Intertwining and the Markov uniqueness problem on path spaces

K. D. Elworthy & Xue-Mei Li

Abstract

Techniques of intertwining by Itô maps are applied to uniqueness questions for the Gross-Sobolev derivatives that arise in Malliavin calculus on path spaces. In particular claims in our article [Elworthy-Li3] are corrected and put in the context of the Markov uniqueness problem and weak differentiability. Full proofs in greater generality will appear in [Elworthy-Li2].

1 Malliavin calculus on $\mathcal{C}_0\mathbb{R}^m$ and $\mathcal{C}_{x_0}M$.

1.1 Notation

Let M be a compact Riemannian manifold of dimension n. Fix $T > 0$ and x_0 in M. Let $\mathcal{C}_{x_0}M$ denote the smooth Banach manifold of continuous paths

$$\sigma : [0,T] \to M \text{ such that } \sigma_0 = x_0$$

furnished with its Brownian motion measure μ_{x_0}. However most of what follows works for a class of more general, possible degenerate, diffusion measures.

Let $\mathcal{C}_0\mathbb{R}^m$ be the corresponding space of continuous $\mathbb{R}^m$- valued paths starting at the origin, with Wiener measure $\mathbb{P}$, and let H denote its Cameron-Martin space: $H = L_0^{2,1}\mathbb{R}^m$ with inner product $\langle \alpha, \beta\rangle_H = \int_0^T \langle \dot\alpha(s), \dot\beta(s)\rangle_{\mathbb{R}^m} ds$.

As a Banach manifold $\mathcal{C}_{x_0}M$ has tangent spaces $T_\sigma M$ at each point σ, given by

$$T_\sigma M = \{v : [0,T] \to TM \,|v(0) = 0, v \text{ is continuous }, v(s) \in T_{\sigma(s)}M, s \in [0,T]\}.$$

Each tangent space has the uniform norm induced on it by the Riemannian metric of M. As an analogue of H there are the 'Bismut tangent spaces' $\mathcal{H}_\sigma$ defined by

$$\mathcal{H}_\sigma = \{v \in T_\sigma \mathcal{C}_{x_0}M \,\Big|\, /\!/_s^{-1} v(s) \in L_0^{2,1} T_{x_0}M, 0 \leqslant s \leqslant T\}$$

where $/\!/_s$ denotes parallel translation of $T_{x_0}M$ to $T_{\sigma(s)}M$ using the Levi-Civita connection.

K.D. Elworthy, Mathematics Institute, Warwick University, Coventry CV4 7AL, U.K. Email: kde@maths.warwick.ac.uk.

Xue-Mei Li, Department of Computing and mathematics, The Nottingham Trent University, Nottingham NG1, 4BU, U.K. Email: xuemei.li@ntu.ac.uk

1.2 Malliavin Calculus on $\mathcal{C}_0\mathbb{R}^m$.

To have a calculus on $\mathcal{C}_0\mathbb{R}^m$ the standard method is to choose a dense subspace, Dom(d^H), of Fréchet differentiable functions (or elements of the first chaos) in $L^2(\mathcal{C}_0\mathbb{R}^m;\mathbb{R})$. By differentiating in the H-directions we obtain the H-derivative operator d^H : Dom(d^H) $\to$ L^2 $(\mathcal{C}_0\mathbb{R}^m;\ H^*)$. By the Cameron -Martin integration by parts formula this operator is closable. Let d : Dom(d) $\to L^2(\mathcal{C}_0\mathbb{R}^m; H^*)$ be its closure and write $\mathbb{D}^{2,1}$ for its domain with its graph norm and inner product.

From work of Shigekawa and Sugita, [Sugita], $\mathbb{D}^{2,1}$ does not depend on the (sensible) choice of initial domain Dom(d^H) and moreover if a function is weakly differentiable with weak derivativative in L^2, in a sense described below, then it is in $\mathbb{D}^{2.1}$. In particular if Dom(d^H) consists of the polygonomial cylindrical functions then $\mathbb{D}^{2,1}$ contains the space BC^1 of bounded functions with bounded continuous Fréchet derivatives.

