Integration by parts formulae for degenerate diffusion measures on path spaces and diffeomorphism groups

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Abstract

Integration by parts formulae are given for a class of measures on the space of paths of a smooth manifold M determined by the laws of degenerate diffusions. The mother of such formulae, on the path space of diffeomorphism group of M is shown to arise from a quasi-invariance property of measures determined by stochastic flows. From this the other formulae are derived by filtering out redundant noise using an associated LeJan-Watanabe connection.

Resumé: Des formules d'intégration par parties sont données pour les mesures sur l'espace des chemins associeés des E.D.S, eventuellement dégénérées. La formule mére est donnée sur l'espace des chemins dans le groupe des difféomorphismes de la variété. Elle procéde de la quasi invariance des lois du flot stochastique. La formule sur l'espace des chemins qui en dérive est obtenue en filtrant le bruit redondant à l'aide de la connexion intraduite par S. Watanabe et le deuxième auteur.

A. Consider the equation

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt \tag{1}$$

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on a compact C^∞ manifold M. Here $\{B_t : t \geq 0\}$ is a Brownian motion on ^{m} and A is a C^∞ vector field while $X : M \times {}^m \to TM$ is C^∞ and gives a linear map $X(x, \cdot) = X(x)(\cdot) : {}^m \to T_x M$ for each $x \in M$.

Let $\{\xi_t : t \geq 0\}$ be the solution flow to (1) with $T_x\xi_t : T_x M \to T_{\xi_t(x)} M$ its derivative map at $x \in M$. For fixed $T > 0$ suppose k is an element of the Cameron-Martin space $H = L_0^{2,1}([0,T]; {}^m)$ and consider the random, time dependent O.D.E. on M, parametrized by $\tau \in$:

$$\begin{array}{rcll} \frac{\partial}{\partial t} H_t^\tau(x) & = & \tau \left(T_x\xi_t\right)^{-1} X\left(\xi_t(H_t^\tau(x))\right) \dot{k}_t, & 0 \leq t \leq T, \\ H_0^\tau(x) & = & x, & x \in M. \end{array} \tag{2}$$

The solution exists for $0 \leq t \leq T$ and we perturb our flow to ξ_t^τ given by

$$\xi_t^\tau(x) = \xi_t(H_t^\tau(x)), \qquad 0 \leq t \leq T.$$

Then $\{\xi_t^\tau(x) : 0 \leq t \leq T\}$ is a semi-martingale satisfying

$$d\xi_t^\tau(x) = X\left(\xi_t^\tau(x)\right) \circ dB_t + A(\xi_t^\tau(x))dt + \tau X(\xi_t^\tau(x))\dot{k}_t dt \tag{3}$$

(It is given by replacing $B.$ by $B. + \tau k$ in (1)) and so the laws of $\xi_\cdot^\tau(x)$, $\tau \in$, are mutually equivalent by the Cameron-Martin theorem. By differentiating the resulting two expressions for $F(\xi_\cdot^\tau(x))$ with respect to τ at $\tau = 0$ in the usual way, e.g. as in [EL] we see

Let $F : C_x([0,T]; M) \to$ be C^1 with bounded derivative. Then

$$dF\left(T\xi_\cdot \int_0^\cdot (T\xi_s)^{-1} \left(X(\xi_s(x)\dot{k}_s\right) ds\right) = F(\xi_\cdot(x)) \int_0^T < \dot{k}_s, dB_s >_{m}. \tag{4}$$

This formula could also be deduced from the usual integration by parts formula on $C_0([0,T]; {}^m)$ as in [Bis81]. However the above argument also shows that *the law $\mu_{\mathcal{D}}$ of $\xi_\cdot$ on the space of paths $C_{id}([0,T]; DiffM)$ into the group of diffeomorphisms of M is quasi-invariant under the transformations $\xi_\cdot \to \xi_\cdot(H_\cdot^\tau)$ with $H_\cdot^\tau$ as above, $\tau \in$, $k_\cdot \in H$.* Let $\mathcal{H}$ be the reproducing kernel Hilbert space of vector fields on M for the laws of the random field $X(\cdot)B_1$. Then $K_\cdot(-) = X(-)k_\cdot$ is an arbitrary element of $L_0^{2,1}([0,T]; \mathcal{H})$. Let $\{W_t : 0 \leq t \leq T\}$ be the martingale part of $\{\int_0^t d\xi_s \circ \xi_s^{-1} : 0 \leq t \leq T\}$. Then $dW_t = X(\cdot)dB_t$ and we obtain:

