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Strong p-completeness of stochastic differential equations and the existence of smooth flows on noncompact manifolds

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Summary. Here we discuss the regularity of solutions of SDE's and obtain conditions under which a SDE on a complete Riemannian manifold M has a global smooth solution flow, in particular improving the usual global Lipschitz hypothesis when $M = R^n$. There are also results on non-explosion of diffusions.

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1 Introduction

Let M be a n-dimensional connected smooth manifold and B_t an m-dimensional Brownian motion on a probability space $\{\Omega, \mathscr{F}, P\}$ with filtration $\{\mathscr{F}_t\}$. Consider the (Stratonovich) stochastic differential equation (SDE) on M:

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt\,. \tag{1}$$

Here X is C^2 from $R^m \times M$ to the tangent bundle TM with $X(x): R^m \to T_xM$ a linear map for each x in M, and A is a C^2 vector field on M. The pair (X,A) is called a stochastic dynamical system (SDS). Let $\{e_1, e_2, \ldots, e_m\}$ be an orthonormal basis for R^m. Set $X^i(x) = X(x)(e^i)$, and write $B_t = (B_t^1, \ldots, B_t^m)$. Then (1) can be written as:

$$dx_t = \sum_{i=1}^m X^i(x_t) \circ dB_t^i + A(x_t)dt\,.$$

Let $\{F_t(x)\}$ be the solution to (1) starting from x with explosion time $\xi(x)$.

A SDE on a Riemannian manifold is called a Brownian system with drift Z if it has (i.e. its associated semigroup has) generator $\frac{1}{2}\Delta + L_Z$. Here Δ is the Laplacian, Z is a vector field and L_Z is the Lie derivative in the direction Z.

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Its solution is called a Brownian motion with drift Z. Let h be a C^3 function on M. The Bismut-Witten Laplacian is $\Delta^h =: \Delta + 2L_{\nabla h}$. A SDE with generator $\frac{1}{2}\Delta^h$ is called a h-Brownian system. Its solution is called a h-Brownian motion.

Recall that a SDE is called *complete* if its explosion time $\xi(x) = \infty$ for each x; it is *strongly complete* if the solution can be chosen to be jointly continuous in time and space for all time. Such a solution is called a *continuous flow.*

The known results on the existence of a continuous flow are mostly on R^n and on compact manifolds. On R^n results are given in terms of global Lipschitz conditions. See Blagovescenskii and Friedlin [3]. The problems concerning the diffeomorphism property of flows have been discussed by e.g. Kunita [15], Carverhill and Elworthy [4]. See Taniguchi [22] for discussions on the strong completeness of a stochastic dynamical system on an open set of R^n. For discussions of higher derivatives of solution flows on R^n, see Krylov [14] and Norris [20].

On a compact manifold, a SDE with C^2 coefficients is strongly complete. In fact the solution flow is C^{r-1} if the coefficients are C^r. Moreover the flow consists of diffeomorphisms. See Kunita [15], Elworthy [9], and Carverhill and Elworthy [4]. For discussions in the framework of diffeomorphism groups see Baxendale [2] and Elworthy [10].

In the article, we discuss the regularity of solution flows from a new approach. We introduce the notions of "strong p-completeness". Roughly speaking a SDE is strongly p-complete, if the map $F.(-)$ is continuous in time and space for all time while restricted to a smooth p-dimensional submanifold of M. This concept reveals the complicated regularity property of the flow. For example the flow $x + B_t$ on $R^n - \{0\}$ is strongly (n − 2)-complete but not strongly (n − 1)-complete (see example 2 in Sect. 2); on $R^n - \ell$, for ℓ a smoothly immersed curve it is only strongly (n − 3)-complete, $n \geq 3$.

Besides this, strong 1-completeness turns out to be a powerful tool for obtaining results on differentiating semigroups (Sect. 9), for getting formulae for the derivatives of the logarithms of the heat kernels [12], or for obtaining related topological and geometrical properties of the underlying manifolds[17][18] via moment stability. The moment stability part is illustrated in theorem 2.4 below.

Main Results

Theorem 2.3 *A stochastic dynamical system on a smooth manifold is strongly complete if strongly* $(n-1)$*-complete.*

Now consider M furnished with a complete Riemannian metric and associated Levi-Civita connection.

Theorem 3.1/4.1 *A SDE on a complete connected Riemannian manifold is strongly p-complete if it is complete at one point and its derivative flow* $\{T_xF.\}$ *satisfies: for each compact set K and each* $t > 0$,

$$\sup_{x\in K} E\left(\sup_{s\leq t}|T_xF_s|^{p+\delta}\right) < \infty$$

for some $\delta > 0$ *(*δ *can be taken to be zero for* $p = 1$*).*

Note for $p = 1$ we only require the first moment of $|T_x F_t|$, so do better than a Sobolev type theorem.

Following from these, we obtain Theorem 5.1 giving criterion for the existence of a global smooth flow in terms of the coefficients of the stochastic differential equations. A straightforward application of Theorem 5.1 extends the standard global Lipschitz result on R^n (Corollary 5.2): denote by $\mathscr{A}$ the differential generator for (1), which is given by

$$\mathscr{A} f(x) = \frac{1}{2}\sum_1^m \nabla^2 f(X^i(x), X^i(x)) + A^X(f)(x).$$

Here $A^X = \frac{1}{2}\Sigma_1^m \nabla X^i(X^i) + A$ is the first order part of the generator.

Theorem 5.3/Corollary 5.2 *A complete SDE on a complete Riemannian manifold is strongly 1-complete if*

$$\begin{aligned} H_1(x)(v,v) = & 2\langle \nabla A^X(v), v\rangle + \sum_1^m \langle R(X^i, v)(X^i), v\rangle \\ & + \sum_1^m |\nabla X^i(v)|^2 - \sum_1^m \frac{1}{|v|^2}\langle \nabla X^i(v), v\rangle^2 \end{aligned}$$

is bounded above. Here R is the curvature tensor. It is strongly complete if $|\nabla X|$ is bounded and if for some constant c

$$2\langle \nabla A^X(v), v\rangle + \sum_1^m \langle R(X^i, v)(X^i), v\rangle \leqq c|v|^2.$$

There are also more refined results:

Theorem 6.2 *Let $M = R^n$ with its flat metric. Suppose the coefficients of the SDE have linear growth, then its solution flow consists of diffeomorphisms if the first derivatives of its coefficients have sub-logarithmic growth.*

Let $r(x)$ denote the distance between x and a fixed point in M.

Theorem 8.2 *A Brownian motion with drift Z is complete if the Ricci curvature is bounded from below by $-c(1 + r^2(x))$, and $dr(Z) \leqq c(1 + r(x))$. It is strongly complete if both $|\nabla X|^2$ and $2\langle \nabla Z(x)(-), -\rangle - Ric_x(-,-)$ have sub-logarithmic growth in the distance function r.*

2 Strong p-completeness: definition

Let S_p be the space of the images of all smooth (smooth in the sense of extending over an open neighbourhood) singular p-simplices. Recall that a singular p-simplex in M is a map from the standard p-simplex to M. For convenience we also use the term singular p-simplex for the image of a singular p-simplex map.

Before giving the definition, here is an example:

Example 1 [9], [10] Let $X(x)(e) = e$, and $A = 0$. Consider the following stochastic differential equation $dx_t = dB_t$ on $R^n - \{0\}$ for $n > 1$. The solution is: $F_t(x) = x + B_t$, which is complete since for a fixed starting point

x, $F_t(x)$ almost surely never hits 0. But it is not strongly complete. However for any n − 2 dimensional hyperplane (or a submanifold) H in the manifold, $\inf_{x\in H}\xi(x,\omega)=\infty$ a.s., since a Brownian motion does not charge a set of codimension 2.

This leads to the following definition suggested by D. Elworthy:

Definition 2.1 *A SDE on a manifold is called strongly p-complete if its solution can be chosen to be jointly continuous in time and space a.s. for all time when restricted to a set $K \in S_p$.*

Example 2 The example above on $R^n - \{0\}$ (for $n > 2$) gives us a SDS which is strongly (n − 2)-complete, but not strongly (n − 1)-complete. It is not strongly $(n-1)$-complete from Proposition 2.3. We shall show it is strongly (n − 2)-complete.