1.3 Malliavin Calculus on $\mathcal{C}_{x_0}M$.

If $f : \mathcal{C}_{x_0}M \to \mathbb{R}$ is Fréchet differentiable with differential $(df)_\sigma : T_\sigma\mathcal{C}_{x_0}M \to R$ at the point σ, define $(d^H f)_\sigma : \mathcal{H}_\sigma \to \mathbb{R}$ by restriction. Choosing a suitable domain Dom(d^H) in L^2 the integration by parts results of [Driver] imply closability and we obtain a closed operator d : Dom(d) $\subset L^2(\mathcal{C}_{x_0}M;\mathbb{R}) \to L^2\mathcal{H}$*, for $L^2\mathcal{H}^*$ the space of L^2-sections of the dual 'bundle' $\mathcal{H}^*$ of $\mathcal{H}$. Let $\mathbb{D}^{2.1}$ or $\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M;\mathbb{R})$ denote the domain of this d furnished with its graph norm and inner product. Possible choices for the initial domain Dom(d^H) include the following:

(i) C^∞ Cyl, the space of C^∞ cylindrical functions;

(ii) BC^1, the space of BC^1 bounded functions with first Fréchet derivatives bounded;

(iii) BC^∞, the space of infinitely Fréchet differentiable functions all of whose derivatives are bounded .

One fundamental question is whether such different choices of the initial domain lead to the same space $\mathbb{D}^{2,1}$. At the time of writing this question appears to still be open. There is a gap in the proof suggested in [Elworthy-Li3] as will be described in §2.3 below. However the techniques given there do show that choices (i) and (iii) above lead to the same $\mathbb{D}^{2,1}$.

From now on we shall assume that choice (i) has been taken. We use ∇ : Dom(d) $\to$ $L^2\mathcal{H}$ defined from d using the canonical isometry of $\mathcal{H}_\sigma$ with its dual space $\mathcal{H}^*_\sigma$. This requires the choice of a Riemannian structure on $\mathcal{H}$; for this see below. Let div : Dom(div) $\subset L^2\mathcal{H} \to L^2(\mathcal{C}_{x_0}M;\mathbb{R})$ denote the adjoint of $-\nabla$. Then if $f \in$ Dom(d) and $v \in$ Dom(div) we have

$$\int df(v)d\mu_{x_0} = -\int f\,\mathrm{div}(v)d\mu_{x_0} = \int \langle \nabla f, v\rangle.d\mu_{x_0}.$$

Using these we get the self-adjoint operator Δ defined to be div ∇. Another basic open question is whether this is essentially self-adjoint on C^∞ Cyl. From the point of view of stochastic analysis it would be almost as good for it to have Markov Uniqueness. Essentially this means that there is a unique diffusion process on $\mathcal{C}_{x_0}M$ whose generator

$\mathcal{A}$ agrees with Δ on C^∞ cylindrical functions, see [Eberle]. Another characterisation of this is given below.

Finally there is the question of the existence of 'local charts' for $\mathcal{C}_{x_0}M$ which preserve, at least locally, this sort of differentiability. The stochastic development maps $\mathfrak{D} : \mathcal{C}_0\mathbb{R}^m \to \mathcal{C}_{x_0}M$ appear not to have this property, [XD-Li]. The Itô maps we use seem to be the best substitute for such charts.

2 The approach via Itô maps and main results.

2.1 Itô maps as a charts

As in [Aida-Elworthy] and [Elworthy-LeJan-Li] take an SDE on M

$$dx_t = X(x_t) \circ dB_t, \quad 0 \le t \le T \tag{2.1}$$

with our given initial value x_0. Here $(B_t, 0 \le t \le T)$ is the canonical Brownian motion on $\mathbb{R}^m$ and $X(x)$ is a linear map from $\mathbb{R}^m$ to the tangent space T_xM for each x in M, smooth in x. Choose the SDE with the properties:

SDE1 The solutions to (1) are Brownian motions on M.

