})$ denote the path in $\overline{U}^{r,l}_{a,b}$ which is restricted to a continuous path on the space $U^{r,l}_{a,b,\triangle}$. Define

$$M^{r,l}_{a,b}(z) := \inf_{p\in \mathbf{I}^{r,l}_{a,b}} \sup_{s\in[0,1]} E^l_{a,b}(p(s)) \quad a,b\in M, \tag{4.47}$$

where $\mathbf{I}^{r,l}_{a,b} = \{p\in\widetilde{C}([0,1];\overline{U}^{r,l}_{a,b});\ p(0)=z,\ p(1)=z_0\}$ and z_0 is a minimum point of $E^l_{a,b}$ on $\overline{U}^{r,l}_{a,b}$. And define

$$m^{r,l}_{a,b} := \sup_{\overline{U}^{r,l}_{a,b}}\left(M^{r,l}_{a,b} - E^l_{a,b}\right) \tag{4.48}$$

In fact, if we take the supremum only among the local minimum points of $E^l_{a,b}$ on $\overline{U}^{r,l}_{a,b}$ in the above definition, the value of $m^{r,l}_{a,b}$ will not change, see lemma 2.1 in [12].

According to the proof of Theorem 2.2 in [12], for each $x,a\in M$, if $\frac{l}{N}$ is less than some $T(x,a,l)$,

$$\lambda\left(U^{r,l}_{x,a};\overline{\nu}^{l,\frac{l}{N}}_{x,a}\right) \geqslant C(x,a,l)\left(\frac{l}{N}\right)^{3(l-1)d-2}\exp\left(-\frac{Nm^{r,l}_{x,a}}{l}\right),\quad x\in M,$$

where d is the dimension of M.

Now our goal is to confirm that the constants $T(x,a,l)$, $C(x,a,l)$ above can be chosen to be independent of $x,a\in M$. From step by step checking the proof Theorem 2.2 in [12], if the following three conditions are true, then we can find such constants:

(a) Uniform estimate on the gradient of the energy function: there exists a constant $C(l) > 0$ depending only on l such that

$$\sup_{x,a\in M}\sup_{z\in\overline{U}^{r,l}_{x,a}} |\nabla E^l_{x,a}(z)|^2 \leqslant C(l).$$

(b) A lower bound on the size of the tube $U^{r,l}_{x,a}$: there exists a constant $\theta(l) > 0$, such that for all $R < 1$,

$$\sup_{x,a\in M}\sup_{z\in\partial U^{r,l}_{x,a}} \frac{Vol(B_R(z)/U^{r,l}_{x,a})}{Vol(B_R(z))} \geqslant \theta(l),$$

where $Vol(A)$ denotes the Riemannian volume of a subset A of M^{l-1}.

(c) If T is smaller than some $T(l) > 0$, there are finite subsets $\Sigma_T^0(x,a) \subset \partial U_{x,a}^{r,l}$ and $\Sigma_T(x,a) \subset \overline{U}_{x,a}^{r,l}$ such that
- $\Sigma_T^0(x,a) \subset \Sigma_T(x,a)$
- $\Sigma_T(x,a)$ contains a minimum point $z_0(x,a)$ of $E_{x,a}^l$.
- $\partial U_{x,a}^{r,l} \subseteq \bigcup_{z\in\Sigma_T^0(x,a)} B_T(z)$, $\overline{U}_{x,a}^{r,l} \subseteq \bigcup_{z\in\Sigma_T(x,a)} B_T(z)$.
- $\sup_{x,a\in M} \#\Sigma_T(x,a) \leqslant C(l)T^{-(l-1)d}$ for some constants $C(l)$.

where # means the number of elements in a finite set.

Since R_0 from proposition 3.2 is less than the injective radius of compact manifold M, when $r \in (0, R_0)$, $E_{x,a}^l$ is differentiable in the domain $U_{x,a}^{r,l}$ and condition (a) can be checked by direct computation. From the proof of Corollary 3.3 in [12], condition (b) is true.

