Derivative flows of stochastic differential equations: moment exponents and geometric properties

K. D. Elworthy and X.-M. Li

1 Introduction

A stochastic differential equation can arise either as a meaningful quantity in itself (e.g. as a model of physical, biological, or economic behaviour, or as something of geometric significance), or as a way of constructing or representing a diffusion process. The study of the solution flow is of obvious importance in the first situation. In the second the precise form of the flow will depend on the choice of stochastic differential equation used, and different choices can lead to flows with radically different properties. However the flow does contain all the information about the diffusion processes and so about its associated generator and Markov semigroup; and conversely properties of the semigroup restrict the qualitative behaviour of the flow [Elw82], [Sch89],[Li94a], [Elw92]. This is also true of the 'linearization' of the flow, the derivative flow. The extension of Ruelle's ergodic theory of diffeomorphisms to stochastic flows [Car85] has shown how the linearization reflects the stability properties of the flow. In [Elw88], [ER96], [EY93], [Li95]

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it is shown that the behaviour of the linearization is intimately connected with the topology of the underlying state space. Some of these results and their geometric consequences are described in §§1,2,4 below, while in §3 we describe criteria in terms of the coefficients of the equation which give the relevant properties, and even existence in the non-compact case, of the flows.

The derivative flow arises naturally from differentiation under the expectation sign in the Feyman-Kac formula. This leads to useful, and simple, formulae exhibiting the smoothing properties of non-degenerate Markov semigroups [EL94] extending Bismut's well known formula (which uses the Ricci curvature of the associated metric). There are also analogous formulae for the heat equation for q-forms on a Riemannian manifold, and these are described in §5.

The 'Ricci flow' which appears in Bismut's formula comes from the Markov generator of the diffusion; in §1 D we give a result from [EY93] which in particular shows that the Ricci flow can be considered as a predictable projection of the derivative flow for the gradient stochastic differential equation. This emphasizes the fact that the derivative flow involves additional structure. This can also be put in a topological context: for example, in the compact case, strong moment stability for the Ricci flow implies that the fundamental group is finite [Li95] while for the derivative flow it implies that the fundamental group vanishes, see §1B, (with analogous results for the non-compact case).

2 The moment exponents: compact manifolds

A. Consider a stochastic differential equation

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt \tag{1}$$

on an n-dimensional complete Riemannian manifold, with $\circ$ denoting the Stratonovich stochastic integral. Here $\{B_t : t \geq 0\}$ is Brownian motion on $\mathbb{R}^m$, $X(x) : \mathbb{R}^m \to T_xM$ is a linear map into the tangent space T_xM at x to M for each x in M and A is a vector field on M. Assume both X and A are C^∞.

For M compact (1) has a solution flow

$$F_t\colon M \times \Omega \to M, \qquad t \geq 0$$

where $\{\Omega, \mathcal{F}, P\}$ is the probability space of $\{B_t : t \geq 0\}$ consisting, almost surely, of C^∞ diffeomorphisms $F_t(-, \omega) : M \to M$. These have derivatives $T_{x_0}F_t(-, \omega) : T_{x_0}M \to T_{x_t}M$ for $x_t = F_t(x_0)$. The sample Lyapunov exponents of (1) occur as almost sure limits

$$\lim_{t\to\infty} \frac{1}{t} \log |TF_t(v_0)|, \qquad v_0 \in TM.$$

If all negative they imply samplewise stability, and (1) is said to be stable. Stronger forms of stability are determined by the moment exponents:

$$\mu_K(p) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log \sup_{x\in K} \mathbb{E}|T_xF_t|^p$$

for $K \subset M$, (see e.g. [Arn84] [BS88]) and their generalization:

$$\mu_K^q(p) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log \sup_{x\in K} \mathbb{E}| \wedge^q T_xF_t|^p$$

where $\wedge^q T_{x_0}F_t(-, \omega)\colon \wedge^q T_{x_0}M \to \wedge^q T_{x_t}M$ is the linear map determined by

$$v^1 \wedge \ldots \wedge v^q \mapsto T_{x_0}F_t(-, \omega)v^1 \wedge \ldots \wedge T_{x_0}F_t(-, \omega)v^q$$

for $\omega \in \Omega$.

B. It turns out that these quantities can only be negative given strong topological and geometrical conditions on the manifold M. The simplest result, which implies that periodic strongly moment stable s.d.e. cannot exist on R, is [Elw93]:
Suppose M is compact and (1) has $\mu_M(1) < 0$, then M is simply connected.