SDE2 For each $e \in \mathbb{R}^m$ the vector field $X(-)e$ has covariant derivative which vanishes at any point x where e is orthogonal to the kernel of $X(x)$.

This can be achieved, for example, by using Nash's theorem to obtain an isometric immersion of M into some $\mathbb{R}^m$ and taking $X(x)$ to be the orthogonal projection onto the the tangent space; see [Elworthy-LeJan-Li].

Let $\mathcal{I} : \mathcal{C}_0\mathbb{R}^m \to \mathcal{C}_{x_0}M$ denote the Itô map $\omega \mapsto x_\cdot(\omega)$ with $\mathcal{I}_t(\omega) = x_t(\omega)$. Then $\mathcal{I}$ maps $\mathbb{P}$ to μ_{x_0}. Set

$$\mathcal{F}^{x_0} = \sigma\{x_s : 0 \le s \le T\}$$

$$\mathbb{D}^{2,1}_{\mathcal{F}^{x_0}} = \{f : \mathcal{C}_0\mathbb{R}^m \to \mathbb{R} \ s.t. \ f \in \mathbb{D}^{2,1} \text{ and } f \text{ is } \mathcal{F}^{x_0}\text{-measurable}\}.$$

Also consider the isometric injection $\mathcal{I}^* : L^2(\mathcal{C}_{x_0}M;\mathbb{R}) \to L^2(\mathcal{C}_0\mathbb{R}^m;\mathbb{R})$ given by $f \mapsto f \circ \mathcal{I}$.

2.2 Basic results.

Theorem 1 *[Elworthy-Li1] The map $\mathcal{I}^*$ sends $\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M;\mathbb{R})$ to $\mathbb{D}^{2,1}_{\mathcal{F}^{x_0}}$ with closed range.*

Theorem 2 *Markov uniqueness holds if and only if $\mathcal{I}^*[\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M;\mathbb{R})] = \mathbb{D}^{2,1}_{\mathcal{F}^{x_0}}$.*

Theorem 3 *If $f : \mathcal{C}_0\mathbb{R}^m \to \mathbb{R}$ is in $\text{Dom}(\Delta)$ and $\mathcal{F}^{x_0}$-measurable then f belongs to $\mathcal{I}^*[\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M;\mathbb{R})]$.*

From Theorem 3 we see that $\text{BC}^2 \subset \mathbb{D}^{2,1}$ on $\mathcal{C}_{x_0}M$. Theorem 2 is a consequence of Theorem 4 below.

Problem 1 Is the set $\{f : \mathcal{C}_0\mathbb{R}^m \to \mathbb{R}$ s.t. f is in $\text{Dom}(\Delta)$ and $\mathcal{F}^{x_0}$-measurable$\}$ dense in $\mathbb{D}^{2,1}_{\mathcal{F}^{x_0}}$?

Problem1 is open. An affirmative answer would imply Markov uniqueness by the theorems above.

2.3 A stronger possibility.

Problem 2 If $f \in \mathbb{D}^{2,1}$ does $\mathbb{E}\{f|\mathcal{F}^{x_0}\} \in \mathbb{D}^{2,1}$?

Problem 2 is open: there is a gap in the 'proof' in [Elworthy-Li3]. It is true for f an exponential martingale or in a finite chaos space. An affirmative answer would imply an affirmative answer to Problem 1 and Markov uniqueness.