For condition (c), note that there is a $T(l) > 0$, for each $T < T(l)$, due to the compactness of M^{l-1}, we can find a finite subset $\widetilde{\Sigma}_T \subseteq M$ such that $M^{l-1} \subseteq \bigcup_{z\in\widetilde{\Sigma}_T} B_T(z)$ and $\#\widetilde{\Sigma}_T \leqslant C(l)T^{-(l-1)d}$. Now since $M^{l-1} \subseteq \bigcup_{z\in\widetilde{\Sigma}_{T/2}} B_{T/2}(z)$, we start to construct the set $\Sigma_T(x,a)$ as following:

(i) if $z \in \widetilde{\Sigma}_{T/2}$ and $B_{T/2}(z) \subset U_{x,a}^{r,l}$, then add such z into $\Sigma_T(x,a)$;

(ii) if $z \in \widetilde{\Sigma}_{T/2}$ and $B_{T/2}(z) \cap \partial U_{x,a}^{r,l} \neq \varnothing$, then take a point $\tilde{z} \in B_{T/2}(z) \cap \partial U_{x,a}^{r,l}$ and add this point $\tilde{z}$ into $\Sigma_T(x,a)$.

(iii) add a minimum point $z_0(x,a)$ of $E_{x,a}^l$ on $\overline{U}_{x,a}^{r,l}$ into $\Sigma_T(x,a)$.

Since in (ii), $B_T(\tilde{z}) \supseteq B_{T/2}(z)$, we have

$$\bigcup_{\tilde{z}\in\Sigma_T(x,a)} B_T(\tilde{z}) \supseteq \bigcup_{z\in\widetilde{\Sigma}_{T/2}} B_{T/2}(z) \supseteq M^{l-1} \supseteq \overline{U}_{x,a}^{r,l}$$

$$\bigcup_{\tilde{z}\in\Sigma_T(x,a)\cap\partial U_{x,a}^{r,l}} B_T(\tilde{z}) \supseteq \bigcup_{z\in\widetilde{\Sigma}_{T/2};\ B_{T/2}(z)\cap\partial U_{x,a}^{r,l}\neq\varnothing} B_{T/2}(z) \supseteq \partial U_{x,a}^{r,l}, \quad x,a \in M$$

and $\#\Sigma_T(x,a) \leqslant \#\widetilde{\Sigma}_{T/2} + 1 \leqslant 2^{(l-1)d}C(l)T^{-(l-1)d}$, so condition (c) is satisfied.

By the above argument, we can find constants $T(l)$ and $C(l)$, which are independent of x, a and N, such that if $\frac{l}{N} < T(l)$, then

$$\lambda\left(U_{x,a}^{r,l}; \overline{\nu}_{x,a}^{l;\frac{l}{N}}\right) \geqslant C(l)\left(\frac{l}{N}\right)^{3(l-1)d-2} \exp\left(-\frac{Nm_{x,a}^{r,l}}{l}\right). \tag{4.49}$$

Step 3 In the following, we try to give some uniform estimate about $m_{x,a}^{r,l}$. As in [12], define the energy of a path $\gamma \in \Omega_{a,b}$ (possibly infinite) as:

$$E(\gamma) := \frac{1}{2}\sup\sum_{i=0}^{k-1}\frac{d(\gamma(s_i),\gamma(s_{i+1}))^2}{s_{i+1}-s_i}$$

where the supremum is obtained over all partitions $0 = s_0 < s_1 < \cdots s_k = 1$. Assume $a, b \in M$ and a is not conjugate to b, let $\Xi_{a,b}$ denote the set of all geodesics (i.e. critical points of E on $\Omega_{a,b}$), and let $\Xi_{a,b}^{\min}$ denote the subset of all local energy minimum. Fix a global energy minimum geodesic $\gamma_{a,b} \in \Omega_{a,b}$, then for each geodesic $\gamma \in \Xi_{a,b}$, we define:

$$M_{a,b}(\gamma) := \inf_{H \in \mathbf{I}} \sup_{s \in [0,1]} E \circ H(s)$$

where $\mathbf{I} = \{H \in C([0, 1], \Omega_{a,b});\ H(0) = \gamma,\ H(1) = \gamma_{a,b}\}$. And define

$$m_{a,b} := \sup \left\{ M_{a,b}(\gamma) - E(\gamma);\ \gamma \in \Xi_{a,b}^{\min} \right\}.$$