$$
\begin{aligned}
&\int_{C_{id}\left([0,T];\mathrm{Diff}_M\right)} dF\left(T\theta. \int_0^{\cdot} ad(\theta_s^{-1})\dot{K}_s ds\right) \mu_{\mathcal{D}}(d\theta) \\
&= \int_{C_{id}\left([0,T];\mathrm{Diff}_M\right)} F(\theta.) \int_0^T < \dot{K}_s, dW_s >_{\mathcal{H}} \mu_{\mathcal{D}}(d\theta)
\end{aligned}
\tag{5}
$$

for any C^1 map $F : C_{id}([0,T]; DiffM) \to$ with suitably bounded derivative dF. Here $ad(\theta_s^{-1})$ refers to the adjoint action of θ_s^{-1} on the space of vector fields of M considered as the Lie algebra of Diff M.

Remarks: (i): In (4), (5) and below we could equally well take $k.$ or $K.$ to be the adapted processes with sample paths in $L_0^{2,1}$, and in $L^{1+\epsilon}$ as $L_0^{2,1}$-valued functions for some $\epsilon > 0$, c.f [EL].

(ii). for a general stochastic flow of diffeomorphisms of adequate regularity formula (5) remains valid but $\mathcal{H}$ may be infinite dimensional. Indeed there is an analogous formula for any right invariant system on a Hilbert manifold with sufficiently regular group structure.

(iii). Equation (2) defining $H_t^\tau, 0 \le t \le T$, is just the right invariant equation on Diff$M$ corresponding to the element $ad(T\xi_t)^{-1}(\dot{K}_t)$. In general it is a function of $\xi.$, but if there exists V in $\mathcal{H}$ which commutes with all other elements of $\mathcal{H}$ then setting $K_t = tv$ we obtain an $H_\cdot^\tau$, right multiplication by which leaves $\mu_{\mathcal{D}}$ quasi-invariant .

B. Equation(4) is not an integration by parts formula on $C_x([0,T]; M)$, strictly speaking, since we do not have a vector field on $C_x([0,T]; M)$ and there is possibly redundant noise coming from lack of injectivity of X. To put (4) in correct form we need the conditional expectation

$$
\bar{V}_t^k; = \quad \left\{ T\xi_t \int_0^t (T\xi_s)^{-1} \left(X(\xi_s(x)\dot{K}_s\right) ds \,\middle|\, \sigma\left\{\xi_s(x) : 0 \le s \le T\right\} \right\}. \tag{6}
$$

For simplicity here we assume that $X(x)$ has constant rank (non-singularity). Then X has image a subbundle $I(X)$ of TM on which X induces (i) a Riemannian metric $\{<,>_x, x \in M\}$ and (ii) an affine connection $\check{\nabla}$, which is a metric connection for this metric (the LeJan-Watanabe connection). If $Y(x) : I(X)_x \to \quad ^m$ is the adjoint of $X(x) : \quad ^m \to I(X)_x$ these satisfy

$$
X(x)Y(x)v = v, \quad v \in T_x M
$$

and

$$
\check{\nabla}_v Z = X(x) d\left(Y(\cdot)Z(\cdot)\right) v
$$

for all $v \in T_xM$ and Z a section of $I(X)$. There is also an 'adjoint semi-connection' $\hat{\nabla}$ to $\check{\nabla}$ which gives a derivative $\hat{\nabla}_{Z_2} Z_1$ of a vector field Z_1 in the direction of a section Z_2 of $I(X)$ by

$$\hat{\nabla}_{Z_2} Z_1 = \check{\nabla}_{Z_1} Z_2 - [Z_1, Z_2].$$

Let $v_0 \in T_xM$ and set

$$\bar{v}_t = \{T\xi_t(v_0) | \, \sigma \{\xi_s(x) : 0 \leq s \leq T\}\}.$$

Then, from [ELL] following [ELJL95] and [EY93], if $A(x) \in I(X)_x$ for each $x \in M$ then $\{\bar{v}_t : 0 \leq t \leq T\}$ satisfies

