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A Poincaré inequality on loop spaces

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Abstract

We show that the Laplacian on the loop space over a class of Riemannian manifolds has a spectral gap. The Laplacian is defined using the Levi-Civita connection, the Brownian bridge measure and the standard Bismut tangent spaces.

Keywords: Path space; Loop space; Brownian bridge measure; Poincaré inequalities; Malliavin calculus

1. Introduction

Let M be a complete Riemannian manifold and $x_0 \in M$ fixed. For $T > 0$ let $\mathcal{C}_{x_0}M$ and $L_{x_0}M \equiv \mathcal{C}_{x_0,x_0}M$ denote the space of continuous paths or the space of continuous loops based at x_0, so

$$\mathcal{C}_{x_0}M = \{\sigma : [0,T] \to M \mid \sigma \text{ is continuous}, \ \sigma(0) = x_0\}$$

and $L_{x_0}M = \{\sigma \in \mathcal{C}_{x_0}M, \ \sigma(T) = x_0\}$. We are concerned in establishing an L^2 theory which relates to the geometry and the topology of the path space and its subspaces. For the Wiener space there is the canonically defined Cameron–Martin space of the Gaussian measure and the associated gradient operator. The corresponding gradient operator gives rise to the Ornstein–Uhlenbeck operator and a well formulated L^2 theory. The challenge for a general path space is

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the lack of a natural choice of subspaces of the tangent spaces with Hilbert space structure and a natural measure. The aim here is to analyse properties of a suitably defined Laplacian d^*d for d on an L^2 space of functions with range in the dual space of a suitably defined sub-bundle of the tangent bundle and d^* its L^2 dual. The exponential map which defines the manifold structure depends on the choice of linear connections on the underlying space. This will determine an unbounded linear operator:

$$d : L^2(\mathcal{C}_{x_0}M; \mathbf{R}) \to L^2\Gamma H^*$$

to the space of L^2 differential H one-forms. Despite that the question of the uniqueness of the operator $\mathcal{L}$ remains open we study here the spectrum properties of the operator with the initial domain the space of smooth cylindrical functions.

A Poincaré inequality on a space N, $\int_N (f - \bar{f})^2 \mu(dx) \leqslant \frac{1}{C} \int_N |\nabla f|^2 \mu(dx)$, depends on the gradient like operator ∇, an admissible set of real valued functions on N and a finite measure μ on N which is normalized to have total mass 1. Here $\bar{f} = \int f \, d\mu$. For a compact Riemannian manifold and the usual gradient the Poincaré constant C, the first non-trivial eigenvalue of the Laplacian, is related to the isoperimetric constant in Cheeger's isoperimetric inequality $h = \inf_A \frac{\mu(\partial A)}{\min\{\mu(A), \mu(M/A)\}}$, where the infimum is taken over all open subsets of M. Standard isoperimetric inequalities say that for an open bounded set A in $\mathbf{R}^n$, the ratio between the area of its boundary ∂A and the volume of A to the power of $1 - \frac{1}{n}$ is minimized by the unit ball. In relation to Poincaré inequality, especially in infinite dimensions, the more useful form of isoperimetric inequality is that of Cheeger. It was shown by Cheeger [15] that $h^2 \leqslant 4\lambda_1$ and if K is the lower bound of the Ricci curvature Buser [12] showed that $\lambda_1 \leqslant C(\sqrt{K} h + h^2)$ for which M. Ledoux [35] has a beautiful analytic proof. For such inequalities for Gaussian measures see e.g. Ledoux [36] and Ledoux–Talagrand [37].

As Wiener measure and the Brownian bridge measure on the Wiener space are Gaussian measures, logarithmic Sobolev inequalities (L.S.I.) hold. The classical approach to this is to use the symmetric property, rotation invariance, of the Gaussian measure or the commutation relation of the Ornstein–Uhlenbeck semi-group. A number of simple proofs have since been given. It is L. Gross, [31], who obtained the logarithmic Sobolev inequality and remarked on its validity in an infinite dimensional space and its relation with Nelson's hypercontractivity. A L.S.I. was shown to hold for the Brownian motion measure on the path space over a compact manifold or for manifolds with suitable conditions on the Ricci curvature and using gradient operator defined by the Bismut tangent space. See Aida–Elworthy [7], Fang [27], and Hsu [34]. However, as noted in Gross [32] for non-Euclidean spaces there is a fundamental difference between Brownian motion measure and the Brownian bridge measure, for example Poincaré inequalities do not hold for the Brownian bridge measure on the Lie group S^1 due to the lack of connectedness of the loop space. We now consider the loop space with the standard Brownian bridge measure and the standard Bismut tangent space structure. There are few positive results, apart from one in [5] which states that L.S.I. holds on loop spaces over manifolds which are diffeomorphic to $\mathbf{R}^n$ and whose Riemannian metric is asymptotically flat and whose curvature satisfies certain additional conditions. In Aida [1] it was shown that the kernel of the OU operator, on a simply connected manifold, contains only constant functions. In [23] A. Eberle gave a criterion for the Poincaré inequality, for the gradient operator restricted to a single homotopy class of loops, to fail. Let γ be a closed geodesic belonging to the trivial homotopy class of a compact connected manifold (which in general may or may not exist) and let U be an ϵ-neighbourhood of γ, where ϵ is sufficiently small. Let Ω_n the homotopy class of U containing γ_n, the closed geodesic γ with n times of the speed of γ. He showed that the weighted Poincaré inequality

$$\int_{L_{x_0}M} (f-\bar{f})^2\mu(d\omega) \leqslant \int_{L_{x_0}M} C(\omega)|\nabla f|^2(\omega)\mu(d\omega)$$

fails if the curvature of M is constant and strictly negative in U and if $\int_{\Omega_n/\Omega_n^\epsilon} C(\omega)\,d\mu(\omega)$ grows sufficiently slowly. Here Ω_n^ϵ is the subspace of Ω_n whose elements consisting of loops whose distance from γ_n is smaller than ϵ. In particular he constructed a compact simply connected manifold diffeomorphic to the sphere on whose loop space the Poincaré inequality fails to hold, by modifying the Riemannian structure around a big circle so that it has constant negative curvature.

On the other hand, a weighted logarithmic Sobolev inequality was shown in Aida [3] to hold for a class of underlying manifolds including the hyperbolic space. In this article we show that Poincaré inequality holds on loop spaces over the hyperbolic space. This demonstrates that the existence of a closed geodesic in the trivial homotopy class plays an essential role in Eberle's argument.

The other natural measures on loop spaces are heat kernel measures, induced by Brownian motions in infinite dimensional spaces, see Malliavin–Malliavin [40] for the construction of a Brownian motion on the loop space over a compact Lie group where they used the Ad-invariant structure. In a series of papers by Driver [19] and Fang [28], quasi-invariance property and the subsequent integration by parts theory were established. In [21] Driver–Lohrenz proved logarithmic Sobolev inequalities for the heat kernel measure on loop spaces over compact type Lie groups. Heat kernel measure is shown to be equivalent to the Brownian bridge measure for simply connected Lie groups, see Aida–Driver [6] and Airault–Malliavin [9], although no suitable bounds on the density is yet obtained to deduce the corresponding inequality for the Brownian bridge measure, see Driver–Srimurthy [22]. In a recent paper, Driver–Gordina [20] showed that on a Heisenberg like extension, $\Omega \times V$ for V a finite vector space, to the Wiener space there is a L.S.I. with respect to the heat kernel measure, i.e. the law induced from the Brownian motion on $\Omega \times V$ together with the Heisenberg group structure. See also [30].