First note every singular n − 2 simplex has an extension to a bounded Lipschitz map from the cube $[0,\ 1]^{n-2}$ to M. Let U be a subset of R^{n-2} containing a ball radius $\varepsilon > 0$. Let f be a bounded Lipschitz map from U to R^n. We only need to show that the capacity Cap(f) of $f(U)$ is zero. For this, the author is grateful to Dr P. Kröger for the following proof. Let $a=\inf_{x\in U}f(x)$. Clearly $\mathrm{Cap}(f(U))=0$ is equivalent to $\mathrm{Cap}(2a+f(U))=0$. Thus we may assume $a>0$. Define $h: R^n \to R\cup\{\infty\}$ as follows:

$$h(y)=\int_U \frac{dx}{|f(x)-y|^{n-2}}\,.$$

Clearly $h(y)$ is superharmonic. Thus $h(B_t)$ is a supermartingale. By the maximal inequality for positive supermartingales, we have:

$$P\left\{\sup_{0\leqq s} h(B_s)\geqq n\right\}\leqq \frac{1}{n}Eh(0)\,.$$

So $P\{\sup_{0\leqq s}h(B_s)=\infty\}=0$. This proves $\mathrm{Cap}(h^{-1}(\infty))=0$. Next we show $f(U)\subset h^{-1}(\infty)$. Let $y=f(z)$ for $z\in U$, then for some constant c,

$$\begin{aligned} h(y)&=\int_U \frac{dx}{|f(x)-f(z)|^{n-2}}\geqq c\int_U\frac{dx}{|x-z|^{n-2}}\\ &\geqq c\int_{B_\varepsilon}\frac{dx}{|x|^{n-2}}=\infty\,. \end{aligned}$$

Thus $\mathrm{Cap}(f(U))=0$ as wanted. ∎

For further discussions, we need the following theorem on the existence of a partial flow, taken from [10] based on [15]. See also [4].

Theorem 2.1 *Suppose X and A are C^r, for $r\geqq 2$. Then there is a partially defined flow $(F_t(\,\cdot\,),\xi(\,\cdot\,))$ which is a maximal solution to (1) such that if $M_t(\omega)=\{x\in M, t<\xi(x,\omega)\}$, then there is a set Ω_0 of full measure such that for all $\omega\in\Omega_0$:*

1. $M_t(\omega)$ is open in M for each $t>0$, i.e. $\xi(\,\cdot\,,\omega)$ is lower semicontinuous.

2. $F_t(\cdot,\omega): M_t(\omega) \to M$ *is in* C^{r-1} *and is a diffeomorphism onto an open subset of* M. *Moreover the map:* $t \mapsto F_t(\cdot,\omega)$ *is continuous into* $C^{r-1}(M_t(\omega))$, *with the topology of uniform convergence on compacta of the first* $r-1$ *derivatives.*

3. Let K *be a compact set and* $\xi^K = \inf_{x\in K}\xi(x)$. *Then*

$$\lim_{t\nearrow\xi^K(\omega)} \sup_{x\in K} d(x_0, F_t(x)) = \infty \tag{2}$$

almost surely on the set $\{\xi^K < \infty\}$. *(Here* x_0 *is a fixed point of* M *and* d *is any complete metric on* M.*)*

From now on, we shall use (F_t, ξ) for the partial flow defined in Theorem 2.1 unless otherwise stated. Note that if the solution can be chosen to be continuous in time and space for all time on a compact set K, then the explosion time ξ^K in the above lemma is infinite (Elworthy [10]). Thus strong p-completeness of a SDE is equivalent to $\xi^K = \infty$ a.s. for all $K \in S_p$.

Proposition 2.2 *If the SDE considered is strongly p-complete, then* $\xi^N = \infty$ *a.s. for any* p *dimensional smooth submanifold* N *of* M. *In particular* F *can be chosen to be* C^{r-1} *on any given smooth p-dimensional submanifold.*

Proof. Let N be a p dimensional submanifold. Since all smooth differential manifolds have a smooth triangulation (Munkres [19]), we can write: $N = \cup V_i$. Here V_i are smooth singular p-simplices. But $\xi^{V_i} = \infty$ a.s. for each i from the assumption. Thus $F.(\cdot)|V_i$ is continuous a.s. and so is $F|_N$ itself. This gives $\xi^N = \infty$ almost surely. The existence of a C^{r-1} version comes from a uniqueness result from [10]. ∎

Note that if $\sigma: \Delta^p \to M$ is a smooth p-simplex, then by [10], strong p-completeness implies that $F_t \circ \sigma$ has a C^{r-1} version.

If p equals the dimension of M, strong p-completeness gives back the usual definition of strong completeness, i.e. the partial flow defined in Theorem 2.1 satisfies $\inf_{x\in M}\xi(x) = \infty$ almost surely. In this case we will continue to use strong completeness for strong n-completeness.

Theorem 2.3 *A stochastic dynamical system on a n-dimensional manifold is strongly complete if strongly* $(n-1)$*-complete.*

Proof. Since we have strong completeness for compact manifolds, we shall assume M is not compact in the following proof. Let B be a geodesic ball centered at some point p in M with radius smaller than the injectivity radius at p. Since M can be covered by a countable number of such balls, we only need to prove $\xi^B = \infty$ almost surely.

Let B be such a ball. Clearly $M - \partial B$ consists of two parts, one K_0 say bounded and the other N_0 unbounded. Fix $T > 0$. By the ambient isotopy theorem there is a diffeomorphism H from $[0,T]\times M$ to $[0,T]\times M$ given by: $(t,x) \mapsto (t, h_t(x))$ for h_t some diffeomorphism from M to its image, and satisfying:

$$h_t|_{\partial B} = F_t|_{\partial B}\,.$$

Set $K_t = h_t(K_0), N_t = h_t(N_0)$. Then

$$M = K_t \cup F_t(\partial B) \cup N_t ,$$

and

$$F_t(\mathring{B}) \subset K_t \tag{3}$$

on $\{\omega : t < \xi^B(\omega)\}$. Now

$$\bigcup_{0 \leqq t \leqq T} \bar{K}_t = \mathrm{Proj}^1[H(\bar{K}_0 \times [0,T])] ,$$

here Proj^1 denotes the projection to M. Thus $\cup_{0 \leqq t \leqq T} \bar{K}_t$ is compact. By (3), $F_t(B) = F_t(K_0) \cup F_t(\partial B)$, for $0 \leqq t \leqq T \wedge \xi^B$, stays in a compact region. So $\xi^B \geqq T$ almost surely from part 3 of Theorem 2.1. ■

Application of strong p-completeness

Let $C^\infty(\Omega^p)$ be the space of C^∞ smooth p forms on $M, H^p(M,R)$ the p^{th} *de Rham cohomology* group, and $H^p_K(M,R)$ the de Rham cohomology group for compactly supported p-forms. Recall that a SDS is said to be strongly p^{th}-moment stable if for all $K \subset M$ compact,

$$\mu_K(p) = \overline{\lim_{t \to \infty}} \sup_{x \in K} \frac{1}{t} \log E|T_x F_t|^p < 0 .$$

The following theorem follows from an approach of [8] for compact manifolds. For a discussion of such topological consequences of strong moment stability on noncompact manifolds, see [17].

Theorem 2.4 *Let M be a Riemannian manifold and assume there is a strongly p-complete SDS with strong p^{th}-moment stability. Then all bounded closed p-forms are exact. In particular the natural map from $H^p_K(M,R)$ to $H^p(M,R)$ is trivial.*

Proof. Let σ be a singular p-simplex, and ϕ a bounded closed p-form. We shall not distinguish a singular simplex map from its image. Denote by $F_t^*\phi$ the pull back of the form ϕ and $(F_t)_*\sigma = F_t \circ \sigma$. Then

$$\int_{(F_t)_*\sigma} \phi = \int_\sigma (F_t)^*\phi .$$

But $(F_t)_*\sigma$ is homotopic to σ by the strong p-completeness. Thus:

$$\int_\sigma \phi = \int_{(F_t)_*\sigma} \phi = \int_\sigma (F_t)^*\phi .$$

Using strong p^{th} moment stability,

$$\begin{aligned} E\left|\int_\sigma \phi\right| &= \lim_{t \to \infty} E\left|\int_\sigma (F_t)^*\phi\right| \leqq |\phi|_\infty \overline{\lim_{t \to \infty}} \int_\sigma E|TF_t|^p \\ &\leqq |\phi|_\infty \overline{\lim_{t \to \infty}} \sup_{x \in \sigma} E|T_x F_t|^p = 0 . \end{aligned}$$

So $\int_\sigma \phi = 0$, and ϕ is exact by de Rham's theorem. ■

Theorem 4.1 below suggests that strong p-completeness is not a major restriction given strong moment stability.