$$\frac{\hat{D}\bar{v}_t}{\partial t} = -\frac{1}{2}\check{\mathrm{Ric}}^{\#}(\bar{v}_t) + \check{\nabla} A(\bar{v}_t) \tag{7}$$

where $< \check{\mathrm{Ric}}^{\#}(u_1), u_2 >= \sum_{i=1}^r < \check{R}(e_i, u_1)u_2, e_i >_x$ for

$$\check{R} : TM \bigoplus TM \to \mathcal{L}\left(I(X); I(X)\right)$$

the curvature tensor of $\check{\nabla}$ and $e_1, \ldots, e_r$ an orthonormal basis for $I(X)_x$ where $u_1 \in T_xM$, $u_2 \in I(X)_x$ and $I(X)_x$ is identified with its image in T_xM. This can also be written as

$$D\bar{v}_t = \nabla X(\bar{v}_t)\left(\tilde{//}_t \circ d\tilde{B}_t\right) + \nabla A(\bar{v}_t)dt - \frac{1}{2}\mathrm{Ric}(\bar{v}_t)dt \tag{8}$$

using the Levi-civita connection ∇ for any Riemannian metric on M which extends that of $I(X)$. Here $\tilde{//}_t \circ d\tilde{B}_t = Y\left(\xi_t(x)\right) X\left(\xi_t(x)\right) \circ dB_t$ and can be represented by a parallel translation of the differential $\mathrm{d}\check{B}_t$ of the martingale part of the stochastic anti-development of $\{\xi_t(x) : 0 \leq t \leq T\}$. Finally we have, from (4): *For any given $v_0 \in T_xM$ let $\{W_t^A(v_0) : 0 \leq t \leq T\}$ be the solution of (8), or equivalently of (7) if $A(y) \in I(y)$ for each $y \in M$. Let μ_x be the law of $\{\xi_t(x) : 0 \leq t \leq T\}$ on $C_x([0,T]; M)$. Then*

$$\begin{aligned}&\textstyle\int_{C_x([0,T];M)} dF\left(W_\cdot^A \int_0^\cdot \left(W_s^A\right)^{-1}\left(X(\sigma(s))\dot{K}_s\right) ds\right)\mu_x(d\sigma))\\ &= \textstyle\int_{C_x([0,T];M)} F(\sigma) \int_0^T \left\langle X(\sigma(s))\dot{K}_s, \check{//}_s d\check{B}_s\right\rangle_{T_{\sigma(s)}M} \mu_x(dx)\end{aligned} \tag{9}$$

for any C^1 map $F : C_x([0,T]; M) \to$ with bounded derivative and any $K_\cdot \in L_0^{2,1}([0,T]; \;^m)$.

Remarks: (i) For the law of $\xi_\cdot(x)$ for a more general stochastic flow of diffeomorphisms, as described in §A formula (9) remains valid with $X(\sigma(s))\dot{k}_s$ replaced by $\dot{K}_s(\sigma(s))$.

(ii). k (or K) can be non-anticipating processes as in Remark (i) of §A.

(iii). For an analogous approach for non-degenerate equations see [EL], for LeJan-Watanabe connections in the non-degenerate case see [ELJL95]; the degenerate case with detailed discussions of these results and of the singular case will appear in [ELL].

(iv) Following on from remark (ii) in section §A note this formulation extends to loop spaces if one adapts the point of view used by Driver in [Dri95] for loops on Lie groups. Replacing B_t by a centred Gaussian process $B_{t,\alpha}, t \in \ ^+, \alpha \in (0,1)$ with covariance $\left(B^i_{t,\alpha} B^j_{s,\beta}\right) = \delta_{i.j} s \wedge t(\alpha \wedge \beta - \alpha\beta)$, one gets a process $\xi_{t,\alpha}$ which represents a Brownian motion in the group of based loops in the diffeomorphism group, acting naturally on pinned loops of M. Taking into account the α dependence, integration by parts formulae (4), (5) and (9) extend naturally to this situation.

Version francaise abreégeé

A. Considerons l'equation diffèrentielle stochastique sur une variété compacte M. $\{B\}$ designe un mouvement brownien sur $\ ^m$ et A et un champ de vecteurs C^∞ tandis que $X : \ ^m \times M \to TM$ est C^∞ et donne une application lineaire $X(x) : \ ^m \to T_x M$ pour tout $x \in M$.