The use of integral currents to represent integral homology classes yields extensions of this [ER96], for example:

If M is compact and (1) has $\mu_M^q(1) < 0$ then $H_q(M, Z) = 0$. Consequently for $k = 1, 2, \ldots, n-1$, $\mu_M(k) < 0$ implies $\pi_i(M) = 0$ for $i = 1$ to k and in particular if $\mu_M([\frac{n+1}{2}]) < 0$, then M is a homotopy sphere.

A more direct extension gives [ER96]: *If M admits an equation (1) which is strongly 2 moment stable then every C^1 map $\alpha : N \to M$ of finite energy of Riemannian manifold N into M is homotopic to a map with arbitrary small energy, as is every C^1 map $\beta : M \to N$.*

The corresponding result is given for p-energy assuming strong p-th moment stability where the p-energy $E_p(\alpha)$ of a map $\alpha : M \to N$ is defined by

$$E_p(\alpha) = \int_M ||T_y\alpha||^p dy,$$

(the usual energy corresponding to the case $p = 2$).

An interesting class of examples are those of the *gradient systems*. Here M is isometrically immersed in some Riemannian space R^m and for each $x \in M$ the map $X(x) : R^m \to T_xM$ is just the orthogonal projection. With $A = 0$ the solutions to (1) are Brownian motions on M. When M is the circle S^1 the standard embedding of S^1 in R^2 gives the gradient Brownian system

$$d\theta_t = \cos\theta_t dB_t^1 + \sin\theta_t dB_t^2$$

parametrizing S^1 by angle θ, with $B_t = (B_t^1, B_t^2)$ Brownian motion on R^2. For this $\mu_M(1) = 0$, and for $M = S^n, n > 1$ with standard embedding in R^{n+1}, $\mu_M(k) < 0$ for $1 \le k \le n-1$ [Elw88].

C. For M compact when (1) and the derivative equation satisfy certain hypoellipticity conditions on M and the sphere bundle of M it was shown in [BS88] that *strong moment stability* $\mu_M(1) < 0$ is implied by *moment stability*, $\mu_x(1) < 0$, for $x \in M$. This possibly weaker notion implies vanishing of the real cohomology group $H^1(M, R)$, when the differential generator $\mathcal{A}$ of the solution to (1) is $\mathcal{A} = \frac{1}{2}\triangle + \nabla h$ for some smooth $h: M \to R$; and when (1) is a gradient Brownian system with a drift ∇h, so $\mathcal{A} = \frac{1}{2}\triangle + \nabla h$ again, then $\mu_x^q(1) < 0$ for ¸some x implies $H^q(M; R) = 0$ [Elw92]. In fact in this case

$$\lambda^{h,q} \le \mu_x^q(1) \le \mu_x(q) \tag{2}$$

where $\lambda^{h,q}$ is the highest eigenvalue of the Bismut-Witten Laplacian on q-forms:

$$\frac{1}{2}\triangle^{h,q} \equiv \frac{1}{2}\triangle^q + \mathcal{L}_{\nabla h}$$

where $\mathcal{L}_{\nabla h}$ denotes Lie-differentiation by ∇h and $\triangle^q$ is the Hodge Laplacian on q-forms.

D. The special property of these gradient systems is that [Elw92], [Kus88]

$$e^{\frac{1}{2}t\triangle^{h,q}}\phi = \mathbb{E}F_t^*(\phi) \tag{3}$$

where $F_t^*(\phi)(v^1,\ldots,v^q) = \phi(T_{x_0}F_t(v^1),\ldots,T_{x_0}F_t(v^2))$ for $v^1,\ldots,v^q$ in $T_{x_0}M$; a fact which holds for arbitrary Brownian systems in general only when $q=1$ and ϕ is closed (i.e. $d\phi = 0$). In fact, [EY93],

For a gradient Brownian system with drift A on a compact manifold, the conditional expectation of $\wedge^q TF_t(v_0)$ given $\{x_s : 0 \leq s \leq t\}$ satisfies

$$\mathbb{E}\left\{\wedge^q TF_t(v_0)|\sigma\{x_s : 0 \leq s \leq t\}\right\} = W_{x_0,t}^{A,q}(v_0) \tag{4}$$

where $v_t \equiv W_{x_0,t}^{A,q}(v_0)$ satisfies the covariant equation along the solution of (1)

$$\frac{Dv_t}{\partial t} = -\frac{1}{2}\mathcal{R}_{x_t}^q(v_t) + (d\wedge^q)\nabla A(v_t)$$

for $v_0 \in \wedge^q T_{x_0}F_t$, with $\mathcal{R}_x^q : \wedge^q T_xM \to \wedge^q T_xM$ the Weitzenbock curvature.