3 Strong 1-completeness

Take a sequence of nested relatively compact open sets $\{U_i\}$ such that it is a cover for M and $\bar{U}_i \subset U_{i+1}$. Let λ^i be a standard smooth cut off function such that:

$$\lambda^i = \begin{cases} 1 & x \in U_{i+1} \\ 0, & x \notin U_{i+2}. \end{cases}$$

Let $X^i = \lambda^i X, A^i = \lambda^i A$, and F^i. the solution flow to the SDS (X^i, A^i). Then F^i can be taken smooth since both X^i and A^i have compact support. Let $S_i(x)$ be the first exit time of $F_t^i(x)$ from $\bar{U}_i$ and $S_i^K = \inf_{x\in K} S_i(x)$ for a compact set K. Thus S_i^K is a stopping time. Furthermore $F_t^i(x) = F_t(x)$ before S_i^K. Clearly $S_i^K \leqq \xi^K$, and in fact $\lim_{i\to\infty} S_i^K = \xi^K$ as proved in [4].
Let

$$K_1^1 = \{\text{Image}(\sigma) | \sigma : [0,\ell] \to M \text{ is } C^1, \ell < \infty\}.$$

Suppose M is given a complete Riemannian metric. Denote by $|-|$ the norm with respect to this metric. Let $TF_t(v)$ be the derivative of F_t in the direction v, whenever it exists. Note it always exists in probability up to explosion time. See [10]. We shall call $\{TF_t(-) : t \geqq 0\}$ the derivative flow.

Theorem 3.1 *Let M be a complete connected Riemannian manifold. Assume there is a point $\bar{x} \in M$ with $\xi(\bar{x}) = \infty$ almost surely. Then $\xi^H = \infty$ for all $H \in K_1^1$, if*

$$\varliminf_{j\to\infty} \sup_{x\in K} E\left(|T_x F_{S_j^K}| \chi_{S_j^K < t}\right) < \infty \tag{4}$$

for every compact set $K \in K_1^1$ and each $t > 0$. In particular when (4) holds we have strong 1-completeness, and strong completeness if the dimension of M is less or equal to 2.

Proof. Let $y_0 \in M$. Let σ_0 be a piecewise C^1 curve parametrized by arc length with end points: $\sigma_0(0) = \bar{x}$, and $\sigma_0(\ell_0) = y_0$. Denote by K_0 the image set of the curve. Let $K_t = \{F_t(x) : x \in K_0\}$, and $\sigma_t = F_t \circ \sigma_0$ be the composed curve with length $\ell(\sigma_t)$. Then $\sigma_t(\omega)$ is a piecewise C^1 curve on $\{\omega : t < \xi^{K_0}(\omega)\}$. Let T be a stopping time such that $T < \xi^{K_0}$, then for each $t > 0$,

$$E\ell(\sigma_T)\chi_{T<t} \leqq E\int_0^{\ell_0} \left|\frac{d}{ds}(F_{T(w)}(\sigma(s),(w))\right| ds\, \chi_{T<t} \tag{5}$$

$$\leqq \int_0^{\ell_0} E(\chi_{T<t}|T_{\sigma(s)}F_T|)ds \leqq \ell_0 \sup_{x\in K_0} E(|T_x F_T| \chi_{T<t}) . \tag{6}$$

Assume $P\{\xi^{K_0} < \infty\} > 0$. There is a number T_0 with $P\{\xi^{K_0} < T_0\} > 0$. On the other hand there is also a number $R(\omega)$ such that $R(\omega) < \infty$ a.s. and

$$\sup_{0 \leqq t \leqq T_0} d(F_t(\bar{x}, \omega), \bar{x}) \leqq R(\omega) \tag{7}$$

following from $\xi(\bar{x}) = \infty$ a.s. But by Theorem 2.1,

$$\lim_{t \nearrow \xi^{K_0}} \sup_{x \in K_0} d(\bar{x}, F_t(x, \omega)) = \infty \tag{8}$$

almost surely on $\{\xi^{K_0} < \infty\}$. So the triangle inequality combined with (7) and (8) yield:

$$\begin{aligned} \underline{\lim}_{t \nearrow \xi^{K_0}} \sup_{x \in K_0} d(F_t(x,\omega), F_t(\bar{x},\omega)) &\geqq \underline{\lim}_{t \nearrow \xi^{K_0}} \left[\sup_{x \in K_0} d(F_t(x,\omega),\bar{x}) - d(\bar{x}, F_t(\bar{x},\omega)) \right] \\ &\geqq \underline{\lim}_{t \nearrow \xi^{K_0}} \sup_{x \in K_0} d(F_t(x,\omega),\bar{x}) - \sup_{0 \leqq t \leqq T_0} d(\bar{x}, F_t(\bar{x},\omega)) = \infty \end{aligned}$$

on $\{\omega : \xi^{K_0} < T_0\}$. Therefore on this set,

$$\underline{\lim}_{t \nearrow \xi^{K_0}} \ell(\sigma_t(\omega)) \geqq \underline{\lim}_{t \nearrow \xi^{K_0}} \sup_{x \in K_0} d(F_t(x,\omega), F_t(\bar{x},\omega)) = \infty \tag{9}$$

almost surely for $t \leqq T_0$. Let $T_j =: S_j^{K_0}$ be as defined in the beginning of the section, which converge to ξ^{K_0}. Then there is a subsequence, still denoted by $\{T_j\}$, s.t. on $\{\xi^{K_0} < T_0\}$,

$$\lim_{j \to \infty} \ell(\sigma_{T_j}) \chi_{\xi^{K_0} < T_0} = \infty, \text{ a.s.} \tag{10}$$

However by equation (6), hypothesis (4) and Fatou's lemma:

$$\begin{aligned} E \underline{\lim}_{j \to \infty} \ell(\sigma_{T_j}) \chi_{\xi^{K_0} < T_0} &\leqq \underline{\lim}_{j \to \infty} E \ell(\sigma_{T_j(\omega)}(\omega)) \chi_{T^j < T_0} \\ &\leqq \ell_0 \underline{\lim}_{j \to \infty} \sup_{x \in K_0} E |T_x F_{T_j}| \chi_{T_j < T_0} < \infty , \end{aligned}$$

contradicting (10). Thus $\xi^{K_0} = \infty$. In particular $\xi(y) = \infty$ for all $y \in M$.

Next take $K \in K_1^1$, and replace K_0 by K in the above proof to get $\xi^K = \infty$, for we only used the fact that there is a point $\bar{x}$ in K_0 with $\xi(\bar{x}) = \infty$ and $|TF_t|$ satisfies (4).

To see strong 1-completeness, just notice the set of smooth singular 1-simplices S_1 is contained in K_1^1. ■

Example 3 A. The requirement for the manifold to be complete is necessary. e.g. example 1 on $R^2 - \{0\}$ in section 2 satisfies equation (4) but is not strongly complete. In fact if we apply the inversion map $z \mapsto \frac{1}{z}$ in complex form as in [4]. The resulting system on R^2 is $(\hat{X}, B)$ where

$$\hat{X}(x,y) = \begin{bmatrix} y^2 - x^2 & 2xy \\ -2xy & y^2 - x^2 \end{bmatrix} .$$

The transformed flow $F_t(z) = \frac{z}{1+zB_t}$ on R^2 by inverting does not satisfy the condition of the theorem on its derivative and it is not strongly 1-complete.

B. Theorem 3.1 is sharp in the sense it does not work if equation (4) is replaced by $\sup_x E|T_xF_t| < \infty$. This can be seen by using the above example on $M = R^2 - \{0\}$ but with the following Riemannian metric:

$$|v|^{\#} = \frac{|v|}{|x|}, v \in T_xM .$$

This is a complete metric since $\int_0^1 \frac{ds}{s} = \infty$ so the point $\{0\}$ is 'at infinity'. But for each compact set K and $t > 0$

$$\sup_{x\in K} E|T_xF_t|^{\#} = \sup_{x\in K} E\frac{1}{|x+B_t|} < \infty .$$

We say a SDE is *complete at one point* if there is a point x_0 in M with $\xi(x_0) = \infty$. From the theorem we have the following corollary, which is known for elliptic diffusions without condition (4).

Corollary 3.2 *The SDE* (1) *is complete if it is complete at one point and satisfies condition* (4) *of theorem* 3.1.

4 Strong p-completeness, flows of diffeomorphisms

Denote by L_p the space of all the image sets of Lipschitz maps from $[0,1]^p$ to M. As in the last section, we assume that M is connected and is given a complete Riemannian metric.

Theorem 4.1 *Assume that the SDE (1) is complete at one point. Let $1 \leqq p \leqq n$. Then $\xi^K = \infty$ for each $K \in L_p$, if for each positive number t and compact set K there is a number $\delta > 0$ such that:*

$$\sup_{x\in K} E\left(\sup_{s\leqq t}|T_xF_s|^{p+\delta}\chi_{s<\xi}\right) < \infty . \tag{11}$$

In particular this implies strong p-completeness.

Proof. Let σ be a Lipschitz map from $[0,1]^p$ to M with image set K. Take a compact set $\hat{K}$ with the following property: for any two points of K, there is a piecewise C^1 curve lying in $\hat{K}$ connecting them.