2.4 Markov uniqueness and weak differentiability

Let $\mathbb{D}^{2,1}\mathcal{H}$ and $\mathbb{D}^{2,1}\mathcal{H}^*$ be the spaces of $\mathbb{D}^{2,1}$-H-vector fields and H-1-forms on $\mathcal{C}_{x_0}M$, respectively, with their graph norms (see details below). Write:

$$\begin{aligned} \mathrm{Cyl}^0\mathcal{H}^* &= \text{linear span } \{gdk | g, k : \mathcal{C}_{x_0}M \to \mathbb{R} \text{ are in } \mathcal{C}^\infty \text{ Cyl}\} \\ W^{2,1} &= \mathrm{Dom}(d^* \mid \mathbb{D}^{2,1}\mathcal{H}^*)^* \\ {}^0W^{2,1} &= \mathrm{Dom}(d^* \mid \mathrm{Cyl}^0\,\mathcal{H}^*)^*. \end{aligned}$$

Then $\mathbb{D}^{2,1} \subseteq W^{2,1} \subseteq {}^0W^{2,1}$. From [Eberle] we have:

$$\text{Markov uniqueness} \Longleftrightarrow \mathbb{D}^{2,1} = {}^0W^{2,1} \tag{2.2}$$

We claim:

Theorem 4 *A.* $f \in W^{2,1}$ *on* $\mathcal{C}_{x_0}M \Longleftrightarrow \mathcal{I}^*(f) \in W^{2,1}$ *on* $\mathcal{C}_0\mathbb{R}^m$.

B.

$$W^{2,1} = {}^0W^{2,1}.$$

If $f \in W^{2,1}$ it has a "weak derivative" $df \in L^2\Gamma\mathcal{H}$ defined by $\int df(V)d\mu_{x_0} = -\int f \,\mathrm{div}\, V d\mu_{x_0}$ for all $V \in \mathbb{D}^{2,1}\mathcal{H}$. See §3.4 below where the proof of Proposition 9 also demonstrates one of the implications of Theorem 4A.

An important step in the proof of part B is the analogue of a fundamental result of [Kree-Kree] for $\mathcal{C}_0\mathbb{R}^m$:

Theorem 5 *The divergence operator on* $\mathcal{C}_{x_0}M$ *restricts to give a continuous linear map* div : $\mathbb{D}^{2,1}\mathcal{H} \to L^2(\mathcal{C}_{x_0}M;\mathbb{R})$.

3 Some details and comments on the proofs.

We will sketch some parts of the proofs. The full details will appear, in greater generality, in [Elworthy-Li2].

3.1 To prove Theorem 3.

For $f : \mathcal{C}_0\mathbb{R}^m \to \mathbb{R}$ in $\mathbb{D}^{2,1}$ take its chaos expansion

$$f = \sum_{k=1}^{\infty} I^k(\alpha^k) = \sum_{k=1}^{N} I^k(\alpha^k) + R_{N+1} \tag{3.1}$$

say. This converges in $\mathbb{D}^{2,1}$ as is well known, eg see [Nualart].

Set $\mathbb{E}\{I^k(\alpha^k)|\mathcal{F}^{x_0}\} = J^k(\alpha^k)$. Then

$$\mathbb{E}\{f|\mathcal{F}^{x_0}\} = \sum_{k=1}^{\infty} J^k(\alpha^k) \tag{3.2}$$

The right hand side converges in L^2. An equivalent probem to Problem 2 is:

Problem 3 Does the right hand side of equation (4) always converge in $\mathbb{D}^{2,1}$?

If f is $\mathcal{F}^{x_0}$-measurable and in the domain of Δ it is not difficult to show that there is convergence in $\mathbb{D}^{2,1}$, using the Lemma below. Moreover $\sum_{k=1}^N J^k(\alpha^k) \in \mathcal{I}^*[\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M;\mathbb{R})]$. Therefore by Theorem 1 we see $f \in \mathcal{I}^*[\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M;\mathbb{R})]$. Again this uses the basic result (c.f. [Elworthy-Yor], [Aida-Elworthy], [Elworthy-LeJan-Li]).