The item $m_{a,b}$ can be viewed as an infinite dimensional version of the item (4.48). Furthermore, every point $z \in U_{a,b}^{r,l}$ corresponds to a piecewise geodesic in M, so intuitively we may have more choices to take supremum in defining $M_{a,b}$ than in defining $M_{a,b}^{r,l}$ as (4.47). In fact, according to the proof of Corollary 1.5 in [12], we have, if a is not conjugate to b,

$$m_{a,b}^{r,l} \leqslant m_{a,b}, \quad r \in (0, \operatorname{inj} M), \quad l \in \mathbb{N}^+. \tag{4.50}$$

For $0 < r < R_0$, choose a $\varepsilon > 0$, satisfying with $r + \varepsilon < \operatorname{inj} M$. For any $x \in M$, $a \in M$ and $\tilde{x} \in B_\varepsilon(x)$, $\tilde{a} \in B_\varepsilon(a)$, if $z = (z_1, \ldots z_{l-1}) \in U_{\tilde{x},\tilde{a}}^{r,l}$, then

$$d(z_1, x) \leqslant d(x, \tilde{x}) + d(z_1, \tilde{x}) < r + \varepsilon \quad d(z_{l-1}, a) \leqslant d(a, \tilde{a}) + d(z_{l-1}, \tilde{a}) < r + \varepsilon$$
$$\text{and } d(z_i, z_{i+1}) < r,\ 1 \leqslant i \leqslant l - 2$$

which means $z \in U_{x,a}^{r+\varepsilon,l}$, hence we have $\overline{U}_{\tilde{x},\tilde{a}}^{r,l} \subseteq U_{x,a}^{r+\varepsilon,l}$.

Suppose $z_0(\tilde{x}, \tilde{a})$ be a minimum point of $E_{\tilde{x},\tilde{a}}^l$ on $\overline{U}_{\tilde{x},\tilde{a}}^{r,l}$, and $z_0(x, a)$ be a minimum point of $E_{x,a}^l$ on $\overline{U}_{x,a}^{r+\varepsilon,l}$, by the definition of $M_{a,b}^{r,l}$ in (4.47), for each $\delta > 0$ and each $z \in \overline{U}_{\tilde{x},\tilde{a}}^{r,l} \subseteq U_{x,a}^{r+\varepsilon,l}$, there exists a path $q_1 \in \widetilde{C}([0, 1]; \overline{U}_{x,a}^{r+\varepsilon,l})$, such that $q_1(0) = z$, $q_1(1) = z_0(x, a)$, and

$$E_{x,a}^l \circ q_1(s) \leqslant E_{x,a}^l(z) + m_{x,a}^{r+\varepsilon,l} + \delta, \quad 0 \leqslant s \leqslant 1 \tag{4.51}$$

As the same reason, we can find a a path $q_2 \in \widetilde{C}([0, 1]; \overline{U}_{x,a}^{r+\varepsilon,l})$ with $q_2(0) = z_0(\tilde{x}, \tilde{a})$, $q_2(1) = z_0(x, a)$ and

$$E_{x,a}^l \circ q_2(s) \leqslant E_{x,a}^l(z_0(\tilde{x}, \tilde{a})) + m_{x,a}^{r+\varepsilon,l} + \delta, \quad 0 \leqslant s \leqslant 1. \tag{4.52}$$