Let $x = \sigma(\underline{s})$ and $y = \sigma(\underline{t})$ and α be a piecewise C^1 curve in $\hat{K}$ connecting them. Denote by H_α the image set of α and ℓ its length. By proposition 3.1, $\xi^{H_\alpha} = \infty$. Thus for any $T_0 > 0$ we have:

$$\begin{aligned}
E\sup_{t\leqq T_0}[d(F_t(x),F_t(y))]^{p+\delta} &\leqq E\left(\int_0^\ell \sup_{t\leqq T_0}|T_{\alpha(s)}F_t|ds\right)^{p+\delta} \\
&\leqq \ell^{p+\delta-1}E\int_0^\ell\left(\sup_{t\leqq T_0}|T_{\alpha(s)}F_t|^{p+\delta}\right)ds \\
&\leqq \ell^{p+\delta}\sup_{x\in\hat{K}}\left(E\sup_{t\leqq T_0}|T_xF_t|^{p+\delta}\right) .
\end{aligned}$$

Taking infimum over a sequence of such curves which minimizing the distance between x and y, we get:

$$E\left(\sup_{t\leqq T_0} d(F_t(x),F_t(y))^{p+\delta}\right) \leqq d(x,y)^{p+\delta}\sup_{x\in\hat{K}} E\left(\sup_{t\leqq T_0}|T_xF_t|^{p+\delta}\right).$$

The Lipschitz property of the map σ gives

$$E\left(\sup_{t\leqq T_0} d(F_t(\sigma(\underline{s}),F_t(\sigma(\underline{t}))))^{p+\delta}\right) \leqq c|\underline{s}-\underline{t}|^{p+\delta}\sup_{x\in\hat{K}} E\left(\sup_{t\leqq T_0}|T_xF_t|^{p+\delta}\right).$$

Thus we have a modification $\tilde{F}.(\sigma(-))$ of $F.(\sigma(-))$ which is jointly continuous from $[0,T_0]\times[0,1]^p \to M$, according to a generalized Kolmogorov's criterion (see e.g. [10]). So for a fixed point x_0 in M:

$$\sup_{t\in[0,T_0]}\sup_{\underline{s}\in[0,1]^p} d(F_t(\sigma(\underline{s}),\omega),x_0) < \infty .$$

On the other hand on $\{\xi^K<\infty\}$, $\lim_{t\nearrow\xi^K}\sup_{x\in K} d(F_t(x,\omega),x_0)=\infty$ almost surely. So ξ^K has to be infinity.

Finally strong p-completeness follows from the fact that every singular p-simplex has an extension to a Lipschitz map from the cube $[0,1]^p$ to M (by squashing one half of the cube to the diagonal). ■

Remarks. 1) As a consequence, we get that a SDS is strongly complete if it is complete at one point and satisfies:

$$\sup_{x\in K} E\sup_{s\leqq t}|T_xF_s|^{n-1+\delta} < \infty$$

for some $\delta>0$ and for each compact subset K of M. On the other hand, any direct application of a Sobolev type inequality would require that the above integrability condition holds for a p^{th} power ($p>n$) of $|T_xF_t|$.
2) Note condition (11) in the theorem cannot be replaced by $\sup_{x\in K}E|T_xF_t|^{p+\delta}$ is finite, since the flow $x+B_t$ with the complete Riemannian metric $\langle,\rangle^{\#}$ in example 3 (section 3) satisfies: for $p<n, \sup_{x\in K}E(|T_xF_t|^{\#})^p<\infty$.

Flows of diffeomorphisms

For the diffeomorphism property, we only need to look at the "adjoint" system of (1):

$$dy_t = X(y_t)\circ dB_t - A(y_t)dt. \tag{12}$$

A strongly complete SDE has a flow of diffeomorphisms if and only its adjoint equation is also strongly complete. See Kunita [15]. See also Carverhill and Elworthy [4]. Suppose there is a uniform cover for (X,A). Then its flow consists of diffeomorphisms if for each compact set K,

$$\sup_{x\in K} E \sup_{s\leq t}\left(|T_xF_s|^{n-1+\delta} + \left(|T_{F_s^{-1}(x)}F_s|^{-1}\right)^{n-1+\delta}\right) < \infty\,,$$

since in this case both equation (1) and (12) are strongly complete by the previous theorem. In this case we also have the C_0-property, i.e. the associated semigroup preserves $C_0(M)$, the space of continuous functions vanishing at infinity. See [11].

5 Existence of smooth flows

Let M be a Riemannian manifold with Levi-Civita connection ∇. There is the stochastic covariant differential equation

$$dv_t = \nabla X(v_t)\circ dB_t + \nabla A(v_t)dt\,. \tag{13}$$

Denote by $T_xF_t(v)$ its solution starting from v. It is in fact the derivative of $F_t(x)$ in measure. See [10]. Let $x_0 \in M, v_0 \in T_{x_0}M$. We shall write $x_t = F_t(x_0)$, and $v_t = T_{x_0}F_t(v_0)$.

The expectations of the norms of $|v_t|$ can be estimated through the following equation (see e.g. Elworthy [7], or [6]):

$$\begin{aligned}|v_t|^p =&|v_0|^p + p\sum_{i=1}^m\int_0^t|v_s|^{p-2}\langle\nabla X^i(v_s), v_s\rangle dB_s^i\\ &+\frac{p}{2}\int_0^t|v_s|^{p-2}H_p(x_s)(v_s,v_s)ds\end{aligned} \tag{14}$$

on $\{t<\xi\}$. Here

$$\begin{aligned}H_p(x)(v,v) =&2\langle\nabla A(x)(v),v\rangle + \sum_{i=1}^m\langle\nabla^2X^i(X^i,v),v\rangle\\ &+\sum_1^m\langle\nabla X^i(\nabla X^i(v)),v\rangle + \sum_1^m|\nabla X^i(v)|^2\\ &+(p-2)\sum_1^m\frac{1}{|v|^2}\langle\nabla X^i(v),v\rangle^2\,,\end{aligned} \tag{15}$$

for all $x\in M$ and $v\in T_xM$. To simplify notation, let

$$M_t^p = \sum_1^m p\int_0^t\frac{\langle\nabla X^i(v_s),v_s\rangle}{|v_s|^2}dB_s^i\,, \tag{16}$$

$$a_t^p = \frac{p}{2}\int_0^t\frac{H_p(x_s)(v_s,v_s)}{|v_s|^2}ds\,. \tag{17}$$

Here M_t^p and a_t^p depends on the point $(x_0,v_0)\in TM$. We shall omit the superscript p if no confusion is caused. Then Eq. (14) gives:

$$|v_t|^p = |v_0|^p e^{M_t^p - \frac{\langle M^p,M^p\rangle_t}{2}+a_t^p} \tag{18}$$

as used by Taniguchi [22]. Let $|X(x)|^2 = \sum_1^m|X^i(x)|^2$, and let $|\nabla X(x)|^2 = \sum_1^m|\nabla X^i(x)|^2$. We have:

Theorem 5.1 *Let M be a complete connected Riemannian manifold. Suppose the SDE* (1) *is complete at one point. Let $p > 0$. Assume there is a function $f : M \to [0, \infty)$ such that:*

1. $\sup_{x\in K} E\left(e^{6p^2 \int_0^t f(F_s(x))\chi_{s<\xi(x)}ds}\right) < \infty$, *for all $t > 0, K$ compact.*
2. $|\nabla X(x)|^2 \leqq f(x)$.
3. $H_p(x)(v, v) \leqq 6pf(x)|v|^2$ *for all $x \in M$ and $v \in T_xM$.*

Then the system is complete and

$$E\left(\sup_{s\leqq t}|T_xF_s|^p\right) < cE\left(e^{6p^2 \int_0^t f(F_s(x))ds}\right) .$$

In particular the system is strongly d-complete for $d < p$.

Proof. First we assume that the SDE is complete. Applying Schwartz's inequality to equation (18), we get for each $p > 0$:

$$E\left(\sup_{s\leqq t}|v_s|^p\right) \leqq |v_0|^p \left(E\sup_{s\leqq t} e^{2M_s-\langle M,M\rangle_s}\right)^{\frac{1}{2}} \left(E\sup_{s\leqq t} e^{2a_s}\right)^{\frac{1}{2}} .$$

Since

$$E\left(e^{6\langle M,M\rangle_s}\right) \leqq E\left(e^{6p^2 \int_0^t f(x_s)ds}\right) < \infty ,$$

$e^{2M_s-\frac{\langle M,M\rangle_s}{2}}$ is a martingale by Novikov's criterion [21]. Consequently

$$\begin{aligned} E\left(\sup_{s\leqq t} e^{2M_s-\langle M,M\rangle_s}\right) &\leqq 4\sup_{s\leqq t} E\left(e^{2M_s-\langle M,M\rangle_s}\right) \\ &= 4\sup_{s\leqq t} E\left(e^{2M_s-4\langle M,M\rangle_s}e^{3\langle M,M\rangle_s}\right) \leqq 4\left[E\left(e^{6\langle M,M\rangle_t}\right)\right]^{\frac{1}{2}} , \end{aligned}$$

by Cauchy Schwartz and using the fact that $e^{4M_s-8\langle M,M\rangle_s}$ is a supermartingale. Also

$$E\left(\sup_{s\leqq t} e^{2a_s}\right) = E\left(\sup_{\alpha\leqq t} e^{p\int_0^\alpha \frac{H_p(v_s,v_s)}{|v_s|^2}ds}\right) \leqq E\left(e^{6p^2 \int_0^t f(x_s)ds}\right)$$

giving

$$E\left(\sup_{s\leqq t}|v_s|^p\right) \leqq 2|v_0|^p \left[E\left(e^{6p^2 \int_0^t f(x_s)ds}\right)\right]^{\frac{3}{4}} < \infty .$$