2. Poincaré inequality on loop spaces

2.1. Preliminaries

There are a number of standard approaches to Poincaré inequalities. On a compact manifold, Poincaré inequality for the Laplace–Beltrami operator is proved by the Rellich–Kondrachov compact embedding theorem of $H^{1,q}$ into L^p. For Gaussian measures there are special techniques. Let B be a Banach space and μ a mean zero standard Gaussian measure with B its topological support and covariance operator Γ. The Gaussian Sobolev space structure can be given to any mean zero standard Gaussian measures and a Poincaré inequality related to the gradient can be shown to be valid for all functions in the Sobolev space with Poincaré constant 1. Indeed the Cameron–Martin space H is the intersection of all vector subspaces of B of full measure and it is a dense set of B of null measure. The Gaussian measure μ is quasi translation invariant precisely in the directions of vectors of H. Let $f: B \to \mathbf{R}$ be an L^2 function differentiable in the directions of H and let $\nabla f \equiv \nabla_H f$, an element of H, be the gradient of f defined by $\langle \nabla f, h\rangle_H = df(h)$. The square of the H-norm of the gradient f is precisely

$\sum_i |df(h_i)|^2$ where h_i is an orthonormal basis of H. There is a corresponding quadratic form: $\int_B |\nabla f|_H^2(x)\mu(dx)$. When B is a Hilbert space the Cameron–Martin space is the range of $\Gamma^{\frac{1}{2}}$ and Γ can be considered as a trace class linear operator on B. If f is a BC^1 function, $\nabla_B f$ is defined and $\nabla_H f = \Gamma \nabla_B f$. The associated quadratic form is $\int_B |\Gamma^{-1/2}\nabla_H f|_B^2\, d\mu(x)$ and the Poincaré inequality becomes, for f with zero mean,

$$\int f^2(x)\mu(dx) \leqslant \frac{1}{C}\int_B \left|\Gamma^{-1/2}\nabla_H f\right|_B^2 d\mu(x).$$

To the quadratic form $\int_B |\Gamma^{-\frac{1}{2}}\nabla_H f|_B^2\, d\mu(x)$ there associates a linear operator $\mathcal{L}$ given by $\int f\mathcal{L}g\, d\mu = \int \langle \nabla_H f, \Gamma^{-1}\nabla_H g\rangle_B\, d\mu$.

The other standard approach is the Clark–Ocone approach used by Capitaine–Hsu–Ledoux [13] and the dynamic one of Bakry–Emery [10] which we will now explain. Both approaches leads usually to the stronger logarithmic Sobolev inequality while we are doubtful that logarithmic Sobolev inequalities hold in our case. For simplicity we assume that the measure concerned is on a finite dimensional complete connected Riemannian manifold M. For $x_0 \in M$ let $(F_t(x_0,\omega), t \geqslant 0)$ be the solution flow to an elliptic stochastic differential equation. The system induces a Riemannian metric and the infinitesimal generator is of the form $\frac{1}{2}\Delta + A$ for Δ the Laplace–Beltrami operator for the corresponding Levi-Civita connection and A a vector field called the drift. Suppose that the drift is of gradient form given by a potential function h. Then the system has an invariant measure $\mu(dx) = e^{2h}\, dx$ which is finite if $\sup_{x\in K} G_t(x) < \infty$, where $G_t(x) = \int_0^\infty \mathbf{E} e^{-\int_0^t \rho^h(F_s(x,\omega))\, ds}\, dt$ and K is any compact subset and$\rho^h(x) = \inf_{|v|=1} \mathrm{Ric}_x(v,v) - 2\,\mathrm{Hess}_x(h)(v,v)$ for Ric the Ricci curvature [39,38]. This condition essentially gives control for $d(P_t f)$ of the form of exponential decay. Formally,

$$\begin{aligned}\int_M (f-\bar f)^2\, d\mu &= \lim_{t\to\infty}\int_M \big(f^2 - (P_t f)^2\big)(x)\, d\mu(x) = -\lim_{t\to\infty}\int_M\int_0^t \frac{\partial}{\partial s}(P_s f)^2\, ds\, d\mu \\ &= \lim_{t\to\infty}\int_0^t\int_M (dP_s f)^2\, d\mu\, ds = \int_0^\infty\int_M (dP_s f)^2\, d\mu\, ds \\ &\leqslant \int_0^\infty\int_M \mathbf{E}|df|^2\big(F_t\big((x,\omega)\big)\, d\mu\, e^{-\rho s}\, ds = \|f\|_{L^2}^2 \int_0^\infty \sup_{x\in K} G_t(x)\, dt.\end{aligned}$$

This shows that there is a spectral gap if $\rho^h > C > 0$. In the case of $M = \mathbf{R}^n$ the Bakry–Emery condition is exactly the log-convexity condition on measures. In the case of a Hilbert space the dynamic of the corresponding semi-group is given by the solution of the Langevin equation $du_t = dW_t - \frac{1}{2}u_t\, dt$, where W_t is a cylindrical Wiener process on H and hence the dynamic approach works. See e.g. Da Prato [16].

2.2. The loop spaces

Let M be a smooth finite dimensional complete connected Riemannian manifold which is stochastically complete. The path space $\mathcal{C}_{x_0}M$ is an infinite dimensional manifold modelled on a Banach space. Its tangent space at a path σ can be considered to be the collection of continuous vector field along the given path:

$$T_\sigma \mathcal{C}_{x_0}M = \{V : [0,T] \to TM \text{ continuous: } v(t) \in T_{\sigma(t)}M,\ v(0) = 0\}.$$

The based path space is a complete separable metric space with distance function ρ given by:

$$\rho(\sigma_1, \sigma_2) = \sup_t d\big(\sigma_1(t), \sigma_2(t)\big).$$

Both $\mathcal{C}_{x_0,y_0}M$ and $L_{x_0}M$ are closed subspaces of $\mathcal{C}_{x_0}M$ viewed as a metric space.

The standard Wiener measure $\mathbf{P}$ on $\Omega = \mathcal{C}_0\mathbf{R}^m$ is a Gaussian measure with Covariance

$$\Gamma(l_1, l_2) = \int_0^T \int_0^T (s \wedge t)\, d\mu_{\ell_1}(s)\, d\mu_{\ell_2}(t)$$

where μ_{ℓ_i} are measures on $[0,T]$ associated to $\ell_i \in \Omega^*$. Its associated Cameron–Martin space is the Sobolev space on $\mathbf{R}^n$ consisting of paths in Ω with finite energy

$$\mathbf{H} = \left\{ h : [0,T] \to \mathbf{R}^m \text{ such that } \int_0^T |\dot{h}_t|^2\, dt < \infty \right\}.$$

The Brownian motion measure μ_{x_0} on $\mathcal{C}_{x_0}M$ is the pushed forward measure of the Wiener measure $\mathbf{P}$ by the Brownian motion.

Let $ev_t : \mathcal{C}_{x_0}M \to M$ be the evaluation map at time t. The conditional law of the canonical process $(ev_t, t \in [0,T])$ on $\mathcal{C}_{x_0}M$ given $ev_T(\sigma) = y_0$ is denoted by μ_{x_0,y_0}, hence for a Borel set A of $\mathcal{C}_{x_0}M$,

$$\mu_{x_0,y_0}(A) = \mu_{x_0}(\sigma \in A \mid \sigma_T = y_0). \tag{2.1}$$

Restricted to $\mathcal{F}_t$ for $t < T$ the Brownian motion measure and the Brownian bridge measure are absolutely continuous with respect to each other with Radon Nikodym derivative given by $\frac{p_{T-t}(y_0,\sigma_t)}{p_T(x_0,y_0)}$. Assume that for $\beta > 0$, $\delta > 0$,

$$\iint d(y,z)^\beta \frac{p_s(x_0,y)p_{t-s}(y,z)p_{T-t}(z,y_0)}{p_T(x_0,y_0)}\, dy\, dz \leqslant C|t-s|^{1+\delta}. \tag{2.2}$$

Define Cyl_t to be the set of smooth cylindrical functions,

$$\mathrm{Cyl}_t = \{F \mid F(\sigma) = f(\sigma_{s_1}, \dots, \sigma_{s_k}),\ f \in C_K^\infty(M^k),\ 0 < s_1 < \cdots < s_k \leqslant t < T\}.$$

Following Einstein, the Brownian bridge measure is defined by its action on cylindrical functions. For $F \in \mathrm{Cyl}_t$,

$$\int_{\mathcal{C}_{x_0}M} f(\sigma_{s_1}, \dots, \sigma_{s_n})\, d\mu_{x_0,y_0}(\sigma)$$

$$= \frac{1}{p_T(x_0, y_0)} \int_{M^n} f(x_1, \dots, x_n) p_{s_1}(x_0, x_1) \dots p_{s_n - s_{n-1}}(x_{n-1}, x_n) p_{T-s_n}(x_n, y_0) \prod_{i=1}^n dx_i.$$

That this defines a measure on $\mathcal{C}_{x_0}M$ is due to Kolmogorov's theorem and (2.2).