Let

$$q(s) = \begin{cases} q_1(2s) & \text{if } 0 < s \leqslant \frac{1}{2}, \\ q_2(2 - 2s) & \text{if } \frac{1}{2} < s \leqslant 1 \end{cases}$$

and $\tau = \inf\{s; q(s) \in \partial U^{r,l}_{\tilde{x},\tilde{a}}\} \wedge 1$, $\hat{\tau} = \sup\{s; q(s) \in \partial U^{r,l}_{\tilde{x},\tilde{a}}\} \vee 1$. Define

$$\widetilde{q}(s) = \begin{cases} q(s) & \text{if } s \in [0,\tau) \cup (\hat{\tau}, 1], \\ q(\hat{\tau}) & \text{if } s \in [\tau, \hat{\tau}]. \end{cases}$$

Then $\tilde{q} \in \widetilde{C}([0,1]; \overline{U}^{r,l}_{\tilde{x},\tilde{a}})$ and $\widetilde{q}(0) = z$, $\widetilde{q}(1) = z_0(\tilde{x}, \tilde{a})$. Note that for each $z \in \overline{U}^{r,l}_{\tilde{x},\tilde{a}}$,

$$\begin{aligned} &|E^l_{\tilde{x},\tilde{a}}(z) - E^l_{x,a}(z)| \\ &\quad = \left| \frac{l(d(z_1,x)^2 - d(z_1,\tilde{x})^2)}{2} + \frac{l(d(z_{l-1},a)^2 - d(z_{l-1},\tilde{a})^2)}{2} \right| \\ &\quad \leqslant (d(a,\tilde{a}) + d(x,\tilde{x}))Dl \leqslant 2lD\varepsilon \end{aligned} \tag{4.53}$$

where D is the diameter of the manifold M. Then, by (4.51), (4.52), (4.53) and the definition of $\tilde{q}$, we have

$$\begin{aligned} E^l_{\tilde{x},\tilde{a}} \circ \widetilde{q}(s) &\leqslant E^l_{x,a} \circ \widetilde{q}(s) + 2lD\varepsilon \\ &\leqslant \max\left\{E^l_{x,a}(z), E^l_{x,a}(z_0(\tilde{x},\tilde{a}))\right\} + m^{r+\varepsilon,l}_{x,a} + \delta + 2lD\varepsilon \\ &\leqslant \max\left\{E^l_{\tilde{x},\tilde{a}}(z), E^l_{\tilde{x},\tilde{a}}(z_0(\tilde{x},\tilde{a}))\right\} + m^{r+\varepsilon,l}_{x,a} + \delta + 4lD\varepsilon \\ &= E^l_{\tilde{x},\tilde{a}}(z) + m^{r+\varepsilon,l}_{x,a} + \delta + 4lD\varepsilon, \quad 0 \leqslant s \leqslant 1. \end{aligned}$$

The equality in the last step above is due to the fact that $z_0(\tilde{x}, \tilde{a})$ is a minimum point of $E^l_{\tilde{x},\tilde{a}}$ on $\overline{U}^{r,l}_{\tilde{x},\tilde{a}}$. Thus, according to the above inequality and the definition of $M^{r,l}_{\tilde{x},\tilde{a}}$, and by the arbitrary of δ, we obtain $M^{r,l}_{\tilde{x},\tilde{a}}(z) \leqslant E^l_{\tilde{x},\tilde{a}}(z) + m^{r+\varepsilon,l}_{x,a} + 4lD\varepsilon$. Hence, by this (4.50) and the definition of $m^{r,l}_{\tilde{x},\tilde{a}}$, when a is not conjugate to x, $d(x,\tilde{x}) < \varepsilon$ and $d(a,\tilde{a}) < \varepsilon$, we have

$$m^{r,l}_{\tilde{x},\tilde{a}} \leqslant m^{r+\varepsilon,l}_{x,a} + 4lD\varepsilon \leqslant m_{x,a} + 4lD\varepsilon. \tag{4.54}$$