Thus for some constant c_2 (depending only on p and n),

$$E\left(\sup_{s\leqq t}|T_xF_s|^p\right) \leqq c_2E\left(e^{(6p^2 \int_0^t f(F_s(x))ds)}\right) .$$

Thus for each compact subset K of the manifold,

$$\sup_{x\in K} E\left(\sup_{s\leqq t}|T_xF_s|^p\right) \leqq c_2 \sup_{x\in K} E\left(e^{\left(6p^2\int_0^t f(F_s(x))ds\right)}\right) < \infty\,.$$

Next assume (1) is complete at one point, we shall show that it is complete everywhere. Let $K \in K_1^1$ be a compact subset of M, and let S_j^K be stopping times as in Theorem 3.1. Then

$$|T_xF_{t\wedge S_j^K}(v_0)| = |v_0| e^{\left(M_{t\wedge S_j^K} - \frac{\langle M,M\rangle_{t\wedge S_j^K}}{2} + a_{t\wedge S_j^K}\right)}\,.$$

Similar calculations as above yield:

$$\sup_{x\in K} E(|T_xF_{S_j^K}|\chi_{S_j^K<t}) \leqq c \sup_{x\in K} E\left(e^{6p^2\int_0^{S_j^K} f(F_s(x))ds}\chi_{S_j^K} < t\right)$$
$$\leqq c \sup_{x\in K} E\left(e^{6p^2\int_0^t f(F_s(x))ds}\right) < \infty\,.$$

Here c is a constant. The completeness follows from Theorem 3.1. The strong completeness follows from Theorem 4.1. ■

Note that the first condition in Theorem 5.1 is a workable condition, since Jensen's inequality gives:

$$E\, e^{\left(6p^2\int_0^t f^2(x_s)\chi_{s<\xi}ds\right)} \leqq \frac{1}{t}\int_0^t E\left[e^{6p^2 t f(x_s)\chi_{s<\xi}}\right] ds\,. \tag{19}$$

For example, take $f \equiv 1$ in Theorem 5.1. Let $A^X = \frac{1}{2}\sum_1^m \nabla X^i(X^i) + A$, and let R be the curvature tensor on M. Recall that the differential generator $\mathscr{A}$ is given by

$$\mathscr{A}f(x) = \frac{1}{2}\sum_1^m \nabla^2 f(X^i(x), X^i(x)) + A^X(f)(x)\,.$$

We next see that the theorem is a direct extension of the global Lipschitz results for R^n:

Corollary 5.2 *The SDE* (1) *is strongly complete if it is complete at one point and satisfies*: $|\nabla X|$ *is bounded and*

$$2\langle \nabla A^X(v), v\rangle + \sum_1^m \langle R(X^i, v)(X^i), v\rangle \leqq c|v|^2$$

for some constant c. In fact under these conditions,

$$\sup_{x\in M} E\left(\sup_{s\leqq t}|T_xF_s|^p\right) < \infty, \text{ for all } p\,.$$

The solution to (1) *consists of diffeomorphisms if also the "adjoint" equation* (12) *is complete at one point and*

$$2\langle \nabla(-A)^X(v), v\rangle + \sum_1^m \langle R(X^i, v)(X^i), v\rangle \leqq c|v|^2\,.$$

Proof. First

$$\nabla A^X(v) = \frac{1}{2}\sum_1^m \nabla^2 X^i(v, X^i) + \frac{1}{2}\sum_1^m \nabla X^i(\nabla X^i(v)) + \nabla A(v) .$$

But by definition of the curvature tensor,

$$\langle \nabla^2 X^i(X^i, v), v\rangle - \langle \nabla^2 X^i(v, X^i), v\rangle = \langle R(X^i, v)(X^i), v\rangle .$$

So

$$\begin{aligned}\langle \nabla A^X(v), v\rangle =& \langle \nabla A(x)(v), v\rangle - \frac{1}{2}\sum_1^m \langle R(X^i, v)(X^i), v\rangle \\ &+ \frac{1}{2}\sum_1^m \langle \nabla^2 X^i(X^i, v), v\rangle + \frac{1}{2}\sum_1^m \langle \nabla X^i(\nabla X^i(v)), v\rangle .\end{aligned}$$

Thus

$$\begin{aligned}H_p(x)(v, v) =& 2\langle \nabla A^X(v), v\rangle + \sum_1^m \langle R(X^i, v)(X^i), v\rangle \\ &+ \sum_1^m |\nabla X^i(v)|^2 + (p-2)\sum_1^m \frac{1}{|v|^2}\langle \nabla X^i(v), v\rangle^2 .\end{aligned}$$

Note the last two terms of H_p are bounded. Thus the conditions of Theorem 5.1 are satisfied, and the SDE is strongly complete. For the diffeomorphism property, note that the 'adjoint' equation has

$$\begin{aligned}H_p(x)(v, v) =& 2\langle \nabla(-A)^Y(v), v\rangle + \sum_1^m \langle R(X^i, v)(X^i), v\rangle \\ &+ \sum_1^m |\nabla X^i(v)|^2 + (p-2)\sum_1^m \frac{1}{|v|^2}\langle \nabla X^i(v), v\rangle^2 ,\end{aligned}$$

and is thus also strongly complete. ■

However for strong 1-completeness, we can do better:

Theorem 5.3 *Assume* (1) *is complete at one point, and* $H_1(x)(v, v) \leqq c|v|^2$ *for some constant c. Then we have strong 1-completeness for* (1). *Furthermore if the dimension of* $M = 2$, *then it is strongly complete.*

Proof. Let $K \in K_1^1$, and S_j^K be the corresponding stopping times as in Theorem 3.1. Then

$$\begin{aligned}|v_{t\wedge S_j^K}| =& |v_0| + \sum_{i=1}^m \int_0^{t\wedge S_j^K} |v_s|^{-1}\langle \nabla X^i(v_s), v_s\rangle dB_s^i \\ &+ \frac{1}{2}\int_0^{t\wedge S_j^K} |v_s|^{-1} H_1(v_s, v_s) ds \end{aligned} \tag{20}$$

from eq. (14) with t replaced by $t \wedge S_j^K$, and letting $p = 1$. On the other hand,

$$|T_xF_{t\wedge S_j^K}(v_o)| = |v_o| e^{\left(M_{t\wedge S_j^K} - \frac{\langle M,M\rangle_{t\wedge S_j^K}}{2} + a_{t\wedge S_j^K}\right)},$$

by (18). But $\langle M,M\rangle_{t\wedge S_j^K}$ and $a_{t\wedge S_j^K}$ are both bounded, since both $|\nabla X^i(x)|$ and $H_1(x)$ are bounded on compact sets. So $|T_xF_{t\wedge S_j^K}(v)|$ is bounded for each j and $v \in T_xM$. Thus

$$E \int_0^{t\wedge S_j^K} |v_s|^{-1}\langle \nabla X^i(v_s), v_s\rangle dB_s^i = 0 .$$

Therefore,

$$\begin{aligned} E|T_xF_{t\wedge S_j^K}(v_0)| &= |v_0| + \frac{1}{2}E \int_0^{t\wedge S_j^K} |v_s|^{-1} H_1(v_s, v_s) ds \\ &\leqq |v_o| + \frac{1}{2}c\int_0^t E|T_xF_{s\wedge S_j^K}(v_o)| ds \end{aligned}$$

Gronwall's inequality gives: $E|T_xF_{S_j^K\wedge t}(v_o)| \leqq |v_o|e^{ct/2}$. So

$$E(|T_xF_{S_j^K}|\chi_{S_j^k<t}) \leqq E|T_xF_{S_j^K\wedge t}| \leqq e^{ct/2} . \tag{21}$$

The strong 1-completeness follows from Theorem 3.1, and the strong completeness for 2-dimensional manifolds follows from Theorem 2.3. ■

It is possible to get a slightly different result from Theorem 5.1 using the fact that

$$|v_t|^p = |v_o|^p e^{M_t^p} e^{\frac{p}{2}\int_0^t \frac{\tilde H(x_s)(v_s,v_s)}{|v_s|^2}} ,$$

where

$$\begin{aligned} \tilde H(x)(v,v) =& 2\langle \nabla A^X(x)(v), v\rangle + \sum_{i=1}^m \langle R(X^i, v)(X^i), v\rangle \\ &+ \sum_1^m |\nabla X^i(v)|^2 - 2\sum_1^m \frac{1}{|v|^2}\langle \nabla X^i(v), v\rangle^2 , \end{aligned} \tag{22}$$

and the fact [21]

$$E \sup_{s\leqq t} e^{\alpha M_s^p} \leqq E\, e^{\sup_{s\leqq t}\alpha M_s^p} < \infty ,$$

if $Ee^{2\alpha^2\langle M^p,M^p\rangle_t} < \infty$. So just as before, if $|\nabla X|$ is bounded, then we have strong completeness if $\tilde H$ is bounded above. This allows consequent variations in the results below.