Denote by H_σ the Bismut tangent space of $\mathcal{C}_{x_0}M$, with the induced inner product,

$$H_\sigma = \left\{ h_t \equiv /\!/ .k.\colon k : [0, T] \to T_{x_0}M \text{ with } \int_0^T |\dot{k}_s|^2\, ds < \infty \right\} \subset T_\sigma \mathcal{C}_{x_0}M.$$

Here $/\!/$ denotes stochastic parallel translation with respect to a torsion skew symmetric connection on *TM* and $H = \bigsqcup H_\sigma$ the vector bundle with fibres H_σ. Let C_K^∞ the space of real valued functions on M with compact support and for $F \in \mathrm{Cyl}$ given by $f(\omega_{t_1}, \dots, \omega_{t_k})$, $h \in H_\omega$,

$$dF(\omega)(h) = \sum_{i=1}^k \partial_i f(\omega_{t_1}, \dots, \omega_{t_k})(h_{t_i}),$$

where $\partial_i f$ stands for differentiation with respect to i-th variable. Hence

$$\nabla F(\omega)(t) = \sum_{i=1}^k /\!/_t(\omega) /\!/_{t_i}^{-1}(\omega) \nabla_i f(\omega_{t_1}, \dots, \omega_{t_k}) \min(t, t_i).$$

The gradient operator, more precisely the associated quadratic form, is associated to the Laplace operator $\mathcal{L} = -\frac{1}{2} d^* d$, where $d^* : L^2(H, \mu_{x_0}) \to L^2(\mathcal{C}_{x_0}M, \mu_{x_0})$ is the adjoint of the unbounded operator d whose initial domain the set of smooth cylindrical functions with compact support has been shown to be a closable operator by Driver's integration by parts formula [17]. We define $\mathbb{D}^{1,2} \equiv \mathbb{D}^{1,2}(\mathcal{C}_{x_0}M)$ to be the closure of smooth cylindrical function Cyl_t, $t < T$ under this graph norm:

$$\sqrt{\int_{\mathcal{C}_{x_0}M} |\nabla f|^2_{H_\sigma}(\sigma) \mu_{x_0}(d\sigma) + \int_{\mathcal{C}_{x_0}M} f^2(\sigma)\, d\mu_{x_0}(\sigma)}.$$

For Brownian bridge measures we take H_σ^0 to be the set of vectors in H with $h(T) = 0$, with H^0 the corresponding bundle. That the Brownian bridge measure is quasi-invariant in the directions of H^0 is proved in [18] and the corresponding gradient will have the green function $\min(t, s)$ replaced by $\min(t, s) - ts$.

2.3. The theorem

Denote by $\mathrm{Ric}_x^\# : T_xM \to T_xM$ the Ricci tensor, and $/\!/(\sigma)$ the stochastic parallel translation along a continuous path σ which is defined by the canonical horizontal vector fields and the time dependent vector field $\nabla \log p_{T-t}(x, y)$. Let σ_t be the canonical process. Let $/\!/(\sigma)h$ be a bounded adapted H-valued vector field with $h(\sigma)(t) \in T_{x_0}M$ and $h_T = 0$. We assume furthermore the integrability condition $\int \int_0^T |\dot{h}_t|^{2+\delta}\, dt\, d\mu_{x_0,x_0} < \infty$ and that the integration by parts formula holds:

$$\int_{\mathcal{C}_{x_0,x_0}M} dF\big(/\!/(\sigma)h\big)\, d\mu_{x_0,x_0}(\sigma) = \int_{\mathcal{C}_{x_0,x_0}M} F(\sigma)\, \mathrm{div}(F)(\sigma)\, d\mu_{x_0,x_0}(\sigma)$$

where

$$\mathrm{div}(F) = \int_0^T \left\langle \dot{h}_s + \frac{1}{2} /\!/_s^{-1} \mathrm{Ric}^{\#}_{\sigma_s}(/\!/_s h_s) - /\!/_s^{-1} \nabla^2 \log p_{T-s}(\sigma_s, x_0)(/\!/_s h_s), db_s \right\rangle$$

for b_t be the martingale part of the stochastic anti-development map. For compact manifolds and for manifolds diffeomorphic to the Euclidean space, this integration by parts formula holds. See [3], also [18,29,26,34]. In this case we have a well defined closable linear differential operator: $d : L^2(\mathcal{C}_{x_0,x_0}M; \mu_{x_0,x_0}) \to L^2\Gamma(H^0)^*$.

Let $X : M \times \mathbf{R}^m \to TM$ be a smooth bundle map induced by an isometric immersion of M in $\mathbf{R}^m$. For each x, $X(x) : \mathbf{R}^m \to T_xM$ is linear and surjective. For an orthonormal basis $\{e_m\}$ of $\mathbf{R}^m$ define $X_i(x) = X(e_i)(x)$. Then $\frac{1}{2}\sum_i L_{X_i} L_{X_i}$ is the Laplacian operator. Consider the stochastic differential equation

$$dy_t = \sum_{i=1}^m X_i(y_t) \circ dB_t^i + \nabla \log p_{T-t}(y_t, x_0)\, dt. \tag{2.3}$$

Let r be the distance function on M from x_0. We define a linear map $W_{s,t} : T_{\sigma(s)}M \to T_{\sigma(t)}M$ as following, denoting $\frac{D}{dt} = /\!/_t \frac{d}{dt} /\!/_t^{-1}$. Consider the stochastic covariant differential equation along the path σ_t with initial point x_0, for $t < T$:

$$\begin{aligned} &\frac{D}{dt} W_{s,t} = -\frac{1}{2} \mathrm{Ric}^{\#}_{\sigma_t}(W_{s,t}) + \nabla^2 \log p_{T-t}(\sigma_t, x_0)(W_{s,t}), \\ &W_{s,s}(v) = v. \end{aligned} \tag{2.4}$$

Its solution $W_{s,t}(\sigma) : T_{\sigma_s}M \to T_{\sigma_t}M$ is a linear map. Write $W_t = W_{0,t}$. For $\delta < 1$, the solution of (2.3) induces a map I^δ from the Wiener paths on $[0, \delta T]$ to continuous paths on the same time interval. Then I^δ is differentiable in the directions of the Cameron–Martin directions and the derivative is denoted as $T\mathcal{I}^\delta$. If J_t^δ is the conditional expectation of $T\mathcal{I}^\delta$ with respect to the filtration of y_t up to time δT. Then

$$J^\delta(h)(t) = W_t(\sigma) \int_0^t W_s^{-1}(\sigma) X(\sigma_s)(\dot{h}_s)\, ds.$$

See [7] and [25]. For $k \in L_0^{2,1}([0, T]; T_{x_0}M)$, the space of finite energy loops on $T_{x_0}M$ based on 0, define $J(\sigma)(k)$ given by

$$J(\sigma)(h)(t)=W_t(\sigma)\int_0^t W_s^{-1}(\sigma)\big(/\!\!/_s(\sigma)\dot{k}_s\big)\,ds,\quad t<T.$$

Definition 2.1. If for each $h\in L_0^{2,1}([0,T];\mathbf{R}^n)$, $\lim_{t\to T}J(h)(t)$ exists almost surely with respect to μ_{x_0,x_0}, we define an alternative tangent space at σ by $\mathcal{H}_\sigma=\{J(h): h\in L_0^{2,1}(\mathbf{R}^n)$ in which case we define

$$\big\langle J(\sigma)(h_1),J(\sigma)(h_2)\big\rangle_{\mathcal{H}_\sigma}=\int_0^T\big\langle \dot{h}_s^1,\dot{h}_s^2\big\rangle\,ds.$$