By [12, Theorem 1.4], when M is a compact simply connected manifold with strict positive Ricci curvature, we have $m_{a,b} < \infty$ for each pair of $a, b \in M$ if a is not conjugate to b. Since for any $\varepsilon > 0$, $a \in M$, there exists a finite set $\Theta_{\varepsilon,a} \subseteq \{x \in M : x \text{ is not conjugate to } a\}$ such that $\bigcup_{x\in\Theta_{\varepsilon,a}} B_\varepsilon(x) \supseteq M$, then by (4.54), for each $a, b \in M$ with $d(a,b) < \varepsilon$,

$$\sup_{y\in M} m^{r,l}_{y,b} \leqslant \sup_{x\in\Theta_{\varepsilon,a}} m_{x,a} + 4lD\varepsilon. \tag{4.55}$$

As the same way, there is a finite set Θ_ε, such that $\bigcup_{x\in\Theta_\varepsilon} B_\varepsilon(x) \supseteq M$, by (4.54) and (4.55),

$$\sup_{y,b\in M} m^{r,l}_{y,b} \leqslant \sup_{a\in\Theta_\varepsilon} \sup_{x\in\Theta_{\varepsilon,a}} m_{x,a} + 4lD\varepsilon. \tag{4.56}$$

Let

$$L(\varepsilon) := \sup_{a\in\Theta_\varepsilon} \sup_{x\in\Theta_{\varepsilon,a}} m_{x,a} < +\infty.$$

So, by (4.46), (4.49) and (4.56), if $\frac{l}{N}$ less than some $T(l,r)$, then

$$\inf_{x,a\in M} \lambda\left(U^{r,l}_{x,a}; \overline{\mu}^{l,\frac{l}{N}}_{x,a}\right) \geqslant \frac{C(l,r)}{N^{C(l,r)}} \exp\left(-\left(\frac{L(\varepsilon)}{l} + 4D\varepsilon\right)\cdot N\right). \tag{4.57}$$

where constant $C(l,r)$ only depends on l, r, by now we have completed the proof. □

5 The main theorem

Theorem 5.1 *Let M be a simply connected compact manifold with strict positive Ricci curvature. For any small $\alpha > 0$, there exists a constant $s_0 > 0$ such that the following weak Poincaré inequality holds, i.e.*

$$\mathbf{Var}(F;\mathbf{P}_{a,a}) \leqslant \frac{1}{s^\alpha}\mathscr{E}_{a,a}(F,F) + s||F||^2_\infty, \quad s\in(0,s_0), \quad F\in\mathscr{D}(\mathscr{E}_{a,a}). \tag{5.58}$$

The constants s_0 does not depend on the starting point $a \in M$.

Proof It suffices to show that (5.58) holds for $F \in \mathcal{F}C^\infty_b(\Omega_{a,a})$. Let $\omega_i(s) := \omega(\frac{i-1+s}{N})$ for each $\omega \in \Omega_{a,a}$. For a function $F \in \mathcal{F}C^\infty_b(\Omega_{a,a})$, as in the proof of Proposition 3.2, there is a unique function $F^{[N]}$ defined on $\bigcup_{(x_1,\ldots,x_{N-1})\in M^{N-1}} \Pi^N_{i=1}\Omega_{x_{i-1},x_i}$ such that,

$$F^{[N]}(\omega_1,\omega_2,\ldots,\omega_N) = F(\omega), \quad \omega\in\Omega_{a,a},$$

Step 1 We fix $N > N_2(\eta,l,r)$ with $l > N_1(\eta,r)$, for $N_1(\eta,r)$ and $N_2(\eta,l,r)$ as given in Proposition 4.1. We first assume $F(\omega) = 0$ if $(\omega(1/N),\omega(2/N),\ldots\omega(1-1/N))$ is not in $U^{r,N}_{a,a}$. For $(x_1,\ldots,x_{N-1})\in M^{N-1}$,

$$\begin{aligned} f^{[N]}(x_1,x_2,\ldots x_{N-1}) &:= \int F^{[N]}\prod_{i=1}^N \mathbf{P}^{\frac{1}{N}}_{x_{i-1},x_i}(d\omega_i) \\ &= \mathbf{E}_{a,a}\left[F(\omega)\,\middle|\,\omega\left(\frac{1}{N}\right)=x_1,\ldots,\omega\left(\frac{N-1}{N}\right)=x_{N-1}\right]. \end{aligned}$$