6 Existence of flows on R^n

In this section we shall show some direct consequences of Theorem 5.1. The usual global Lipschitz condition is improved to allow some growth of the derivatives of the coefficients (see Theorem 6.2). Consider on R^n

$$\text{(Itô)} \quad dx_t = X(x_t)\, dB_t + A(x_t)\, dt\,. \tag{23}$$

It can be rewritten in Stratonovich form:

$$dx_t = X(x_t) \circ dB_t + \bar{A}(x_t)dt\,,$$

where $\bar{A} = A - \frac{1}{2}\sum_1^m DX^i(X^i)$. So

$$H_p(v,v) = 2\langle DA(v), v\rangle + |DX(v)|^2 + (p-2)\sum_1^m \frac{1}{|v|^2}\langle DX^i(v), v\rangle^2\,. \tag{24}$$

Thus the second derivative of X is not involved. Let $g : R^n \to [0,\infty)$ be a C^2 function. Then by Itô's formula, on $\{t < \xi\}$

$$e^{g(x_t)} = e^{g(x_o)+N_t-\frac{\langle N,N\rangle_t}{2}+b_t}\,, \tag{25}$$

where $N_t = \int_o^t Dg(X(x_s)dB_s)$ and

$$b_t = \int_0^t \frac{1}{2}\sum_1^m \left([(Dg)(x_s)(X_i(x_s))]^2 + (D^2g)(x_s)(X^i(x_s),X^i(x_s))\right) ds$$
$$+ \int_0^t (Dg)(x_s)(A(x_s))ds\,.$$

Lemma 6.1 *Let c be a constant. Let τ be a stopping time with $\tau < \xi$ on $\{\xi < \infty\}$. Then for some constant k*

$$Ee^{(cg(x_{t\wedge\tau}))} \leqq e^{c(g(x_0)+kt)}\,,$$

provided that

$$\frac{1}{2}\sum_1^m |Dg(X^i)|^2 + \frac{1}{2}\sum_1^m D^2g(X^i,X^i) + Dg(A) \text{ is bounded above}\,.$$

Proof. Replacing t by $t \wedge \tau$ in (25), and g by cg, then taking expectations on both sides of the inequality above, we get the required inequality.

Theorem 6.2 *The SDE* (23) *on R^n with C^2 coefficients is strongly complete if its coefficients have linear growth* (*in an extended sense*), *i.e.*

$$|X(x)| \leqq c(1+|x|^2)^{\frac{1}{2}}$$

$$\langle x, A(x)\rangle \leqq c(1+|x|^2)\,,$$

and the derivatives of the coefficients have sub-logarithmic growth, i.e.

$$|\nabla X(x)|^2 \leqq c[1+\ln(1+|x|^2)] \tag{26}$$

$$\langle \nabla A(x)(v), v\rangle \leqq c[1+\ln(1+|x|^2)]|v|^2 \tag{27}$$

for all x and $v \in R^n$. Here c is a constant. In fact under these conditions we have:

$$E|x_t|^{2p} \leqq c_{1,p}(1+|x_0|^2)^p e^{c_{2,p}t}$$

for some constant $c_{1,p}$ *and* $c_{2,p}$ *depending only on* p *and* $\sup_{x\in K} E \sup_{s\leqq t} |T_xF_s|^p$ *is finite for all* p *and compact sets* K.

Proof. Let $f(x) = [1+\ln(1+|x|^2)], g(x) = \ln(1+|x|^2)$. Then

$$Df(x)(A(x)) = Dg(x)(A(x)) = \frac{2\langle x, A(x)\rangle}{1+|x|^2},$$

and

$$D^2f(x)(X^i(x), X^i(x)) = \frac{2\langle X^i(x), X^i(x)\rangle}{1+|x|^2} - \frac{4\langle x, X^i(x)\rangle^2}{(1+|x|^2)^2}.$$

So by the previous lemma (applied to the function g),

$$E|x_{t\wedge T}|^2 \leqq (1+|x_0|^2)e^{k_1 t} - 1$$

for some constant k_1 and stopping times T with $T < \xi$. Thus the system is complete by a standard argument. Applying the same lemma to cf, we have:

$$Ee^{c[1+\ln(1+|x_t|^2)]} \leqq e^c(1+|x_0|^2)^c e^{kt}$$

for some constant k(k may depend on c). So

$$\sup_{x\in K} E(e^{6p^2 \int_0^t c[1+\ln(1+|x_s|^2)]ds}) = \sup_{x\in K} \frac{1}{t}\int_o^t e^{6p^2ct}(1+|x_0|^2)e^{ks}ds < \infty .$$

The strong completeness follows from Theorem 5.1, using (24) and the assumptions on ∇X and ∇A. ■

For related estimates on $E|x_t|^p$, see [14]. Note that there is a stochastically complete SDE on R^2 with $|\nabla X(x)| \leqq |x|$ but which is not strongly complete: let A $\equiv 0$, and $X(x,y) = \begin{pmatrix} y & 0 \\ 0 & \frac{x^2}{2} \end{pmatrix}$. See Kunita [16].

A different choice of the function f in Theorem 5.1 leads to an improvement of a theorem of Taniguchi [22]:

Corollary 6.3 *The SDE* (23) *on* R^n *is strongly complete if for some* $\varepsilon \geqq 0$:

$$|X^i(x)| \leqq c(1+|x|^2)^{\frac{1}{2}-\varepsilon}$$

$$\langle x, A(x)\rangle \leqq c(1+|x|^2)^{1-\varepsilon}$$

$$|DX^i(x)|^2 \leqq c(1+|x|^2)^{\varepsilon}$$

$$\langle \nabla A(x)(v), v\rangle \leqq c(1+|x|^2)^{\varepsilon}|v|^2 .$$

Proof. Clearly the stochastic differential equation is complete. Take $g(x) = c(1+|x|^2)^{\varepsilon}$ in the lemma for $\varepsilon > 0$($\varepsilon = 0$ gives the usual globally Lipschitz continuous condition). Then

$$Dg(x)(A(x)) = 2c\varepsilon(1+|x|^2)^{\varepsilon-1}\langle x, A(x)\rangle\,,$$

$$\begin{aligned} D^2g(x)(X^i(x),X^i(x)) =& 2c\varepsilon(1+|x|^2)^{\varepsilon-1}\langle X^i(x),X^i(x)\rangle \\ &+ 4c\varepsilon(\varepsilon-1)(1+|x|^2)^{\varepsilon-2}\langle x, X^i(x)\rangle^2\,. \end{aligned}$$

So lemma 6.1 applies to get $Ee^{cg(x_t)} < e^{c(g(x_0))+2kt}$ for some constant k and the result follows from theorem 5.1. ■

This theorem improves a theorem of Taniguchi since: (a) We only need growth conditions on the normal parts of A and ∇A, and (b) we do not assume $\varepsilon > \frac{1}{3}$ as in [22].

7 Existence of flows on manifolds with a pole

A similar argument on the existence of flow (c.f. theorem 6.2) to that on R^n works for general manifolds to allow the coefficients to have unbounded derivatives. We first assume that M is equipped with a Riemannian metric such that there is a pole P in M, i.e. the distance function $r(-): M \to R$ from P is smooth. Recall that $A^X = \frac{1}{2}\sum_1^m \nabla X^i(X^i) + A$.

Theorem 7.1 *Let M be a complete Riemannian manifold with a pole. Assume the sectional curvature is bounded from below by $-L^2(r(-))$. Here L is a nondecreasing function bigger or equal to* 1. *Then the SDE (1.1) is complete and*

$$E[r(x_t)]^p \leqq [1+r(x_0)]^p e^{k_0[1+p^2]t}$$

for some constant k_0, if the following holds for some constant c:

1. $|X(x)|^2 \leqq \frac{c[1+r(x)]}{L(r(x))\coth(r(x)L(r(x)))}$;
2. $dr(A^X(x)) \leqq c[1+r(x)]$.

It is strongly complete and $\sup_{x\in K} E\sup_{s\leqq t}|TF_s|^p < \infty$ *for all p and compact sets K, if we also have:*

3. $|\nabla X(x)|^2 \leqq c[1+\ln(1+r(x))]$;
4. $2\langle\nabla A^X(v), v\rangle + \sum_1^m\langle R(X^i,v)(X^i),v\rangle \leqq c[1+\ln(1+r(x))]|v|^2$.