Soit $\{\xi_t^\tau\}$ le flot stochastique solution de

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt \tag{10}$$

et $T_x\xi_t : T_x M \to T_{\xi_t(x)} M$ sa derivée en $x \in M$. Pour $T > 0$ fixé, soit k un element de l'espace de Cameron Martin $H = L_0^{2,1}([0,T]; \ ^m)$. Considerons l'equation différentielle ordinaire sure M paramétrée par $\tau \in \ $.

$$\begin{array}{rcll} \frac{\partial}{\partial t} H_t^\tau(x) & = & \tau \left(T_x\xi_t\right)^{-1} X\left(\xi_t(H_t^\tau(x))\right) \dot{k}_t, & 0 \le t \le T, \\ H_0^\tau(x) & = & x, & x \in M. \end{array} \tag{11}$$

La solution existe pour $0 \le t \le T$. Definissons le flot perturbé ξ_t^τ par

$$\xi_t^\tau(x) = \xi_t(H_t^\tau(x)), \qquad 0 \le t \le T.$$

Alors $\{\xi_t^\tau\}$ est une semi-martingale v erifiant (3). Les lois des $\xi_\cdot^\tau(x)$ sont equivalentes. En différentiant par rapport à τ en $\tau = 0$ (cf par exemple [EL]) on voit que:

Soit $F : C_x([0,T];M)$ *supposée* C^1 *et à derivées bouneés. Alors*

$$dF\left(T\xi_\cdot \int_0^\cdot (T\xi_s)^{-1}\left(X(\xi_s(x)\dot{k}_s\right) ds\right) = \quad F(\xi_\cdot(x)) \int_0^T < \dot{k}_s, dB_s >_{\ m} . \tag{12}$$

Cette formule peut être deduite de la formule usuelle d'integration par parties sur $C_0([0,T]; \quad ^m)$ comme dans [Bis81]. Cependant, l'argument précédent montre aussi que *la loi* $\mu_{\mathcal{D}}$ *de* $\xi_\cdot$ *sur l'espace des trajectoines* $C_{id}([0,T]; DiffM)$ *dans le groupe des diffeomorphismes de* M *est quasi invariante par les transformations* $\xi_\cdot \to \xi_\cdot(H_\cdot^\tau)$*, où* $H_\cdot$ *est défini ci dessus,* $\tau \in \quad$ *,* $k_\cdot \in H$. Soit $\mathcal{H}$ l'espace auto–reproduisant de champs de vecteurs sur M associé au champ gaussien $X(\cdot)B_1$. $K_\cdot(-) = X(-)k_\cdot$ est alors un element arbitraire de $L_0^{2,1}([0,T];\mathcal{H})$. Soit $\{W_t : 0 \le t \le T\}$ le partie martingale de $\{\int_0^t d\xi_s \circ \xi_s^{-1} : 0 \le t \le T\}$. Alores $dW_t = X(\cdot)dB_t$ et nous obtenons:

$$\begin{array}{l} \int_{C_{id}\left([0,T];\mathrm{Diff}_M\right)} dF\left(T\theta_\cdot \int_0^\cdot ad(\theta_s^{-1})\dot{K}_s ds\right) \mu_{\mathcal{D}}(d\theta) \\ = \int_{C_{id}\left([0,T];\mathrm{Diff}_M\right)} F(\theta_\cdot) \int_0^T < \dot{K}_s, dW_s >_{\mathcal{H}} \mu_{\mathcal{D}}(d\theta) \end{array} \tag{13}$$

pour toute application C^1 $F : C_{id}([0,T]; DiffM) \to \quad$ *possédant une derivée adéquatement bornée.* On désigne par $ad(\theta_s^{-1})$ la représentation adjointe de f θ_s^{-1} sur l'espace des champs de vecteurs sur M considéré comme l'algébre de Lie de Diff M.