Let $\Im_N := \sigma\{\omega(i/N),\ 1 \leqslant i \leqslant N-1\}$, an σ- algebra on $\Omega_{a,a}$. Then we have,

$$\begin{aligned}
&\mathbf{Var}(F;\mathbf{P}_{a,a})\\
&\quad= \mathbf{E}_{a,a}\left[(F-\mathbf{E}_{a,a}[F|\Im_N])^2\right] + \mathbf{E}_{a,a}\left[(\mathbf{E}_{a,a}[F|\Im_N]-\mathbf{E}_{a,a}[F])^2\right]\\
&\quad= \int\limits_{M^{N-1}} \mathbf{Var}\left(F^{[N]};\otimes_{i=0}^{N-1}\mathbf{P}^{\frac{1}{N}}_{x_i,x_{i+1}}\right)d\mu^{N,1}_{a,a} + \mathbf{Var}\left(f^{[N]},\mu^{N,1}_{a,a}\right)\\
&\quad\leqslant \int\limits_{U^{r,N}_{a,a}} \left\{\sum_{j=1}^{N}\int \mathbf{Var}_j\left(F^{[N]};\mathbf{P}^{\frac{1}{N}}_{x_{j-1},x_j}\right)\prod_{i\neq j}\mathbf{P}^{\frac{1}{N}}_{x_{i-1},x_i}(d\omega_i)\right\}\mu^{N,1}_{a,a}(dx)\\
&\qquad+\mathbf{Var}\left(f^{[N]};\mu^{N,1}_{a,a}\right).
\end{aligned} \tag{5.59}$$

Here $\mathbf{Var}_j$ indicates the variance is with respect to the jth subpath. Note that $f^{[N]}$ is smooth with support in $\overline{U}^{r,N}_{a,a}$ and $||f^{[N]}||_\infty \leqslant ||F||_\infty$. From Proposition 4.1, if $N > N_2(\eta,l,r)$, then

$$\begin{aligned}
&\mathbf{Var}\left(f^{[N]};d\mu^{N,1}_{a,a}\right)\\
&\quad\leqslant C(l,r)N^{C(l,r)}\mathrm{e}^{2N\eta r^2}\sum_{i=1}^{N-1}\int\limits_{M^{N-1}}|\nabla_i f^{[N]}|^2\mathrm{d}\mu^{N,1}_{a,a}\\
&\qquad+C(l,\eta,r)N^{C(l,r)}\mathrm{e}^{\frac{-N(1-8\eta)r^2}{2}}||F||^2_\infty.
\end{aligned} \tag{5.60}$$

According to the proof of lemma 6.1 and lemma 6.2 in [11] (since the support of $f^{[N]}$ is in $\overline{U}^{r,N}_{a,a}$, we can choose some vector with better asymptotic property in the estimate of the derivative of expectation with pinned Wiener measure as in Lemma 2.2), there exists a constant $C(r)$, such that

$$\begin{aligned}
&\sum_{i=1}^{N-1}\int|\nabla_i f^{[N]}|^2\mathrm{d}\mu^{N,1}_{a,a} \leqslant C(r)N\mathbf{E}_{a,a}|\mathbf{D}_0F|^2_{\mathbf{H}^0_\omega}\\
&\quad+C(r)N\int\limits_{U^{r,N}_{a,a}}\left\{\sum_{j=1}^{N}\int\mathbf{Var}_j\left(F^{[N]};\mathbf{P}^{\frac{1}{N}}_{x_{j-1},x_j}\right)\prod_{i\neq j}\mathbf{P}^{\frac{1}{N}}_{x_{i-1},x_i}(d\omega_i)\right\}\mu^{N,1}_{a,a}(dx).
\end{aligned} \tag{5.61}$$