Here $(d\wedge^q)(\nabla A)$ is defined to be linear and given on primitive elements by

$$(d\wedge^q)(\nabla A)(v^1,\ldots,v^q) = \nabla A(v^1)\wedge v^2\ldots\wedge v^q + v^1\wedge\nabla A(v^2)\wedge v^3\ldots\wedge v^q + \ldots.$$

Note that the 'conditional expectation' used in (3) does not make immediate sense since $\wedge^q TF_t(v_0)$ may lie in different linear spaces $\wedge^q T_{x_t}M$ for different elements ω of our underlying probability space. To define it first parallel translate back along $\{x_s : 0 \leq s \leq t\}$ to $\wedge^q T_{x_0}M$, then take the classical conditional expectation, and then parallel translate back to $\wedge^q T_{x_t}M$.

If $A = \nabla h$, it is a standard result that

$$e^{\frac{1}{2}t\triangle^{q,h}}\phi(v_0) = \mathbb{E}\phi(W_{x_0,t}^{h,q}(v_0)) \tag{5}$$

for a q-form ϕ and $v_0 \in \wedge^q T_{x_0}M$. Thus in this case (3) follows from (4).

From (4) it also follows that for gradient systems [ER93]

$$\mu_{x_0}^q(1) \geq \overline{\lim_{t\to\infty}} \frac{1}{t} \log \mathbb{E}||W_{x_0,t}^{A,q}||. \tag{6}$$

Based on (5) there are strong topological consequences of the right hand side of (6) being negative e.g. [ER88] [ER91]. In particular (6) together with results from [ER88] [ER91] gives [ER93].

For a gradient Brownian flow on a compact manifold
(i) If dim M=3, then moment stability implies $\pi_2(M) = 0$,
(ii) If dim M=4 and M is simply connected then $\mu_x^2(1) < 0$ implies M is diffeomorphic to the sphere S^4.

E. Such vanishing results are usually associated with positivity conditions on the curvature, and for Brownian systems we have [EY93]:

On a compact manifold M with non positive Ricci curvature no Brownian system is moment stable. If the Ricci curvature is negative then $\mu_x(1) > 0$ for all such systems.

3 Non-compact manifolds

A. For noncompact manifolds the situation becomes more complicated from the start. First of all the solutions to (1) may only exist up to an explosion time ξ; but even if they exist for all time there may be no continuous version of the flow [Elw78], [Kun90], [Jak89]. However the derivative in probability $v_t = TF_t(v_0)$ does exist for $v_0 \in T_{x_0}M$ and is given by the covariant equation

$$Dv_t = \nabla X(v_t) \circ dB_t + \nabla A(v_t)dt \tag{7}$$

up to the explosion time of (1). Nevertheless $|v_t|$ need not to be integrable so that the moment exponents may not exist, and also the associated semigroup δP_t on 1-forms given by

$$\delta P_t(\phi)(v_0) = \mathbb{E}\phi(v_t)$$

with its extensions to q forms may not exist, and there is no reason to believe we can differentiate under the expectation sign to obtain $d(P_t f) = \delta P_t(df)$

for P_t the Markov semigroup associated to (1). It turns out [Li92] that integrability conditions on $|v_t|$ are the key to all of these questions, at least when $\mathcal{A} = \frac{1}{2}\triangle + \nabla h$, and to most of them in general. Criteria for such integrability can, in turn, be obtained in terms of the coefficients of (1) and (7). For nonexplosion a modification of a proof by Bakry [Bak86] shows that [Li92]:

Let $\{F_t(x)\}$ be a h-Brownian motion, i.e. it has $\frac{1}{2}\triangle + \nabla h$ as generator, on a complete Riemannian manifold with $d(P_t f) = \delta P_t(df)$ for f in C_0^∞, the space of smooth functions with compact support. Suppose there is an open set U of M and $t_0 > 0$ such that $\mathbb{E}(|T_x F_t| \chi_{t<\xi(x)}) < \infty$ for all $x \in U$ and $t < t_0$ where $\xi(x)$ is the explosion time from x. Then $\xi(x) = \infty$ a.s. for all x in M, i.e. there is no explosion.

Moreover if (1) is strongly moment stable, i.e. $\mu_K(1) < 0$ for all compact sets K, then M has finite volume (with respect to $\mathrm{e}^{2h} dx$ for dx the Riemannian volume element).