Lemma 6 *Let $K^\perp(x) : \mathbb{R}^m \to \mathbb{R}^m$ denote the orthogonal projection onto the orthogonal complement of the kernel of $X(x)$ for each x in M. Suppose $(\alpha_s, 0 \le s \le T)$ is progressively measurable, locally square integrable and $\mathbb{L}(\mathbb{R}^m;\mathbb{R}^p)$-valued. Then*

$$\mathbb{E}\left\{ \int_0^T \alpha_s(dB_s) \middle| \mathcal{F}^{x_0} \right\} = \int_0^T \mathbb{E}\{\alpha_s | \mathcal{F}^{x_0}\} K^\perp(x_s)\, dB_s.$$

3.2 The Riemannian structure for $\mathcal{H}$.

Let $\mathrm{Ric}^\sharp : TM \to TM$ correspond to the Ricci curvature tensor of M, and $W_s : T_{x_0}M \to T_{x_s}M$ the damped, or 'Dohrn-Guerra', parallel translation, defined for v_0 in $T_{x_0}M$ by

$$\begin{aligned} \frac{\mathbb{D} W_s(v_0)}{ds} &= 0 \\ W_0(v_0) &= v_0. \end{aligned}$$

Here $\frac{\mathbb{D}}{ds} = \frac{D}{ds} + \frac{1}{2}\mathrm{Ric}^\sharp$. Define $\langle v^1, v^2 \rangle_\sigma = \int_0^T \langle \frac{\mathbb{D}}{ds} v^1, \frac{\mathbb{D}}{ds} v^2 \rangle_{\sigma_s} ds$ and let $\boldsymbol{\nabla}$ denote the damped Markovian connection of [Cruzeiro-Fang]; see [Elworthy-Li2] for details.

For each $0 \le t \le T$ the Itô map $\mathcal{I}_t : H \to T_{x_t}M$ is infinitely differentiable in the sense of Malliavin Calculus, with derivative $T_\omega \mathcal{I}_t : H \to T_{x_t(\omega)}M$ giving rise to a continuous linear map $T_\omega \mathcal{I} : H \to T_{x_\cdot(\omega)}M$ defined almost surely for $\omega \in \mathcal{C}_0\mathbb{R}^m$. For $\sigma \in \mathcal{C}_{x_0}M$ define $\overline{T\mathcal{I}}_\sigma : H \to \mathcal{H}_\sigma$ by

$$\overline{T\mathcal{I}}_\sigma(h)_s = \mathbb{E}\{T\mathcal{I}_s(h) | x_\cdot = \sigma\}.$$

From [Elworthy-LeJan-Li] this does map into the Bismut tangent space and gives an orthogonal projection onto it. It is given by

$$\frac{\mathbb{D}}{ds}\overline{T\mathcal{I}}_\sigma(h)_s = X(\sigma(s))(\dot{h}_s)$$

and has right inverse $\boldsymbol{Y}_\sigma : \mathcal{H}_\sigma \to H$ given by

$$\boldsymbol{Y}_\sigma(v)_t = \int_0^t Y_{\sigma(s)}(\frac{\mathbb{D}}{ds} v_s) ds,$$

for $Y_x : T_xM \to \mathbb{R}^m$ the right inverse of $X(x)$ defined by $Y_x = X(x)^*$.

It turns out, [Elworthy-Li2], that for suitable H-vector fields V on $\mathcal{C}_{x_0}M$, the covariant derivative is given by $\boldsymbol{\nabla}_u V = \overline{T\mathcal{I}}_\sigma(d(\boldsymbol{Y}_-(V(-)))_\sigma(u))$, for $u \in T_\sigma \mathcal{C}_{x_0}M$, and we define V to be in $\mathbb{D}^{2,1}\mathcal{H}$ iff $\sigma \mapsto \boldsymbol{Y}_\sigma(V(\sigma))$ is in $\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M; H)$.