By Proposition 3.2, if $\frac{1}{N} < T_0(\eta, r)$, then

$$\int\limits_{U_{a,a}^{r,N}} \left\{ \sum_{j=1}^{N} \int \mathbf{Var}_j \left(F^{[N]}; \mathbf{P}_{x_{j-1},x_j}^{\frac{1}{N}} \right) \prod_{i \neq j} \mathbf{P}_{x_{i-1},x_i}^{\frac{1}{N}}(d\omega_i) \right\} \mu_{a,a}^{N,1}(dx_1, \ldots dx_{N-1})$$
$$\leqslant \frac{C(r)}{N} \int \sum_{j=1}^{N} |\mathbf{D}_{0,(j)} F^{[N]}(\omega_1, \ldots \omega_N)|_{\omega_j}^2 \mathbf{P}_{a,a}(d\omega)$$
$$+ NC(\eta, r) \mathrm{e}^{-\frac{(1-4\eta)Nr^2}{2}} ||F||_\infty^2, \tag{5.62}$$

where $\mathbf{D}_{0,(j)}$ means the gradient $\mathbf{D}_0$ of $F^{[N]}$ with respect to the jth subpath. According to (6.16) in the proof of Lemma 6.3 in [11], we have the following relation,

$$\sum_{j=1}^{N} |\mathbf{D}_{0,(j)} F^{[N]}(\omega_1, \ldots \omega_N)|_{\omega_j}^2 \leqslant N |\mathbf{D}_0 F|_{\mathbf{H}_\omega^0}^2, \quad \omega \in \Omega_{a,a}. \tag{5.63}$$

By (5.59)–(5.63), if $N > N_2(\eta, l, r)$ with $l > N_1(\eta, r)$, then

$$\mathbf{Var} \left(F; \mathbf{P}_{a,a}^1 \right) \leqslant C(l, r) N^{C(l,r)} \mathrm{e}^{2N\eta r^2} \mathbf{E}_{a,a} |\mathbf{D}_0 F|_{\mathbf{H}_\omega^0}^2$$
$$+ C(l, \eta, r) N^{C(l,r)} \mathrm{e}^{\frac{-N(1-8\eta)r^2}{2}} ||F||_\infty^2. \tag{5.64}$$

Step 2 Consider a general function F from $\mathcal{F}C_b^\infty(\Omega_{a,a})$. Define a smooth cut-off function on $\Omega_{a,a}$ as,

$$\Psi_N(\omega) := \prod_{i=1}^{N} \varphi \left(d \left(\omega \left(\frac{i-1}{N} \right), \omega \left(\frac{i}{N} \right) \right) \right)$$

where φ is defined as in the proof of Lemma 3.1. By the proof of Lemma 3.1, if $\frac{1}{N} < T_0(\eta, r)$,

$$\mathbf{P}_{a,a}(\Psi_N \neq 1) \leqslant N \exp \left(-\frac{(1-4\eta)Nr^2}{2} \right), \quad |\mathbf{D}_0 \Psi_N(\omega)|_{\mathbf{H}_\omega^0} \leqslant \frac{6N}{\eta r}. \tag{5.65}$$

Then note that $F\Psi_N(\omega) = 0$ when $(\omega(1/N), \omega(2/N), \ldots \omega(1-1/N))$ is not in $U_{a,a}^{r,N}$, hence by (5.64) and (5.65), if $N > N_2(\eta, l, r)$ with $l > N_1(\eta, r)$, we obtain