Let

$$\rho_t(x)=\inf_{|v|=1}\left\langle -\frac{1}{2}\operatorname{Ric}_x^{\#}(v)+\nabla^2\log p_{T-t}(x,x_0)(v),v\right\rangle.$$

Then $\|W_{s,t}\|^2\leqslant\exp(\int_s^t 2\rho_r(\sigma_r)\,dr)$. To achieve that $\lim_{t\to T}J(h)(t)=0$ we need to assume that ρ is negative and blows up to $-\infty$ at $t\to T$. This is not likely to happen for compact manifolds as there is always a closed geodesic. Let us consider the case that the manifold is diffeomorphic to $\mathbf{R}^n$. Under suitable conditions e.g. when the manifold is the hyperbolic space, the map J extends to $t=T$, in which case we have a map to $L_{0,0}^{2,1}([0,T];T_{x_0}M)$. We quote the following theorem from Aida:

Theorem 2.1 *(Aida). Let H^n be the hyperbolic space of constant curvature. For f smooth cylindrical functions,*

$$\int_{\mathcal{C}_{x_0}H^n} f^2\log\frac{f^2}{\mathbf{E}|f|_{L^2}^2}(\gamma)\,d\mu_{x_0,y_0}(\gamma)\leqslant\int_{\mathcal{C}_{x_0}H^n} C(\gamma)|\nabla f|^2_{H_\gamma^0}(\gamma)\,d\mu_{x_0,y_0}(\gamma)\tag{2.5}$$

for $C(\gamma)=C_1(n)+C_2(n)\sup_{0\leqslant t\leqslant 1}d^2(\gamma_t,y_0)$.

Aida's theorem is more general than that stated above. The above inequality (2.5) holds for a weight function C if M is diffeomorphic to $\mathbf{R}^n$ and that the operator $J(\sigma)$ extends to a bounded operator from $L_0^{2,1}([0,T];\mathbf{R}^m)$ to H_σ^0 for almost surely all σ and such that $\int\|Id+/\!\!/^{-1}J\|^2\,d\mu_{x_0,y_0}<\infty$ and $\lim_{t\to T}\int_0^t\int\|W_{s,t}\|^2\,d\mu_{x_0,y_0}\,ds<\infty$. Poincaré inequalities with potentials and a number of modified Poincaré inequalities have been intensively studied for which we refer to the recent work of Gong–Ma [29]. See also Gong–Röckner–Wu who showed the existence of the gap of spectrum for a Schrödinger operators with the potential function given in Gross's paper, for which Aida [4] gave an estimate. Their work builds on the results of Aida [2] and M. Hino [33] on exponential decay estimates of the associated semi-group . Aida's proof begins with the Clark–Ocone formula approach [13]. From an integration by parts formula he obtained the following Clark–Ocone formula by the integral representation theorem:

$$\mathbf{E}^{\mu_{x_0,y_0}}\{F|\mathcal{G}_t\}=\mathbf{E}^{\mu_{x_0,y_0}}F+\int_0^t\big\langle H_s(\gamma),dW_s\big\rangle,$$

$$H(s,\gamma)=\mathbf{E}^{\mu_{x_0,y_0}}\left\{(Id+J)(\gamma)\frac{d}{ds}\nabla F(\gamma)(s)|\mathcal{G}_s\right\}$$

almost surely with respect to the product measure $dt\otimes\mu_{x_0,y_0}$. Here $\mathcal{G}_t$ is the filtration generated by the Brownian filtration and the end point of the Brownian bridge.

Note that on the hyperbolic space, $\int_{\mathcal{C}_{x_0}M} e^{C\sup_t d^2(\sigma_t,x_0)}\,d\mu_{x_0,y_0}(\sigma)<\infty$ for C sufficiently small, by the time reversal of the Brownian bridge and its symmetric property and the concentration property of the Brownian motion measure. We bow state our main theorem, which follows from Lemma 2.3 and Lemma 2.4 of the coming section.

Theorem 2.2. *Poincaré inequality holds on the loop space over the hyperbolic spaces.*

2.4. Proof of Theorem 2.2

Consider $\mathcal{C}_{x_0}M$ with an elliptic diffusion measure, including measures which concentrates on a subspace e.g. the loop space, and corresponding Bismut tangent space with the usual inner product and the corresponding gradient operator. Throughout this section let μ be a probability measure which is absolutely continuous with respect to that measure. When there is no confusion of which measure is used, we denote by the integral of a function f with respect to μ by $\mathbf{E}f$, its variance $\mathbf{E}(f-\mathbf{E}f)^2$ by $\mathbf{Var}(f)$ and its entropy $\mathbf{E}f\log\frac{f}{\mathbf{E}f}$ by $\mathbf{Ent}(f)$.

By restriction to an exhausting relatively compact open sets U_n, local Poincaré inequality always exist for the Laplace–Beltrami operator on a complete finite dimensional Riemannian manifold. Once a blowing up rate for local Poincaré inequalities are obtained, one obtains the so-called weak Poincaré inequality and in the case of Entropy we have the weak logarithmic Sobolev inequality.

$$\mathbf{Var}(f)\leqslant\alpha(s)\int|\nabla f|^2\,d\mu+s|f|_\infty^2,$$

$$\mathbf{Ent}\big(f^2\big)\leqslant\beta(s)\int|\nabla f|^2\,d\mu+s|f|_\infty^2$$

where α and β to be non-decreasing functions from $(0,\infty)$ to $\mathbf{R}_+$. The rate of convergence to equilibrium for the dynamics associated to the Dirichlet form $\int|\nabla f|^2\,d\mu$ is strongly linked to Poincaré inequalities. In the case of weak Poincaré inequalities, exponential convergence is no longer guaranteed. There is a huge literature on these inequalities, see e.g. Aida [2], Aida–Masuda–Shigekawa [8], Mathieu [41], and Röckner–Wang [42]. It is worth noting that the weak Poincaré inequality holds for any α is equivalent to Kusuoka–Aida's weak spectral gap inequality which states that any mean zero sequence of functions f_n in $\mathbb{D}^{1,2}$ with $\mathbf{Var}(f_n)\leqslant 1$ and $\mathbf{E}(|\nabla f|^2)\to 0$ is a sequence which converges to 0 in probability. Note also that the ergodicity that uniformly bounded sequences with $\mathbf{E}(|\nabla f|^2)\to 0$ is a sequence which converges to 0 in probability together with a L.S.I. of the form

$$\mathbf{Ent}\big(f^2\big)\leqslant\beta(s)\int|\nabla f|^2\,d\mu+s|f|_2^2$$

implies the existence of a spectral gap [41].

Lemma 2.3. *Let μ be a probability measure on $\mathcal{C}_{x_0}M$ with the property that there exists a positive function $u \in \mathbb{D}^{1,2}$ such that Aida's type inequality holds*:

$$\mathbf{Ent}(f^2) \leqslant \int u^2|\nabla f|^2\,d\mu, \quad \forall f \in \mathbb{D}^{1,2} \cap L_\infty. \tag{2.6}$$

1. *Assume that $|\nabla u| \leqslant a$ and $\int e^{Cu^2}\,d\mu < \infty$ for some $C, a > 0$. Then for all functions f in $\mathbb{D}^{1,2} \cap L_\infty$*

$$\mathbf{Ent}(f^2) \leqslant \beta(s) \int |\nabla f|^2\,d\mu + s|f|_\infty^2, \tag{2.7}$$

   where $\beta(s) = C|\log s|$ for $s < s_0$ where C and s_0 are constants.
2. *If* (2.6) *holds for $u \in \mathbb{D}^{2,1}$ with the property $|\nabla u| \leqslant a$, $u \geqslant 0$ and*

$$\mu\big(u^2 > r^2\big) < m^2(r),$$

   for a non-increasing function m of the order $o(r^{-2})$, then the weak logarithmic Sobolev inequality holds with $\beta(s)$ of the order of the square of the inverse function of $r^2m(r)$ for s small.

