Large time asymptotics of Barycentres of Brownian motions on Hyperbolic spaces

Marc Arnaudon and Xue-Mei Li

In [**1**] we considered the evolution of barycentres of a measure carried by stochastic flows. Under sutiable conditions the barycentre as a stochastic process was shown to be a semi-martingale and solves a stochastic differential equation. Here we are mainly interested in the evolution of exponential barycentres of independen Brownian motions on a hyperbolic plane.

The expoential barycentre of a mass of unit one on the hyperbolic plane $\mathbb{H}$ is preciesely the minimizer of the distance square averaged under this measure. Let us consider only discrete measures $\mu = \sum_{n=1}^{m} p_i \delta_{x^i}$ where x^i are distinctive points in $\mathbb{H}$. Its barycentre $b(\mu)$ exists and is unique and is given by the solution of the following equation:

$$\sum_{i=1}^{n} \dot\gamma(x^i, b(\mu))(0) = 0 \tag{1}$$

where $\gamma(x,y)(s), 0 \le s \le 1$ is the geodesic starting from x and ends at y at time 1, and $\dot\gamma$ denotes for time derivative of γ. The barycentre of the measure μ exists and is unique. Furthermore it is a smooth function of $(x^1, \dots, x^n)$ by the implicit function theorem and the following estimate.

Lemma 0.1. *For the map $H : \mathbb{H}^{n+1} \to T\mathbb{H}$ defined by*

$$H(z, x^1, \dots, x^n) = \sum_{j=1}^{n} p_j \dot\gamma(z, x^j)(0),$$

$$\left\langle \nabla H(z, x^1, \dots, x^n)(u), u \right\rangle_{\mathbb{H}} \le -\varepsilon^2 \|u\|_{\mathbb{H}}^2 \sum_{1 \le j \le n} p_j \rho(z, x^j), \qquad u \in T_z\mathbb{H}$$

where ∇ denotes covariant derivative with respect to the first variable, and

$$\varepsilon(z, x^1, \dots, x^n) = \sup_{1 \le j \le n} \left| \cos\left(u, \dot\gamma(z, x^j)(0)\right) \right|.$$

Proof. Write $u = u^{L(z,x^j)} + u^{N(z,x^j)}$ as the sum of a vector u^L along the geodesic $\gamma(z, x^j)$ and its complement. Then

$$\begin{aligned} \left\| u^{N(z,x^j)} \right\|_{\mathbb{H}}^2 &= \sin^2\left(u, \dot\gamma(z, x^j)(0)\right) \|u\|_{\mathbb{H}}^2 \\ \left\| u^{L(z,x^j)} \right\|_{\mathbb{H}}^2 &= \cos^2\left(u, \dot\gamma(z, x^j)(0)\right) \|u\|_{\mathbb{H}}^2. \end{aligned}$$

For $x \in \mathbb{H}$ and $u, v \in T_x\mathbb{H}$, let $\langle u, v\rangle_{\mathbb{H}}$ denote the scalar product of u and v. Then

$$\begin{aligned}
&\left\langle \nabla H(z, x^1, \dots, x^n)(u), u\right\rangle_{\mathbb{H}} \\
&\quad = -\sum_{j=1}^n p_j \left(\left\| u^{L(z,x^j)} \right\|_{\mathbb{H}}^2 + \rho(z, x^j) \coth \rho(z, x^j) \left\| u^{N(z,x^j)} \right\|_{\mathbb{H}}^2 \right) \\
&\quad \le -\sum_{j=1}^n p_j \left(\left\| u^{L(z,x^j)} \right\|_{\mathbb{H}}^2 + \rho(z, x^j) \left\| u^{N(z,x^j)} \right\|_{\mathbb{H}}^2 \right) \\
&\quad \le -\|u\|_{\mathbb{H}}^2 \sum_{j=1}^n p_j \left(\cos^2\left(u, \dot\gamma(z, x^j)(0)\right) + \sin^2\left(u, \dot\gamma(z, x^j)(0)\right) \rho(z, x^j) \right)
\end{aligned}$$

Thus for $z \neq x^j$,

$$\left\langle \nabla H(z, x^1, \dots, x^n)(u), u\right\rangle_{\mathbb{H}} \le -\min\left(\min_{1\le j\le n} \rho(z, x^j), 1\right)\|u\|_{\mathbb{H}}^2.$$

If $z = x^{j_0}$ some j_0 then

$$\left\langle \nabla H(z, x^1, \dots, x^n)(u), u\right\rangle_{\mathbb{H}} \le -\min_j p_j \|u\|_{\mathbb{H}}^2.$$

□

However as t goes to infinity a Brownian motion on $\mathbb{H}$ converges to a point on the boundary, as observed by Dynkyn, Prat etc, and so the barycentre of the measure $\mu_t \equiv \sum_{i=1}^n p_i X_t^i$ where X_t^i are independent Brownian motions with initial point x^i cannot be defined through the above equation. On the other hand given a reasonable measue on the boundary, which shall be defined below, there is the concept of Buseman barycentres.