As for differentiation under the expectation sign, for arbitrary generator $\mathcal{A}$ we have [Li94c] [EL94] *for all f bounded with bounded first derivatives, $dP_t f = (\delta P_t)(df)$ provided for each $t > 0$ and compact subset K, $\sup_{x \in K} \mathbb{E}|T_x F_t|^{1+\delta} < \infty$ for some $\delta > 0$ and $\mathbb{E} \sup_{s \le t} |T_x F_s| < \infty$.*

It turns out that it is useful to have weaker notions than the existence of a continuous solution flow, and (1) is said to be *strongly p-complete* if there is a version of the solution flow continuous on any given (smooth) image in M of the standard p-simplex. For example the flow of $dx_t = dB_t$ on $R^n - \{0\}$ is easily seen to be strongly (n-2)-complete, but not strongly (n-1)-complete. Furthermore [Li94c] *if (1) is strongly (n-1)-complete, then it has a global smooth solution flow, i.e. is strongly complete ('strictly conservative').*

From [Li92] we have:

The equation (1) on a complete Riemannian manifold is strongly p-complete if it is complete and if

$$\sup_{x \in K} \mathbb{E} \sup_{s \le t} |T_x F_s|^{p+\delta} < \infty$$

for all K compact and some constant $\delta > 0$ (δ can be zero if $p = 1$).

B. These ideas enable the extension of some of the result of §1 to the noncompact case. The first example applied to a cylinder almost shows that no periodic system (1) on R^n can be strongly moment stable [Li92]:

For a complete Riemannian manifold and (1) with generator $\frac{1}{2}\triangle + \nabla h$, *assume there is no explosion and* $\sup_{x\in K} \mathbb{E} \sup_{s\leq t} |TF_s| < \infty$ *for all compact subsets* K, *then the first homotopy group* $\pi_1(M)$ *vanishes if we have strong moment stability.*

There are also cohomology obstructions to strong moment stability in the non-compact case [Li92]:

If M *is a Riemannian manifold on which there is a strongly p-complete and strongly p-th moment stable (i.e.* $\mu_K(p) < 0$ *for all compact sets* K*) stochastic differential equation, then all bounded closed p-forms are exact. Consequently the natural map from* $H_K^p(M,R)$, *the pth real cohomology with compact supports, to* $H^p(M,R)$, *the pth real cohomology, is trivial for such a manifold.*

4 Conditions on the coefficients

The results mentioned depend on integrability conditions on TF_t. These are controlled by the quadratic field H_p defined for Brownian systems with drift Z, or any system on R^n with $A = Z$, by:

$$\begin{aligned} H_p(x)(v,v) &= -Ric_x(v,v) + 2 < \nabla Z(x)(,v) > + \sum_1^m |\nabla X^i(x)(v)|^2 \\ &\quad +(p-2)\sum_1^m \frac{1}{|v|^2} < \nabla X^i(x)(v), v >^2, \end{aligned}$$

for $v \in T_xM$, $x \in M$. The basic results are as follows, for refinements see [Li94c]:

1. *Suppose (1) is complete, then it is strongly 1-complete if* H_1 *is bounded above; and strongly complete if* $|\nabla X|$ *is bounded and* H_1 *is bounded above.*

2. *If* $H_{1+\delta}$ *is bounded above, then* $dP_t f = \delta P_t(df)$ *for all* $f \in BC^1$.

3. *Strong pth-moment stability holds if for some* $c > 0$,

$$H_p(x)(v,v) \le -c|v|^2, \qquad \textit{for } v \in T_xM, x \in M$$

4. *If* $M = R^n$ *and the Stratonovich differential in (1) is replaced by the Itô differential, suppose that* X *and* A *have linear growth, i.e.*

$$|X(x)| \le c(1+|x|), \ < x, A(x) > \le c(1+|x|)$$

for some constant c. *Then the stochastic differential equation has a smooth solution flow if the derivatives of the coefficients have sub-logarithmic growth, i.e.*

$$|\nabla X(x)| \le c[1 + \ln(1+|x|)],$$

$$< \nabla A(x)(v), v > \le c[1 + \ln(1+|x|)]|v|^2.$$

See also [Tan89].