Proof. First we have:

$$\begin{aligned} r(x_t) =& r(x_0) + \int_0^t dr(X(x_s)dB_s) + \frac{1}{2}\sum_1^m\int_0^t \nabla^2 r(X^i(x_s),X^i(x_s))ds \\ &+ \int_0^t dr(A^X(x_s))ds\,. \end{aligned}$$

But by Hessian comparison theorem in [13] (p.19 and example 2.25 on p.34. The results there is for constant L, but the proof depends only on the behaviour of the manifold around the geodesic from p to x),

$$\nabla^2 r(x) \leqq L(r(x))\coth\,(r(x)L(r(x)))\,.$$

Let $T_n(x)$ be the first exit time of $F_t(x)$ from the geodesic ball $B(p,n)$, centered at p and radius n. Then

$$
\begin{aligned}
Er(x_{t\wedge T_n}) =& r(x_0) + \frac{1}{2}\sum_1^m E\int_0^{t\wedge T_n} \nabla^2 r(X^i(x_s), X^i(x_s))ds \\
&+ E\int_0^{t\wedge T_n} dr(A^X(x_s))ds \\
\leqq& r(x_0) + \frac{k_1}{2}\int_0^t E\chi_{s<T_n}(1 + r(x_s))ds\,.
\end{aligned}
$$

Here k_1 is a constant. Thus

$$
Er(x_{t\wedge T_n}) \leqq \left[r(x_0) + \frac{k_1 t}{2}\right] e^{k_1 t/2}\,.
$$

So

$$
\begin{aligned}
P\{T_n < t\} &= \frac{1}{n}E(r(x_{t\wedge T_n})\chi_{T_n<t}) \\
&\leqq \frac{1}{n}\left[r(x_0) + \frac{k_1 t}{2}\right] e^{k_1 t/2} \to 0\,,
\end{aligned}
$$

as n goes to infinity. Thus there is no explosion. Now

$$
\begin{aligned}
[1 + r(x_t)]^p =& [1 + r(x_0)]^p + p\int_0^t [1 + r(x_s)]^{p-1} dr(X(x_s)dB_s) \\
&+ \frac{p(p-1)}{2}\sum_1^m\int_0^t [1 + r(x_s)]^{p-2}[dr(X^i(x_s))]^2 ds \\
&+ \frac{p}{2}\sum_1^m\int_0^t [1 + r(x_s)]^{p-1}\nabla^2 r(X^i(x_s), X^i(x_s))ds \\
&+ p\int_0^t [1 + r(x_s)]^{p-1} dr(A^X(x_s))ds\,.
\end{aligned}
$$

Let

$$
M_t = \int_0^t p\frac{dr(X(x_s)dB_s)}{1 + r(x_s)}\,,
$$

and let

$$
\begin{aligned}
b_t =& \frac{1}{2}\sum_1^m\int_0^t \left(p(p-1)\frac{[dr(X^i(x_s))]^2}{[1 + r(x_s)]^2} + p\frac{\nabla^2 r(X^i(x_s), X^i(x_s))}{1 + r(x_s)}\right) ds \\
&+ p\int_0^t \frac{dr(A^X(x_s))}{1 + r(x_s)}ds\,.
\end{aligned}
$$

We have:

$$
[1 + r(x_t)]^p = [1 + r(x_0)]^p \mathscr{E}(M_t) e^{bt}\,.
$$

Here $\mathscr{E}(M_t) = e^{M_t - \frac{1}{2}\langle M, M\rangle t}$. But b_t is bounded from the assumptions. So

$$E[1+r(x_t)]^p \leqq [1+r(x_0)]^p e^{k_0[1+p^2]t}$$

for some constant k_0. Thus

$$\sup_{x\in K} E(e^{6p^2\int_0^t c[1+\ln(1+r(F_s(x))]ds}) \leqq \frac{1}{t}\sup_{x\in K}\int_0^t E\left(e^{6p^2ct[1+\ln(1+r(F_s(x)))]}\right)ds$$

$$\leqq \frac{1}{t}e^{6cp^2t}\sup_{x\in K}\int_0^t E\left([1+r(F_s(x))]^{6cp^2s}\right)ds < \infty$$

So Theorem 5.1 applies to the function $f(x)=c[1+\ln(1+r(x))]$ to get the strong completeness. ∎

Remarks.

(i) From the above calculations we also get, for each $p>0$:

$$P\{T_n < t\} \leqq \frac{1}{n^p}[1+r(x_0)]^p e^{k_0[1+p^2]t}.$$

(ii) Note that if the sectional curvatures are nonpositive, then $\nabla^2 r(x) \geqq 0$ and so $\nabla^2 r(x) \leqq \Delta r(x)$. If the Ricci curvature has lower bound $-L^2(r(-))$, where L is as before. Then [13]

$$\Delta r(-) \leqq (n-1)L(r(-)) \coth (rL(r(-)). \tag{28}$$

In this case the theorem holds without further assumptions on the sectional curvatures.
In general, let $g: M \to R$ be a C^2 function, then

$$e^{g(x_t)} = e^{g(x_0)} + \int_0^t e^g dg(X(x_s)dB_s) + \frac{1}{2}\int_0^t e^g \sum_1^m [dg(X^i(x_s))]^2 ds$$

$$+ \int_0^t e^g \left(dg(A^X(x_s)) + \frac{1}{2}\sum_1^m \nabla(dg)(X^i(x_s),X^i(x_s))\right) ds.$$

By Gronwall's inequality $Ee^{g(x_t)} < e^{g(x_0)}e^{kt}$ if $dg(X^i)$ is bounded for each i and $\sum_1^m \nabla dg(X^i,X^i) + dg(A^X)$ is bounded above. Using $g(x)=(1+r(x))^\varepsilon$, a similar proof to that of Corollary 6.3 gives:

Proposition 7.2 *Let M be a complete Riemannian manifold with a pole. Assume its sectional curvature is bounded from below by $-L^2(r(-))$. Here L is a nondecreasing function bigger or equal to* 1. *Then the SDE* (1.1) *is complete if for some $\varepsilon > 0$:*

1. $|X(x)|^2 \leqq \frac{c[1+r(x)]^{2-\varepsilon}}{L(r(x)) \coth (r(x)L(r(x)))}$; $|\nabla X(x)|^2 \leqq c[1+(r(x))]^\varepsilon$;
2. $dr(A^X(x)) \leqq c[1+r(x)]^{2-\varepsilon}$;

It is strongly complete, if also

3. $H_p(x)(v,v) \leqq c[1+(r(x))]^\varepsilon |v|^2$, *for some* $p>0$.

Note that this relaxes the conditions on the derivatives, compared to theorem 7.1 but imposes more stringent bounds on the coefficients.

8 Strong completeness of nondegenerate equations

In this section we shall assume that the SDS considered is a Brownian motion with drift Z, i.e. $X^*X = Id$, and $Z =: A^X = \frac{1}{2}\sum_1^m \nabla X^i(X^i) + A$.
Recall that R is the curvature tensor and Ric is the Ricci curvature. Then

$$\sum_1^m \langle R(X^i, v)(X^i), v\rangle = -\mathrm{Ric}(v, v)\,,$$

giving

$$\begin{aligned} H_p(x)(v,v) =& 2\langle \nabla Z(v), v\rangle_x - \mathrm{Ric}_x(v,v) + \sum_1^m |\nabla X^i(v)|_x^2 \\ &+ (p-2)\sum_1^m \frac{1}{|v|^2}\langle \nabla X^i(v), v\rangle_x^2\,. \end{aligned} \tag{29}$$

Theorem 8.1 *Let M be a complete Riemannian manifold. Assume $|\nabla X|$ is bounded and $\frac{1}{2}\mathrm{Ric}(v,v) - \langle \nabla Z(v), v\rangle \geqq -c|v|^2$ for some constant c. Then the Brownian motion with drift Z is strongly complete if complete.*

Proof. This follows from theorem 5.1 by taking $f \equiv 1$. ■

In particular suppose the drift is ∇h for a smooth function h. Then we have strong completeness if $|\nabla X|$ is bounded and if $\frac{1}{2}$ Ric-Hess(h) is bounded from below, since a h-Brownian motion is complete if $\frac{1}{2}$ Ric-Hess(h) is bounded from below. See [1].

Let p be a point in M. Let $r(x)$ denote the Riemannian distance between p and x. The results in the last section hold for h-Brownian motions without the assumption that there is a pole for the manifold. Let c be a constant.