Denote by $\bar{\mathbb{H}}$ the visibility compactification of $\mathbb{H}$ and let $\partial\mathbb{H}$ be its boundary circle so $\partial\mathbb{H}$ is the set of equivalence classes of geodesic rays. In the disc model $\partial\mathbb{H}$ is the boundary circle and in the half upper plane model it is the union of the boundary line and the point at infinity. In both cases the visibility topology coincides with the usual topology when $\mathbb{H}$ is considered as a subset of R^2.

Let $(\varphi(z, y)(s), s \ge 0)$ be the unit speed geodesic connecting z and y starting at z with $\dot\varphi(z, y)(0) = \frac{d}{ds}\big|_{s=0}\varphi(z, y)(s)$. The Buseman barycentre of the measure $\frac{1}{n}\sum_{j=1}^n p_j \delta_{y^j}$ where $y^j \in \partial\mathbb{H}$ is defined to be the solution of

$$\lim_{t\to\infty} \sum_{j=1}^n p_j \dot\varphi(Z_\infty, y^j)(0) = 0$$

if it exists. We prove that the exponential barycentre of $n \ge 3$ independent Brownian motions on the hyperbolic plane $\mathbb{H}$ converges almost surely to a random variable in $\mathbb{H}$ as time goes to infinity. The limit is exactly the Buseman barycentre of the corresponding limit points of the Brownian motions on the boundary $\partial\mathbb{H}$.

More precisely note that a sequence of points γ_n goes to a boundary point γ in the visibility compactification if and only if the function $\rho(\cdot, \gamma_n) - \rho(o, \gamma_n)$ on $\mathbb{H}$ converges uniformly on compact sets where ρ is the distance function on $\mathbb{H}$. For any point $x, y \in \mathbb{H}$, $\gamma \in \partial\mathbb{H}$ we define the Buseman function:

$$\beta_\gamma(x,y) = \lim_{n\to\infty} \left(\rho(y,\gamma_n) - \rho(x,\gamma_n)\right),$$

where $\gamma_n \in \mathbb{H}$ is a sequence converging to γ. The limit exists and is independent of the choice of the sequence γ_n. Furthermore β_γ satisfies the cocycle property:

$$\beta_\gamma(x,y) + \beta_\gamma(y,z) = \beta_\gamma(x,z), \qquad \forall x, y, z \in \mathbb{H}.$$

See [**4**], [**2**]. In particular for the chosen reference point o in $\mathbb{H}$, there is the (normalized) Buseman function:

$$\beta_\gamma(z) = \lim_{n\to\infty} \left(\rho(z,\gamma_n) - \rho(o,\gamma)\right), \qquad z \in \mathbb{H}, \gamma \in \partial\mathbb{H}.$$

Equivalently

$$\beta_\gamma(z) = \lim_{t\to\infty} \left(\rho(z,\gamma_t) - t\right), \qquad z \in \mathbb{H}, \gamma \in \partial\mathbb{H}$$

where $\gamma(t)$ is the geodesic ray from o to γ parameterized by arc length. In particular $\beta_\gamma(o) = 0$ and for any point $\gamma' \in \partial\mathbb{H}$, $\neq \gamma$, $\lim_{z\to\gamma'} \beta_\gamma(z) = +\infty$. Note by the cocycle property if another reference point o' is chosen, the two Buseman functions differ only by the constant $\beta_\gamma(o, o')$.

For $x_i \in \mathbb{H}$, $1 \le i \le n$ let $\bar\mu$ be the discrete measure given by: $\bar\mu = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$. We associate to $\bar\mu$ a Buseman function as follows:

$$\beta_{\bar\mu}(z) = \frac{1}{n}\sum_{i=1}^n \lim_{t\to\infty} \left(\rho(z, x_i(t)) - t\right). \tag{2}$$

If there is a point in $\mathbb{H}$ which minimizes $\beta_{\bar\mu}$ it is called the Buseman barycentre of the measure $\bar\mu$.