Note. In application the weight function u is a distance function and $|\nabla\rho| = 1$.

Proof of Lemma 2.3. Let $\alpha_n : \mathbf{R} \to [0,1]$ be a sequence of smooth functions approximating 1 such that

$$\alpha_n(t) = \begin{cases} 1 & t \leqslant n-1, \\ \in [0,1], & t \in (n-1, n), \\ 0 & t \geqslant n. \end{cases} \tag{2.8}$$

We may assume that $|\alpha_n'| \leqslant 2$. Define

$$f_n = \alpha_n(u)f$$

for u as in the assumption. Then f_n belongs to $\mathbb{D}^{1,2} \cap L_\infty$ if f does. We may apply Aida's inequality (2.6) to f_n. The gradient of f_n splits into two parts of which one involves f and the other involves ∇f. The part involving the gradient vanishes outside of the region of $A_n := \{\omega\colon u(\omega) < n\}$ and on A_n it is controlled by g and therefore by n. The part involving f itself vanishes outside $\{\omega\colon n-1 < u(\omega) < n\}$ and the probability of $\{\omega\colon n-1 < u(\omega) < n\}$ is very small by the exponential integrability of u. We split the entropy into two terms: $\mathbf{Ent}(f^2) = \mathbf{Ent}(f_n^2) + [\mathbf{Ent}(f^2) - \mathbf{Ent}(f_n^2)]$, to the first we apply the Sobolev inequality (2.6).

$$\begin{aligned} \int f_n^2 \log\frac{f_n^2}{\mathbf{E}f_n^2}\,d\mu &\leqslant \int u^2|\nabla f_n|^2\,d\mu \\ &\leqslant \int u^2\big[|\nabla f|\alpha_n(u) + \big|\alpha_n'\big||\nabla u|f\big]^2\,d\mu \end{aligned}$$

$$\leqslant \int\limits_{u<n} 2u^2|\nabla f|^2\alpha_n^2(u)\,d\mu + 4a^2 \int\limits_{n-1<u<n} u^2 f^2\,d\mu$$

$$\leqslant 2n^2 \int |\nabla f|^2\,d\mu + 4a^2n^2|f|_\infty^2\mu(n>u>n-1).$$

Next we compute the difference between $\mathbf{Ent}(f^2)$ and $\mathbf{Ent}(f_n^2)$.

$$\begin{aligned}
&\mathbf{Ent}\big(f^2\big) - \mathbf{Ent}\big(f_n^2\big)\\
&= \int \left(f^2 \log \frac{f^2}{\mathbf{E}f^2} - f_n^2 \log \frac{f_n^2}{\mathbf{E}f_n^2}\right) d\mu\\
&= \int \big(1-\alpha_n^2(u)\big) f^2 \log \frac{f^2}{\mathbf{E}f^2}\,d\mu + \int f^2\alpha_n^2(u)\left(\log \frac{f^2}{\mathbf{E}f^2} - \log \frac{\alpha_n^2(u)f^2}{\mathbf{E}\alpha_n^2(u)f^2}\right) d\mu\\
&= I + II.
\end{aligned}$$

Observe that

$$\begin{aligned}
I &= \int \big(1-\alpha_n^2(u)\big) f^2 \log \frac{f^2}{\mathbf{E}f^2}\,d\mu\\
&\leqslant \int\limits_{u>n-1} f^2\big(1-\alpha_n^2(u)\big) \log \frac{f^2}{\mathbf{E}f^2}\,d\mu\\
&\leqslant 2|f|_\infty^2 \int\limits_{u>n-1} \left(\log \frac{|f|}{\sqrt{\mathbf{E}f^2}}\right)^+ d\mu.
\end{aligned}$$

By the elementary inequality $\log x \leqslant x$ and Cauchy–Schwartz inequality

$$\begin{aligned}
I &\leqslant 2|f|_\infty^2 \int\limits_{u>n-1} \left(\frac{|f|}{\sqrt{\mathbf{E}f^2}}\right) d\mu\\
&\leqslant 2|f|_\infty^2 \sqrt{\mathbf{E}\left(\frac{|f|}{\sqrt{\mathbf{E}f^2}}\right)^2} \sqrt{\mu(u>n-1)}\\
&\leqslant 2|f|_\infty^2 \sqrt{\mu(u>n-1)}.
\end{aligned}$$

For the second term of the sum, with the convention that $0\log 0 = 0$,

$$II = -\int\limits_{n-1<u<n} f^2\alpha_n^2(u)\log\alpha_n^2(u)\,d\mu + \int\limits_{u<n} f^2\alpha_n^2(u) \log \frac{\mathbf{E}f^2\alpha_n^2(u)}{\mathbf{E}f^2}\,d\mu.$$

Using the fact that $\log \frac{\mathbf{E} f^2 \alpha_n^2(u)}{\mathbf{E} f^2} \leqslant 0$ from $\alpha_n^2(u) \leqslant 1$ and $x \log x \geqslant -\frac{1}{e}$, we see that

$$II \leqslant \frac{1}{e} \int\limits_{n-1<u<n} f^2 \, d\mu \leqslant \frac{1}{e}\big(|f|_\infty\big)^2 \cdot \mu(n-1<u<n).$$

Finally adding the three terms together to obtain

$$\int f^2 \log \frac{f^2}{\mathbf{E} f^2} \, d\mu \leqslant 2n^2 \int |\nabla f|^2 \, d\mu + \left(4a^2 n^2 + \frac{1}{e}\right) |f|_\infty^2 \mu(n-1<u<n) \\ + 2|f|_\infty^2 \sqrt{\mu(u>n-1)}$$

which can be further simplified to the following estimate:

$$\int f^2 \log \frac{f^2}{\mathbf{E} f^2} \, d\mu \leqslant 2n^2 \int |\nabla f|^2 \, d\mu + \left(4a^2 n^2 + \frac{1}{e} + 2\right) |f|_\infty^2 \sqrt{\mu(u>n-1)}. \tag{2.9}$$

The exponential integrability of u will supply the required estimate on the tail probability,

$$\sqrt{\mu(u>n-1)} \leqslant e^{-\frac{C}{2}(n-1)^2} \sqrt{\mathbf{E} e^{Cu^2}}.$$

Define $b(r) = (4a^2 r^2 + \frac{1}{e} + 2) e^{-\frac{C}{2}(r-1)^2}$. Then

$$\int f^2 \log \frac{f^2}{\mathbf{E} f^2} \, d\mu \leqslant 2n^2 \int |\nabla f|^2 \, d\mu + b(n) |f|_\infty^2.$$

For r sufficiently large, $b(r)$ is a strictly monotone function whose inverse function is denoted by $b^{-1}(s)$ which decreases exponentially fast to 0. Define $\beta(s) = 2(b^{-1}(s))^2$. For any s small choose $n(s)$ to be the smallest integer such that $s \geqslant b(n)$. Then

$$\int f^2 \log \frac{f^2}{\mathbf{E} f^2} \, d\mu \leqslant \beta(s) \int |\nabla f|^2 \, d\mu + s|f|_\infty^2.$$

Here $\beta(s)$ is of order $|\log s|$ as $s \to 0$.

For part 2, note that in the above proof we only needed the weak integrability of the function u^2, or the estimate $\mu(u>n-1)$. Take the square of the inverse of $(4a^2 r^2 + \frac{1}{e} + 2) m(r-1)$ as $\frac{1}{2}\beta(s)$ for the desired conclusion. □

In [24] Eberle showed that a local Poincaré inequality holds for loops spaces over a compact manifold. However the computation was difficult and complicated and there wasn't an estimate on the blowing up rate. Although a concrete estimate from Eberle's framework can be obtained, the technique is radically different from that used here we do not make detailed comment here.