B. L'équation (12) n'est pas une formule d'intégration par parties sur $C_x([0,T];M)$, à proprement parler. Pour mettre (12) sous une forme plus appraprieé, il faut prendre l'espérance conditionelle

$$\bar{V}_t^k; = \quad \left\{T\xi_t \int_0^t (T\xi_s)^{-1}\left(X(\xi_s(x)\dot{K}_s\right) ds \,\middle|\, \sigma\left\{\xi_s(x) : 0 \le s \le T\right\}\right\}. \tag{14}$$

Pour simplifier, nous supposerons que le rang de $X(x)$ est constant. Alors, l'image de X est un sous-fibré $I(X)$ de TM sur lequel X induit (i) une métrique riemanniene $\{<, >_x, x \in M\}$ et (ii) une connexion affine $\{\breve{\nabla},$

qui preserve la métrique en question. Si nous notons $Y(x) : I(X)_x \to \quad {}^m$ l'application adjointe de $X(x) : \quad {}^m \to I(X)_x$, on a

$$X(x)Y(x)v = v, \quad v \in T_xM$$

et

$$\check{\nabla}_v Z = X(x)d\left(Y(\cdot)Z(\cdot)\right)v$$

pout tout $v \in T_xM$ et pour toute section Z de $I(X)$. Il y a aussi une 'semi connection adjointe' $\hat{\nabla}$ de $\check{\nabla}$ qui associe une derivee $\hat{\nabla}_{Z_2}Z_1$, à tout champ de vecteur Z_1 dans la direction Z_2 d'une section de $I(X)$, définie par

$$\hat{\nabla}_{Z_2}Z_1 = \check{\nabla}_{Z_1}Z_2 - [Z_1, Z_2].$$

Si $v_0 \in T_xM$ posons

$$\bar{v}_t = \{T\xi_t(v_0)|\,\sigma\{\xi_s(x) : 0 \leq s \leq T\}\}.$$

Alors, d'aprés [ELL] et suite à [ELJL95] et [EY93], si $A(x) \in I(X)_x$ pour tout $x \in M$, $\{\bar{v}_t : 0 \leq t \leq T\}$ verifie

$$\frac{\hat{D}\bar{v}_t}{\partial t} = -\frac{1}{2}\check{\mathrm{Ric}}^{\#}(\bar{v}_t) + \check{\nabla}A(\bar{v}_t) \tag{15}$$

où $< \check{\mathrm{Ric}}^{\#}(u_1), u_2 >= \sum_{i=1}^{r} < \check{R}(e_i, u_1)u_2, e_i >_x$,

$$\check{R} : TM \bigoplus TM \to \mathcal{L}\left(I(X); I(X)\right)$$

étant le tenseur de courbure de $\check{\nabla}$ et $e_1, \ldots, e_r$ une base orthonormale de $I(X)_x$: u_1 est un element de T_xM, u_2 un element de $I(X)_x$ et $I(X)_x$ est identifié à son image dans T_xM. On peut aussi l'écrire sous la forme

$$D\bar{v}_t = \nabla X(\bar{v}_t)\left(\tilde{//}_t \circ d\tilde{B}_t\right) + \nabla A(\bar{v}_t)dt - \frac{1}{2}\check{\mathrm{Ric}}(\bar{v}_t)dt \tag{16}$$

en utilisant la connection de Levi Civita associée à toute metrique riemanniene sur M étendant celle définie sur $I(X)$. $\tilde{//}_t \circ d\tilde{B}_t = Y\left(\xi_t(x)\right)X\left(\xi_t(x)\right) \circ dB_t$ peut être representé par transport parallèle de la differentielle $\mathrm{d}\check{B}_t$ de la partie martingale de l'antideveloppement stochastique de $\{\xi_t(x) : 0 \leq t \leq T\}$. Enfin on deduit de (12) le résultat suivant:

it Pour tout $v_0 \in T_x M$ soit $\{W_t^A(v_0) : 0 \leq t \leq T\}$ la solution de (16) (ou de (15) si $A(y) \in I(y)$ pour tout $y \in M$). Alors

$$\begin{array}{l} \int_{C_x([0,T];M)} dF\left(W_\cdot^A \int_0^\cdot \left(W_s^A\right)^{-1}\left(X(\sigma(s))\dot{K}_s\right) ds\right) \mu_x(d\sigma)) \\ = \int_{C_x([0,T];M)} F(\sigma) \int_0^T \left\langle X(\sigma(s))\dot{K}_s, \check{//}_s d\check{B}_s \right\rangle_{T_{\sigma(s)}M} \mu_x(dx) \end{array} \tag{17}$$

pour toute application C^1 F de $C_x([0,T];M)$ a derivée bornée et tout $K_\cdot \in L_0^{2,1}([0,T]; \quad ^m)$.

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