For gradient Brownian systems, $H_p(x)(v,v)$ can be written in terms of the second fundamental form $\alpha_x : T_xM \times T_xM \to \nu_x M$ where $\nu_x M$ is the normal space to M for its given immersion in R^m:

$$\begin{aligned} H_p(x)(v,v) &= -< \alpha(v,v), \text{trace}\, \alpha > +2 < Hess(h)(v), v > +2|\alpha_x(v,-)|^2_{H,S} \\ &\quad + \frac{(p-2)}{|v|^2}|\alpha_x(v,v)|^2_{\nu_x}. \end{aligned}$$

Here $|-|_{H,S}$ denotes the Hilbert-Schmidt norm, and Hess(h) is the Hessian of h. Let $h_p(x) = p \sup_{|v|\le 1} H_p(x)(v,v)$, and $\underline{h}_p(x) = p \inf_{|v|\le 1} H_p(x)(v,v)$. We have [Li94b]:

For M *isometrically immersed in* R^m, *the induced gradient Brownian system is pth moment stable if*

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log \mathbb{E}\left(e^{\frac{1}{2}\int_0^t h_p(F_s(x))ds} \right) < 0$$

for each $x \in M$ *and p-th moment unstable if*

$$\underline{\lim_{t\to\infty}} \frac{1}{t} \log \mathbb{E}\left(e^{\frac{1}{2}\int_0^t \underline{h}_p(F_s(x))ds} \right) > 0$$

5 Vanishing theorems from spectral positivity

By applying the above to an immersion of M in R^m various relations between the second fundamental form of M and the topology of M are obtained, extending and refining results of Lawson & Simon [LS73] and Howard & Wei [HW86]. A feature of the stochastic approach is that the results only require positivity (of the curvatures etc.) on 'most' of the manifold, e.g. see [ELR93], [RY94]. Here are some results. We let $\underline{\mathrm{Ric}}(\mathrm{x})$ be the infimum of the Ricci curvature at $x \in M$, and $\check{\underline{\mathcal{R}}}^p(x)$ the infimum of the p-th Weitzenbock curvature on primitive vectors:

$$\check{\underline{\mathcal{R}}}^p(x) = \inf\{\mathcal{R}^p(v_1 \wedge \ldots \wedge v_p) : v_1, \ldots, v_p \in T_xM, < v_i, v_j >_x = \delta_{i,j}\},$$

also let $H : M \to R^m$ be the mean curvature $H(x) = \frac{1}{n}\mathrm{tr}\,\alpha_x$.

[ER96] For M compact, if $\triangle - \underline{Ric} + \frac{||\alpha||^2}{2} - \frac{n}{2}|H|^2 < 0$ then $\pi_1(M) = 0$; and if $\triangle - \check{\underline{\mathcal{R}}}^2 + \frac{1}{2}||\alpha||^2 - \frac{n}{2}|H|^2 < 0$, then $\pi_2(M) = 0$. If $\triangle - \check{\underline{\mathcal{R}}}^q + \frac{1}{2}||\alpha||^2 - \frac{n}{2}|H|^2 < 0$, then $H_q(M, Z) = H_{n-q}(M, Z) = 0$.

[Li94b] If $||\alpha_x||^2 \le const.\,(1 + \ln[1 + d(x)])$, $x \in M$, where d is the Riemannian distance from a fixed point, and if $\triangle - \underline{Ric} + \frac{||\alpha||^2}{2} - \frac{n}{2}|H|^2 < 0$ then $\pi_1(M) = 0$.

6 Differentiation of heat semigroups

Let $P_t^q = e^{\frac{1}{2}\triangle^q t}$ be the heat semigroup for q-forms on M. The derivatives of gradient flows can be used to give rather simple formulae for $dP_t^q\phi$ in terms of the given q-forms ϕ when ϕ is not necessarily differentiable. For this define the 1-form valued process Ψ_t by

$$\Psi_{t,x}(v) = \int_0^t < X(x_s)dB_s, TF_s(v) >$$

(so $\{\Psi_{t,x} : t \ge 0\}$ is a local martingale in T_x^*M with tensor quadratic variation $\{\int_0^t ((T_xF_s)^* T_xF_s)\, ds : t \ge 0\}$). In fact for a gradient system, Ψ_t is exact:

$\Psi_t = d\psi_t$ where $\psi_t : M \to \mathbb{R}$ is given by

$$\psi_t(x) = \int_0^t < jF_s(x), dB_s >_{\mathbb{R}^m} ds$$

where $j : M \to \mathbb{R}^m$ is the given immersion.

For a gradient Brownian system,

$$dP_t^{h,q-1}\phi = \frac{1}{t}\mathbb{E}\left\{\Psi_t \wedge F_t^*(\phi)\right\}$$

for ϕ a (q-1) form with $d\phi$ a bounded C^2 form in the domain of $\triangle^{h,q}$.

For $q = 0$, there are corresponding results for more general non-degenerate diffusion equations [Elw92], [EL94]. For applications see [LZ], [DPEZ95], and [PZ93].

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Address:
Mathematics Institute
University of Warwick
Coventry CV4 7AL, U.K.
e-mail: xl@maths.warwick.ac.uk.bitnet