The functional inequalities for a measure describes how the L^2 or other norms of a function is controlled by its derivatives with a universal constant. They describe the concentration of an admissible function around its mean. A well chosen gradient operator is used to give these control. On the other hand concentration inequalities are related intimately with isoperimetric inequalities. For finite dimensional spaces it was shown in the remarkable works of Cattiaux–Gentil–Guillin [14] and Barthe–Cattiaux–Roberto [11] for measures in finite dimensional spaces one can pass from capacity type of inequalities to weak logarithmic Sobolev inequalities and vice versa with great precision. Similar results hold for weak Poincaré inequalities. This gives a great passage between the two inequalities. Although their results are stated for finite dimensional state spaces, they hold true in infinite dimensional spaces. We give here a direct simple proof without going through capacity inequalities. The proof is also inspired by [35]. The constant before the rate function is slightly improved but not optimal.

Lemma 2.4. *Suppose that for all f bounded measurable functions in $\mathbb{D}^{1,2}$, the weak logarithmic Sobolev inequality holds for $0 < s < r_0$, some given $r_0 > 0$,*

$$\mathbf{Ent}\big(f^2\big) \leqslant \beta(s) \int |\nabla f|^2\, d\mu + s|f|_\infty^2$$

where $\beta(s) = C \log \frac{1}{s}$ for some constant $C > 0$. Then Poincaré inequality

$$\mathbf{Var}(f) \leqslant \alpha \int |\nabla f|^2\, d\mu$$

holds for some constant $\alpha > 0$.

Proof. By the minimizing property of the variance for any real number m,

$$\mathbf{Var}(f) \leqslant \int \big((f-m)^+\big)^2\, d\mu + \int \big((f-m)^-\big)^2\, d\mu. \tag{2.10}$$

We choose m to be the median of f such that $\mu(f - m > 0) \leqslant \frac{1}{2}$ and $\mu(f - m < 0) \leqslant \frac{1}{2}$.

Let g be a positive function in $\mathbb{D}^{2,1}$ such that $\int g^2 d\mu = 1$ and $\mu\{g \neq 0\} \leqslant \frac{1}{2}$. Here we take $g = g_1$ or $g = g_2$ for

$$g_1 = \frac{(f-m)^+}{\sqrt{\int ((f-m)^+)^2\, d\mu}} \quad \text{or} \quad g_2 = \frac{(f-m)^-}{\sqrt{\int ((f-m)^-)^2\, d\mu}}. \tag{2.11}$$

For $\delta_0 > 0$ and $\delta > 1$ and $0 < \delta_0 < \delta_1 < \delta_2 < \cdots$ with $\delta_n = \delta_0 \delta^n$,

$$\begin{aligned}
\mathbf{E}g^2 &= \int_0^{+\infty} 2s\mu\big(|g| > s\big)\, ds = \int_0^{\delta_1} 2s\mu\big(|g| > s\big)\, ds + \sum_{n=1}^{\infty} \int_{\delta_n}^{\delta_{n+1}} 2s\mu\big(|g| > s\big)\, ds \\
&\leqslant \int_0^{\infty} 2s\mu\big(|g| \wedge \delta_1 > s\big)\, ds + \sum_{n=1}^{\infty} \int_{\delta_n}^{\delta_{n+1}} 2s\mu\big(|g| > s\big)\, ds.
\end{aligned}$$

Consequently we have

$$\mathbf{E}g^2 \leqslant \mathbf{E}(g \wedge \delta_1)^2 + \sum_{n=1}^{\infty} \int_{\delta_n}^{\delta_{n+1}} 2s\mu\big(|g| > s\big)\,ds. \tag{2.12}$$

Define

$$I_1 := \mathbf{E}(g \wedge \delta_1)^2, \qquad I_2 := \sum_{n=1}^{\infty} \int_{\delta_n}^{\delta_{n+1}} 2s\mu\big(|g| > s\big)\,ds.$$

Recall the following entropy inequality. If $\varphi : \Omega \to [-\infty, \infty)$ is a function such that $\mathbf{E}e^{\varphi} \leqslant 1$ and G is a real valued random function such that φ is finite on the support of G, then

$$\int G^2 \varphi \, d\mu \leqslant \mathbf{Ent}\big(G^2\big).$$

Here we take the convention that $G^2\varphi = 0$ where $G^2 = 0$ and $\varphi = \infty$. Let

$$\varphi := \begin{cases} \log 2 & \text{if } g > 0, \\ -\infty & \text{otherwise.} \end{cases}$$

Then $\int e^{\varphi}\,d\mu = 2\mu(g \neq 0) \leqslant 1$. Hence

$$\mathbf{Ent}\big((g \wedge \delta_1)^2\big) \geqslant \int (g \wedge \delta_1)^2 \varphi \, d\mu$$

so that

$$\mathbf{E}\big((g \wedge \delta_1)^2\big) \leqslant \frac{1}{\log 2} \mathbf{Ent}\big((g \wedge \delta_1)^2\big).$$

We apply the weak logarithmic Sobolev inequality

$$\mathbf{Ent}\big(f^2\big) \leqslant \beta(r)\mathbf{E}|\nabla f|^2 + r|f|_\infty^2$$

to $g \wedge \delta_1$ to obtain, for some $r < r_0$,

$$\mathbf{E}(g \wedge \delta_1)^2 \leqslant \frac{\beta(r)}{\log 2} \cdot \int |\nabla g|^2 1_{g<\delta_1}\,d\mu + \frac{r \cdot \delta_1^2}{\log 2}. \tag{2.13}$$

Now we are going to estimate I_2. For $n = 0, 1, \ldots,$ let

$$g_n = (g - \delta_n)^+ \wedge (\delta_{n+1} - \delta_n).$$

Then $g_n \in \mathbb{D}^{1,2}$, $\mathbf{E}g_n^2 \leqslant 1$ and

$$|\nabla g_n| \leqslant |\nabla g| 1_{\delta_n \leqslant g < \delta_{n+1}}.$$

From $g_n \geqslant (\delta_{n+1} - \delta_n) I_{\{g > \delta_{n+1}\}}$,

$$\mu(g > \delta_{n+1}) \leqslant \frac{\mathbf{E} g_n^2}{(\delta_{n+1} - \delta_n)^2}.$$

Next we observe that for $n \geqslant 1$,

$$\begin{aligned}\int_{\delta_n}^{\delta_{n+1}} 2s\mu\big(|g| > s\big)\, ds &\leqslant \mu(g > \delta_n) \cdot \big(\delta_{n+1}^2 - \delta_n^2\big) \\ &\leqslant \frac{\delta_{n+1}^2 - \delta_n^2}{(\delta_n - \delta_{n-1})^2} \mathbf{E} g_{n-1}^2 = \delta^2 \frac{\delta + 1}{\delta - 1} \mathbf{E} g_{n-1}^2. \qquad (2.14)\end{aligned}$$

Next we compute $\mathbf{E} g_n^2$. We'll chose a function φ_n which can be used to estimate the L^1 norm of g_n^2 by its entropy. Define

$$\varphi_n := \begin{cases} \log \delta_n^2 & \text{if } g > \delta_n, \\ -\infty & \text{otherwise.} \end{cases}$$

Then $\int e^{\varphi_n}\, d\mu = \delta_n^2 \mu(g > \delta_n) \leqslant 1$, hence

$$\mathbf{Ent}\big(g_n^2\big) \geqslant \int g_n^2 \varphi_n\, d\mu.$$

Thus,

$$\mathbf{E} g_n^2 \leqslant \frac{1}{\log \delta_n^2} \mathbf{Ent}\big(g_n^2\big) \leqslant \frac{1}{2\log \delta_0 + 2n \log \delta} \mathbf{Ent}\big(g_n^2\big). \qquad (2.15)$$

By (2.14) and (2.15) the second term in $\mathbf{E} g^2$ is controlled by the entropy of the functions g_n^2 to which we may apply the weak logarithmic Sobolev inequality with constants $r_n < r_0$. The constant r_n are to be chosen later.