Note that each term $\rho(z, x_i(t))$ is differentiable in the first variable and that

$$\nabla\beta_{\bar\mu}(z) = \frac{1}{n}\lim_{t\to\infty}\sum_{i=1}^n \nabla_1(\rho(z,x_i(t)) - t) = \frac{1}{n}\sum_{i=1}^n \lim_{t\to\infty} \frac{\dot\gamma(z, x_i(t)(0)}{\rho(z, x_i(t))}.$$

For $z \in \mathbb{H}$ and $x \in \mathbb{H}\cup\partial\mathbb{H}$ satisfying $x \neq z$, we denote by $\varphi(z,x)(s)$ $(0 \le s)$ the hyperbolic geodesic connecting z and x with hyperbolic speed 1: $\varphi(z,x)(0) = z$ and $\varphi(z,x)(\rho(z,x)) = x$ for $x \in \mathbb{H}$ and $\lim_{s\to\infty}\varphi(z,x)(s) = x$ in Euclidean metric for $x \in \partial\mathbb{H}$. Then

$$\dot\varphi(z, x_i(t))(0) = \lim_{t\to\infty} \frac{\dot\gamma(z, x_i(t))(0)}{\rho(z, x_i(t))}.$$

In [**1**] it was shown that if $n \ge 3$

$$\sum_{i=1}^n \dot\phi(z, x_i)(0) = 0 \tag{3}$$

has a unique solution which shall be denoted by $\bar z$, i.e. $b_{\bar\mu}$ has a unique minimizer $\bar z$. Next consider the equation:

$$\sum_{i=1}^n \dot\gamma(y_t^i, x)(0) \equiv \sum_{i=1}^n \dot\phi(z, x_i(t))(0)\rho(z, x_i(t)) = 0.$$

For each t the equation above has a unique solution $z_t \equiv G_p(x_1(t), \dots, x_n(t))$, as the barycentre of the measure $\frac{1}{n}\sum_{i=1}^n \delta_{x_i(t)}$, where G_p is a Lipschitz continuous function. In fact that

$$\lim_{t\to\infty} G_p(x_1(t), \dots, x_n(t)) = \bar z,$$

each $x_i(t)$ converges to x_i with uniform speed (this includes the case when $x_i(t)$ are Brownian particles.)

The discussion can adapt to the universal cover $\tilde{M}$ of a connected locally symmetric manifold M with negative curvature where the concept of Buseman function and Buseman barycentre are well defined for measures on the boundary. see e.g. [**3**]. Furthermore a measure on $\partial\tilde{M}$ with no atom of weight more than $\frac{1}{3}$ has a unique Buseman barycentre. The convergence of barycentres carried by a stochastic flow, e.g. by an isotropic flow, depends on the speed of convergence to infinity of the flow as can be seen by the observation below which follows from a conversation with V. Kaimanovich: Let μ be a measure on $\partial\tilde{M}$ with no atom of weight more than 1/3. If μ_n is a sequence of measures on $\tilde{M}$ and μ a measure on $\partial\tilde{M}$ such that $\beta_{\mu_n}(x) \equiv \int_M \beta_y(x) d\mu_n(y)$ converges to $\beta_\mu(x) \equiv \int_{\partial\tilde{M}} \beta_\gamma(x)\mu(d\gamma)$ uniformly on compact sets, then the minimizer of β_{μ_n} is unique and converges to the Buseman barycentre of μ. Note that even though the minimizer of β_{μ_n} is different from the exponential barycentre of μ_n, the convergence of exponential barycentres to the corresponding Buseman barycentre for well behaved flows can be deduced using the argument described above.

The asymptotics of the barycentre of two independent Brownian motions is slightly different. Let X_t and Y_t be two independent Brownian motions in $\mathbb{H}$. on a probability space $(\Omega, \mathcal{F}_t, P)$. Then the geodesic $(XY)_t = (X_t Y_t)$ connecting X_t and Y_t converges almost surely, uniformly on compact sets, to a limiting geodesic $(XY)_\infty$. We prove that the barycentre Z_t is a martingale converging to a Brownian motion on $(XY)_\infty$ in the following sense: for each $\mathcal{F}_\infty$ measurable random variable u on $(XY)_\infty$, that is $u(\omega) \in (XY)_\infty(\omega)$ for almost surely all ω, there exists a sequence of stopping times T_n going to infinity such that the law of the process $(Z_{T_n+t})_{t\geq 0}$ conditioned by (X_{T_n}, Y_{T_n}), converges to the law of a Brownian motion of variance 2 on $(XY)_\infty$ starting from u.

References

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Département de mathématiques
Université de Poitiers, Téléport 2 - BP 30179
F–86962 Futuroscope Chasseneuil Cedex, France
E-mail address: arnaudon@mathlabo.univ-poitiers.fr

Department of Computing and Mathematics
The Nottingham Trent University, Nottingham NG1 4BU, U.K.
E-mail address: xuemei.li@ntu.ac.uk