$$\begin{aligned}
&\mathbf{Var}(F;\mathbf{P}_{a,a})\\
&\leqslant \mathbf{Var}(F\Psi_N;\mathbf{P}_{a,a})+3\mathbf{P}_{a,a}(\Psi_N\neq 1)||F||_\infty^2\\
&\leqslant C(l,r)N^{C(l,r)}\mathrm{e}^{2N\eta r^2}\mathbf{E}_{a,a}|\mathbf{D}_0(F\Psi_N)|^2_{\mathbf{H}^0_\omega}+C(l,\eta,r)N^{C(l,r)}\mathrm{e}^{\frac{-N(1-8\eta)r^2}{2}}||F||_\infty^2\\
&\leqslant C(l,r)N^{C(l,r)}\mathrm{e}^{2N\eta r^2}\mathbf{E}_{a,a}|\mathbf{D}_0F|^2_{\mathbf{H}^0_\omega}+C(l,\eta,r)N^{C(l,r)}\mathrm{e}^{-\frac{N(1-8\eta)r^2}{2}}||F||_\infty^2.
\end{aligned} \tag{5.66}$$

Let $s:=C(l,\eta,r)N^{C(l,r)}\mathrm{e}^{-\frac{(1-8\eta)Nr^2}{2}}$ in (5.66), then s tends to zero when N tends to infinity. In particular for any small $\alpha>0$ we can choose a η small enough so that there is a constant $s_0(\eta,r,l,\varepsilon,\alpha)$, s_0 does not depend on the starting point a of the loop space, such that,

$$\mathbf{Var}(F;\mathbf{P}_{a,a})\leqslant\frac{1}{s^\alpha}\mathscr{E}_{a,a}(F,F)+s||F||_\infty^2,\quad s\in(0,s_0),\ F\in\mathscr{D}(\mathscr{E}_{a,a}).$$

By now we have completed the proof. □

Remark (1) The algebraic rate of blowing up in this weak Poincaré inequality is the consequence of the exponential growth in N of the constant $\xi(N)$ for the Dirichlet form item in the variance estimate for the discretized Brownian bridge measure on the product space M^{N-1}. (cf. Proposition 4.1, where $\xi(N)=C(l,r)N^{C(l,r)}\mathrm{e}^{2N\eta r^2}$). In one step of the proof of Proposition 4.1, we derive an estimate by comparison with the path space where Poincaré inequality is known to hold. This estimate is not sharp. In fact, we think that maybe for some manifolds with special properties, a better estimate of $\xi(N)$ can be obtained by using these special properties (e.g. symmetric property of the sphere). A better estimate of $\xi(N)$ would result in a lower blowing up rate in the weak Poincaré inequality.

(2) We take the exhausting local sets $\{O_{r,N}\}$ to prove the weak Poincaré inequality, where

$$O_{r,N}=\left\{\omega\in\Omega_{a,a}:\sup_{0\leqslant i\leqslant N-1}\mathrm{d}\left(\omega\left(\frac{i}{N}\right),\omega\left(\frac{i+1}{N}\right)\right)<r\right\}$$

and $\bigcup_{N=N_0}^\infty O_{r,N}=\Omega_{a,a}$ for each $N_0>0$. Note that these exhausting local sets are slightly different from that taken by Eberle in [11]. We make this choice just for technical reason. It is easier to estimate the probability of the complement of these exhausting local sets.

(3) The positive Ricci curvature condition in the main theorem can be replaced by $m_{x,y}<+\infty$ for each $x,y\in M$ satisfying that x is not conjugate to y. This condition is crucial to obtain the estimate in Proposition 4.1. A better understanding of the geometric meaning of $m_{x,y}$ would be useful to get a sharper estimate for some explicit manifolds. And from the analysis of the Section 1.4 of [12], for a compact simply connected manifold M and x,y in M not conjugate with each other, $m_{x,y}<+\infty$ if there only exist a finite number of geodesics connecting

x and y, which are the local minimum of the energy functional on $\Omega_{x,y}$. In particular, by the proof of Theorem 1.4 in [12], if M is compact simply connected and has strict positive Ricci curvature, the above assumption is satisfied and $m_{x,y} < +\infty$ for any $x, y \in M$ satisfying that x is not conjugate to y.

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