$$\begin{aligned}&\int_{\delta_{n+1}}^{\delta_{n+2}} 2s\mu\big(|g| > s\big)\, ds \\ &\quad\leqslant \delta^2 \frac{\delta + 1}{\delta - 1} \cdot \frac{1}{2\log \delta_0 + 2n \log \delta} \mathbf{Ent}\big(g_n^2\big) \\ &\quad\leqslant \delta^2 \frac{\delta + 1}{\delta - 1} \cdot \frac{1}{2\log \delta_0 + 2n \log \delta} \left(\beta(r_n) \int |\nabla g|^2 I_{\delta_n \leqslant g < \delta_{n+1}}\, d\mu + r_n \cdot |g_n|_\infty^2 \right). \qquad (2.16)\end{aligned}$$

Note that $|g_n|_\infty \leqslant \delta_{n+1} - \delta_n$ and summing up in n we have,

$$I_2 \leqslant \frac{\delta^2(\delta+1)}{2(\delta-1)} \sum_{n=0}^{\infty} \frac{\beta(r_n)}{\log\delta_0 + n\log\delta} \int |\nabla g|^2 I_{\delta_n \leqslant g < \delta_{n+1}}\, d\mu$$
$$+ \frac{\delta^2-1}{2} \sum_{n=0}^{\infty} \frac{\delta_0^2 \cdot \delta^{2n+2}}{\log\delta_0 + n\log\delta} \cdot r_n. \tag{2.17}$$

Denote

$$b_{-1} = \frac{\beta(r)}{\log 2}, \qquad b_n = \frac{\delta^2(\delta+1)}{2(\delta-1)} \frac{\beta(r_n)}{\log\delta_0 + n\log\delta}$$

and

$$c_{-1} = \frac{r\cdot\delta_1^2}{\log 2}, \qquad c_n = \frac{\delta^2-1}{2} \sum_{n=0}^{\infty} \frac{\delta_0^2\cdot\delta^{2n+2}}{\log\delta_0 + n\log\delta} \cdot r_n.$$

Finally combining (2.13) with (2.17) we have

$$\mathbf{E}g^2 \leqslant b_{-1} \int |\nabla g|^2 1_{g<\delta_1} + \sum_{n=0}^{\infty} b_n \int |\nabla g|^2 1_{\{\delta_n \leqslant g < \delta_{n+1}\}}\, d\mu + \sum_{n=-1}^{\infty} c_n.$$

We'll next choose r_n so that $\sum c_n < 1/2$ and that the sequence b_n has an upper bound. This is fairly easy by choosing that r_n of the order $\frac{\delta^{-(2n+2)}}{n}$. Taking $g = g_1$, we see that

$$1 = \mathbf{E}g_1^2 \leqslant 2\sup_n (b_n) \int |\nabla g_1|^2\, d\mu + \sum c_n$$
$$\leqslant 2\sup_n (b_n) \frac{1}{\mathbf{E}[(f-m)^+]^2} \int |\nabla f|^2 1_{\{f>m\}}\, d\mu + \sum c_n.$$

Hence

$$\mathbf{E}\big[(f-m)^+\big]^2 \leqslant 4\sup_n (b_n) \int |\nabla f|^2 1_{\{f>m\}}\, d\mu,$$
$$\mathbf{E}\big[(f-m)^-\big]^2 \leqslant 4\sup_n (b_n) \int |\nabla f|^2 1_{\{f<m\}}\, d\mu.$$

The Poincaré inequality follows. □

Remark 2.5. We could optimise the constant in the Poincaré inequality. For example when $r_0 = 1/2$, we let $\epsilon = 1/8$, $\delta = \sqrt{2}$, $\delta_0 = 2^{\frac{9}{2}}$, the Poincaré constant is approximately $40.82C$, which is smaller than that given in Cattiaux–Gentil–Guillin [14]. However we do not expect to have a sharp estimate on the constant.

Proof. We need to choose the r_n, δ, δ_0 carefully to optimise on the constant. Assume that $\mathbf{E}g^2 = 1$ for simplicity. We choose suitable constants δ_0, δ, ϵ satisfying $\frac{\epsilon}{\delta_0^2\cdot\delta^2} < r_0$ and take

$r = \frac{\epsilon}{\delta_0^2 \cdot \delta^2}$ in b_{-1} and recall that $\beta(s) = C \log \frac{1}{s}$ here. Then

$$I_1 = \mathbf{E}(g \wedge \delta_1)^2 \leqslant \frac{C \cdot \log(\frac{\delta_0^2 \cdot \delta^2}{\epsilon})}{\log 2} \cdot \int |\nabla g|^2 1_{g<\delta_1}\, d\mu + \frac{\epsilon}{\log 2}. \tag{2.18}$$

Next we take

$$r_n := \frac{\log \delta^2}{\delta_0^2 \cdot \delta^{2n+2} (\log \delta_0 + n \log \delta)^2} = \frac{1}{\delta_0^2 \cdot \delta^{2n+2}} \cdot \frac{1}{(\frac{\log \delta_0}{\log \delta} + n)^2}.$$

Choose δ_0, δ so that $r_n < r_0$ for each $n \geqslant 0$, and $\log \log \delta_0 > 0$. For simplicity we denote $A := \frac{\log \delta_0}{\log \delta}$. In (2.17) take $\beta(s) = C \log \frac{1}{s}$ and note that

$$\log \frac{1}{r_n} = 2 \log \delta_0 + 2(n+1) \log \delta + 2 \log(A + n).$$

It follows that

$$\begin{aligned}
I_2 &\leqslant C\delta^2 \cdot \frac{\delta+1}{\delta-1} \sum_{n=0}^{\infty} \left(1 + \frac{1}{n+A} + \frac{\log(n+A)}{n+A} \cdot \frac{1}{\log \delta}\right) \int |\nabla g|^2 I_{\{\delta_n \leqslant g < \delta_{n+1}\}}\, d\mu \\
&\quad + \frac{\delta^2 - 1}{2 \log \delta} \sum_{n=0}^{\infty} \frac{1}{(n+A)^3} \\
&\leqslant C\delta^2 \cdot \frac{\delta+1}{\delta-1} \left(1 + \frac{1}{A} + \frac{\log A}{A} \cdot \frac{1}{\log \delta}\right) \int |\nabla g|^2 I_{\{\delta_0 \leqslant g\}}\, d\mu \\
&\quad + \frac{\delta^2 - 1}{4 \log \delta} \cdot \frac{1}{(A-1)^2}.
\end{aligned} \tag{2.19}$$

Let

$$\begin{aligned}
C_1(\delta, \delta_0, \epsilon) &:= C\delta^2 \cdot \frac{\delta+1}{\delta-1} \left(1 + \frac{1}{A} + \frac{\log A}{A} \cdot \frac{1}{\log \delta}\right), \\
C_2(\delta, \delta_0, \epsilon) &:= \frac{C \cdot \log(\frac{\delta_0^2 \cdot \delta^2}{\epsilon})}{\log 2}, \\
C_3(\delta, \delta_0, \epsilon) &:= \frac{\delta^2 - 1}{4 \log \delta} \cdot \frac{1}{(A-1)^2} + \frac{\epsilon}{\log 2}.
\end{aligned}$$

So from (2.18) and (2.19) and the assumption $\mathbf{E}g^2 = 1$, we have:

$$\mathbf{E}g^2 \leqslant \frac{C_1(\delta, \delta_0, \epsilon) + C_2(\delta, \delta_0, \epsilon)}{1 - C_3(\delta, \delta_0, \epsilon)} \int |\nabla g|^2\, d\mu \tag{2.20}$$

provided we choose suitable constants $\delta, \delta_0, \epsilon$ to make $C_3(\delta, \delta_0, \epsilon) < 1$. Apply the above estimate to g_1, g_2 and these together with (2.10) give the required inequality. $\square$

When the function $\beta(s)$ in weak logarithmic Sobolev inequality is of order greater than $\log\frac{1}{s}$, we no longer have a Poincaré inequality, but a weak Poincaré inequality is expected. In fact there is the following relation. The finite dimensional version can be found in [11]. We give here a direct proof without going through any capacity type inequalities.