3.3 Continuity of the divergence

There is also a continuous linear map $\overline{T\mathcal{I}(-)} : L^2(\mathcal{C}_0\mathbb{R}^m; H) \to L^2\mathcal{H}$ defined by $\overline{T\mathcal{I}(U)}(\sigma)_s = \mathbb{E}\{T_-\mathcal{I}_s(U(-))|x_\cdot(-) = \sigma\}$, [Elworthy-Li1]. Another fundamental and easily proved result is

Proposition 7 *Suppose the H-vector field U on $\mathcal{C}_0\mathbb{R}^m$ is in* Dom(div). *Then $\overline{T\mathcal{I}(U)}$ is in* Dom(div) *on $\mathcal{C}_{x_0}M$ and*

$$\mathbb{E}\{\operatorname{div} U|\mathcal{F}^{x_0}\} = (\operatorname{div} \overline{T\mathcal{I}(U)}) \circ \mathcal{I} \tag{3.3}$$

Theorem 5 follows easily from Proposition 7 by observing that if $V \in \mathbb{D}^{2,1}\mathcal{H}$ then, from Theorem 1, $\mathcal{I}^*(\boldsymbol{Y}_-V(-)) \in \mathbb{D}^{2,1}$. By [Kree-Kree] this implies that $\mathcal{I}^*(\mathbf{Y}_-(V(-))$ is in Dom(div). Since

$$\overline{T\mathcal{I}(\mathcal{I}^*(\mathbf{Y}_-(V(-)))} = V$$

Proposition 7 assures us that $V \in$ Dom(div). Moreover

$$\operatorname{div} V(x_\cdot) = \mathbb{E}\{\operatorname{div} \mathcal{I}^*(\boldsymbol{Y}_-(V(-)))|\mathcal{F}^{x_0}\}. \tag{3.4}$$

Theorem 4A can be deduced from Proposition 7 together with:

Lemma 8 *The set of H-vector fields V on $\mathcal{C}_0\mathbb{R}^m$ such that $\overline{T\mathcal{I}(V)} \in \mathbb{D}^{2,1}\mathcal{H}$ is dense in $\mathbb{D}^{2,1}$.*

3.4 Intertwining and weak differentiability.

To see how weak differentiability relates to intertwining by our Itô maps we have:

Proposition 9 *If $f \in W^{2,1}$ it has weak derivative df given by*

$$(\mathrm{df})_\sigma = \mathbb{E}\{d(\mathcal{I}^*(f))_\omega|x_\cdot(\omega) = \sigma\}\boldsymbol{Y}_\sigma \tag{3.5}$$

Proof Let $V \in \mathbb{D}^{2,1}\mathcal{H}$. Then for $f \in W^{2,1}$, by equation (3.4) and then by Theorem 4A,

$$\begin{aligned}
\int_{\mathcal{C}_{x_0}M} f \operatorname{div}(V) d\mu &= \int_{\mathcal{C}_0\mathbb{R}^m} \mathcal{I}^*(f) \operatorname{div}(V) \circ \mathcal{I}\, d\mathbb{P} \\
&= \int_{\mathcal{C}_0\mathbb{R}^m} \mathcal{I}^*(f) \operatorname{div} \mathcal{I}^*(\boldsymbol{Y}_-(V(-)))\, d\mathbb{P} \\
&= -\int_{\mathcal{C}_0\mathbb{R}^m} d(\mathcal{I}^*(f))_\omega(\boldsymbol{Y}_{x_\cdot(\omega)}(V(x_\cdot(\omega)))\, d\mathbb{P}(\omega) \\
&= -\int_{\mathcal{C}_{x_0}M} \mathbb{E}\{d(\mathcal{I}^*(f))_\omega|x_\cdot(\omega) = \sigma\}\boldsymbol{Y}_\sigma(V(\sigma))\, d\mu_{x_0}(d\sigma)
\end{aligned}$$

as required. □

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