Remark 2.6. If for all bounded measurable functions f in $\mathbb{D}^{1,2}$, the weak logarithmic Sobolev inequality holds for $s < r_0$, some given $r_0 > 0$ and a non-increasing function $\beta : (0, r_0) \longmapsto R^+$,

$$\mathbf{Ent}\big(f^2\big) \leqslant \beta(s)\mathbf{E}|\nabla f|^2 + s|f|_\infty^2.$$

Then there exist constants $r_1 > 0$, c_1, c_2 such that for all $s < r_1$, the weak Poincaré inequality

$$\mathbf{Var}(f) \leqslant \frac{\beta(c_2 s \log\frac{1}{s})}{c_1 \log\frac{1}{s}}\mathbf{E}\big(|\nabla f|^2\big) + s|f|_\infty^2$$

holds.

Proof. As a Poincaré inequality is not expected, we need to cut off the integrand at infinity. We keep the notation of the proof of Proposition 2.4. Let $\delta_n = \delta_0 \cdot \delta^n$ for some $\delta_0 > 1$, $\delta > 1$ and the function g as in (2.11). We have

$$\mathbf{E}g^2 = \mathbf{E}(g \wedge \delta_1)^2 + \sum_{n=1}^{N+1} \int_{\delta_n}^{\delta_{n+1}} 2s\mu(g > s)\,ds + \sum_{n=N+2}^{2N+1} \int_{\delta_n}^{\delta_{n+1}} 2s\mu(g > s)\,ds + \int_{\delta_{2N+2}}^{\infty} 2s\mu(g > s)\,ds. \tag{2.21}$$

First from $\mathbf{E}g^2 = 1$, we have the following tail behaviour:

$$\int_{\delta_{2N+2}}^{\infty} 2s\mu(g > s)\,ds = \mathbf{E}\big(g^2 - \delta_{2N+2}^2\big)_+ \leqslant |g|_\infty^2 \mu(g > \delta_{2N+2}) \leqslant \frac{1}{\delta_0^2 \delta^{4N+4}}|g|_\infty^2. \tag{2.22}$$

We now consider δ^{4N+4} to be of order $1/s$. For the first two terms of (2.21), we use estimates from the previous proof. First recall (2.13),

$$\mathbf{E}(g \wedge \delta_1)^2 \leqslant \frac{\beta(r)}{\log 2} \cdot \int |\nabla g|^2 1_{g<\delta_1}\,d\mu + \frac{r \cdot \delta_1^2}{\log 2}.$$

Next by (2.16), we have:

$$\sum_{n=1}^{N+1}\int_{\delta_n}^{\delta_{n+1}} 2s\mu(g>s)\,ds$$
$$\leqslant C_2\sum_{n=0}^{N}\frac{\beta(r_n)}{n+C_3}\int|\nabla g|^2 I_{\delta_n\leqslant g<\delta_{n+1}}\,d\mu + C_2\sum_{n=0}^{\infty} r_n\cdot\frac{\delta^{2n}}{n+C_3}.$$

Here C_2, C_3 are some constants depending on δ_0 and δ and $C_3=\frac{\log\delta_0}{\log\delta}$. For $n=0,1,\dots,N$, take

$$r_n=\frac{1}{\delta^{2n}\cdot(n+C_3)}.$$

We may assume that $\beta(r)$ is a non-increasing function of order greater than $\log(\frac{1}{r})$ for r small, in which case

$$\frac{\beta(\frac{1}{\delta^{2n}\cdot(n+C_3)})}{n+C_3}$$

is an increasing function of n for n sufficiently large. Hence

$$\sum_{n=1}^{N+1}\int_{\delta_n}^{\delta_{n+1}} 2s\mu(g>s)\,ds$$
$$\leqslant C_2\frac{\beta(\frac{1}{\delta^{2N}\cdot(N+C_3)})}{N+C_3}\int|\nabla g|^2 I_{\delta_0\leqslant g<\delta_{N+1}}\,d\mu+\frac{C_2}{C_3-1}. \tag{2.23}$$

If we apply this estimate to the whole range $n\leqslant 2N$, $\frac{\beta(r_{2N})}{2N}$ would be the order of $\beta(\frac{s}{|\log s|})$. However to make the estimate more precise, we take a different rate function r_n for $N+1\leqslant n\leqslant 2N$. Let $r_n=\frac{N}{\delta^{4N}}$ in (2.24) and we will give a more precise estimate on $|g_n|_\infty$. Apply (2.16) again to the sum from $N+1$ to $2N$ in (2.21)

$$\sum_{n=N+1}^{2N+1}\int_{\delta_n}^{\delta_{n+1}} 2s\mu(g>s)\,ds$$
$$\leqslant C_2\sum_{n=N}^{2N}\frac{\beta(r_n)}{n+C_3}\int|\nabla g|^2 I_{\delta_n\leqslant g<\delta_{n+1}}\,d\mu + C_2\sum_{n=N}^{2N}\frac{r_n}{n+C_3}\cdot|g_n|_\infty^2. \tag{2.24}$$

Since g is bounded, there is k such that $\delta_k<|g|_\infty\leqslant\delta_{k+1}$ for some integer k.

$$\sum_{n=0}^{\infty}|g_n|_\infty^2=\sum_{n=0}^{k-1}(\delta_{n+1}-\delta_n)^2+\left(|g|_\infty-\delta_k\right)^2$$
$$\leqslant\left(\sum_{n=0}^{k-1}(\delta_{n+1}-\delta_n)+|g|_\infty-\delta_k\right)^2=\left(|g|_\infty-\delta_0\right)^2.$$

Hence

$$\sum_{n=0}^{\infty} |g_n|_\infty^2 \leqslant |g|_\infty^2.$$

Recall that $r_n = \frac{N}{\delta^{4N}}$,

$$\begin{aligned} &\sum_{n=N+1}^{2N+1} \int_{\delta_n}^{\delta_{n+1}} 2s\mu(g > s)\,ds \\ &\quad \leqslant C_2 \cdot \frac{\beta(\frac{N}{\delta^{4N}})}{N + C_3} \int |\nabla g|^2 I_{\delta_N \leqslant g < \delta_{2N+1}}\,d\mu + \frac{C_2}{\delta^{4N}} |g|_\infty^2. \end{aligned} \tag{2.25}$$

Now adding estimates to all terms in (2.21) together, (2.22)–(2.25), and rearrange the constants. We also note that $\mathbf{E}g^2 = 1$ and obtain for N large enough

$$1 \leqslant C_1 \frac{\beta(\frac{N}{\delta^{4N}})}{N} \int |\nabla g|^2\,d\mu + \frac{C_2}{\delta^{4N}} |g|_\infty^2 + \frac{r\delta_1^2}{\log 2} + \frac{C_2}{C_3 - 1}. \tag{2.26}$$

Here we use the monotonicity of β: $\beta(\frac{N}{\delta^{4N}}) \geqslant \beta(\frac{1}{\delta^{2N}\cdot(N+C_3)})$. Take r small and δ_0 large so that $\frac{r\delta_1^2}{\log 2} + \frac{C_2}{C_3-1} < 1$. Let $s = \frac{1}{\delta^{4N}}$ in (2.26), the required result follows. $\square$

Corollary 2.7. *Let μ be a probability measure. Suppose that there is a positive function $\mu(u^2 > r^2) \sim m^2(r)$ some increasing function m of order $o(r^{-2})$ for r large and such that $|\nabla u| \leqslant a$, $a > 0$ and for all $f \in \mathbb{D}^{1,2}$,*

$$\mathbf{Ent}\big(f^2\big) \leqslant \int u^2 |\nabla f|^2\,d\mu. \tag{2.27}$$

Let $l(s)$ be the inverse function of $r^2 m(r)$, for s sufficiently small. Then

$$\mathbf{Var}(f) \leqslant c_2 \frac{l^2(c_1 s|\log s|)}{|\log s|} \int |\nabla f|^2\,d\mu + s|f|_\infty^2.$$

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