Theorem 8.2 *Let M be a complete Riemannian manifold. Assume the Ricci curvature is bounded from below by $-c(1 + r^2(x))$. Here c is a constant. Suppose $dr(Z(x)) \leqq c[1 + r(x)]$ outside the cut locus cut(p) of p, then the Brownian motion with drift Z is complete. Furthermore let $p > 1$, then $E[r(x_t)]^p \leqq [1 + r(x_0)]^p e^{k_0(1+p^2)t}$ for some constant k_0. It is strongly complete and*

$$\sup_{x\in K} E\left(\sup_{s\leqq t}|T_xF_s|^p\right) < \infty$$

for each $t > 0$ and compact set K, if the following also holds:
1). $|\nabla X(x)|^2 \leqq c[1 + \ln(1 + r(x))]$,
2). $Ric_x(v,v) - 2\langle \nabla Z(v), v\rangle_x \geqq -c[1 + \ln(1 + r(x))]|v|^2$.

Proof. The proof of theorem 7.1 works here, noticing the following two points:
A. The Ito formula for $[1 + r(x_t)]^p$ (in the proof of Theorem 7.1) holds with a correction term L_t^p:

$$
\begin{aligned}
[1+r(x_t)]^p = & [1+r(x_0)]^p + p\int_0^t [1+r(x_s)]^{p-1} dr(X(x_s)dB_s) \\
& + \frac{p(p-1)}{2}\sum_1^m \int_0^t [1+r(x_s)]^{p-2}[dr(X^i(x_s))]^2 ds \\
& + \frac{p}{2}\sum_1^m \int_0^t [1+r(x_s)]^{p-1} \Delta r(x_s) ds \\
& + p\int_0^t [1+r(x_s)]^{p-1} dr(A^X(x_s)) ds - L_t^p \, .
\end{aligned}
$$

where $L_t^p \geqq 0$ and Δr and dr are defined to be zero on cut(p). See [5].

B. When x does not belong to the cut-locus $C(p)$ of p, there is the following estimate from [13] (p.26 and (2.27) on p.35):

$$
|\Delta r(x)| \leqq (n-1)\sqrt{cL(r(x))} \coth\left(r(x)\sqrt{cL(r(x))}\right) ,
$$

Note also $\sum_1^m \nabla^2(X^i(x), X^i(x)) = \Delta r(x)$. However the cut-locus $C(p)$ has measure zero, and so the Brownian motion spends zero amount of time on the cut-locus by Fubini's theorem, since it has a density with respect to dx for dx the Riemannian volume measure. So the proof of Theorem 7.1 follows through. ∎

Note that this method could also applied to the case of the Ricci curvature is bounded below by $-L^2(r(-))$, where L is a nondecreasing function greater or equal to 1, just as in theorem 7.1.

Gradient Brownian systems

Let $f : M \to R^m$ be an isometric embedding. Let $X(\cdot)(e) = \nabla\langle f(\cdot), e\rangle$. Such systems are called gradient Brownian systems. Let v_x be the space of normal vectors to M at x. There is the second fundamental form:

$$
\alpha_x : T_xM \times T_xM \to v_x
$$

and the shape operator: $A_x : T_xM \times v_x \to T_xM$ related by $\langle \alpha_x(v_1, v_2), w\rangle = \langle A_x(v_1, w), v_2\rangle$. Let $\{e_i\}$ be an orthonormal basis for R^m. If $Y(x) : R^m \to v_x$ is the orthogonal projection, then [10] [6]

$$
\nabla X^i(v) = A_x(v, Y(x)e_i) \, .
$$

Let $f_1, \ldots f_n$ be an o.n.b. for T_xM. Consider $\alpha_x(v, \cdot)$ as a linear map from T_xM to v_x. Denote by $|\alpha_x(v, \cdot)|_{H,S}$ the corresponding Hilbert Schmidt norm, and $|\cdot|_{v_x}$ the norm of a vector in v_x. Accordingly we have:

$$
\begin{aligned}
\sum_1^m |\nabla X^i(v)|_x^2 &= \sum_{i=1}^m \sum_{j=1}^n \langle A_x(v, Y(x)e_i), f_j\rangle^2 = \sum_{i=1}^m \sum_{j=1}^n \langle \alpha_x(v, f_j), Y(x)e_i\rangle^2 \\
&= \sum_{j=1}^m |\alpha_x(v, f_j)|_{v_x}^2 = |\alpha_x(v, \cdot)|_{H,S}^2 ,
\end{aligned}
$$

and

$$\sum_1^m \langle \nabla X^i(v), v\rangle_x^2 = |\alpha_x(v,v)|^2_{\nu_x}\,.$$

This gives

$$H_p(v,v) = -\operatorname{Ric}(v,v) + 2\langle \nabla Z(v), v\rangle + |\alpha_x(v,\ \cdot\ )|^2_{H,S}$$
$$+ \frac{(p-2)}{|v|^2}|\alpha_x(v,v)|^2_{\nu_x}\,. \tag{30}$$

Further, Gauss's theorem: $\operatorname{Ric}(v,v) = \langle \alpha(v,v), \text{ trace } \alpha\rangle - |\alpha(v,\ \cdot\ )|^2_{H,S}$ gives

$$H_p(v,v) = -\ \langle \alpha(v,v), \text{trace } \alpha\rangle + 2|\alpha_x(v,\ \cdot\ )|^2_{H,S}$$
$$+ \frac{1}{|v|^2}(p-2)|\alpha_x(v,v)|^2_{\nu_x} + 2\langle \nabla Z(v), v\rangle_x\,. \tag{31}$$

Thus the completeness and strongly completeness of a gradient Brownian motion rely only on bounds on the second fundamental form and on the drift:

Corollary 8.3 *Let M be a closed immersed submanifold of R^m with its second fundamental form α bounded by $c[1+\ln(1+r(x))]^{\frac{1}{2}}$. Then a gradient Brownian motion on M with drift Z is strongly complete if*

$$dr(Z) \leqq c[1+r(x)]\,,$$

and

$$\langle \nabla Z(v), v\rangle_x \leqq c[1+\ln(1+r(x))]|v|^2\,.$$

It has a flow of diffeomorphisms if also $|Z(x)| \leqq c[1+r(x)]$, and $|\nabla Z(x)| \leqq c[1+\ln(1+r(x))]$.

Proof. The strong completeness is clear from theorem 8.2. The diffeomorphism property comes from the fact that for gradient Brownian systems [7],

$$\sum_1^m \nabla X^i(X^i) = 0\,.$$

So the 'adjoint' Eq. (12) to (1) is also a gradient Brownian system (with drift $-Z$). ■

Let $Z = 0$, we get the following useful corollary:

Corollary 8.4 *Let M be a complete Riemannian manifold isometrically immeresed in R^m with its second fundamental form bounded by $c[1+\ln(1+r(x))]^{\frac{1}{2}}$. Then the gradient Brownian motion on it has a flow of diffeomorphisms.*

See also Baxendale [2] for a discussion of flows on manifolds with second fundamental form bounded and globally Lipschitz.

According to Theorem 5.3, a SDS is strongly 1-complete if it is complete and if $H_1(x)(v,v) \leqq c|v|^2$. But for gradient Brownian systems, we can do better. Let $\mathscr{E}(M_t) = e^{M_t^1} - \frac{\langle M^1, M^1\rangle_t}{2}$, where M_t^1 is as defined before theorem 5.1 and let $f(x) = \sup_{|v|=1} H_1(x)(v,v)$. Then we have:

Proposition 8.5 *Let M be a closed immersed submanifold of R^n. Then a stochastically complete gradient Brownian system is strongly 1-complete if*

$$\sup_{x\in K} E\left(e^{\frac{1}{2}\int_0^T f(F_s(x))ds}\right) < \infty$$

for all compact set K and bounded stopping times T.

Proof. We shall use the notations of theorem 5.3. Let

$$\tilde{B}_t = B_t - \int_0^t Y(x_s)^*\left(\alpha_{x_s}\left(\frac{v_s}{|v_s|}, \frac{v_s}{|v_s|}\right)\right) ds$$

and let $\tilde{x}_t$ and $\tilde{v}_t$ be the solutions to the stochastic differential equation

$$dx_t = X(x_t) \circ d\tilde{B}_t + A(x_t)dt \tag{32}$$

and the stochastic covariant equation:

$$dv_t = \nabla X(v_t) \circ d\tilde{B}_t + \nabla A(v_t)dt$$

respectively. For $x \in M$, choose an o.n.b. $\{e_1, \ldots, e_m\}$ for R^m, such that $\{X(x)(e_i)\}_1^n$ is an o.n.b. for T_xM and $X(x)(e_j) = 0$ for $j > n$. Then it is clear that $X(Y^*(v)) = 0$ for $v \in v_x$. So Eq. (32) is the same as our original stochastic differential equation (1), and thus $\tilde{x}_t$ has the same distribution as x_t and has no explosion. On the other hand, by formula (18):

$$\begin{aligned} E|v_{S_j^K}|\chi_{S_j^K<t} &= |v_0|E\left(\mathscr{E}\left(M_{t\wedge S_j^K}\right) e^{a^1_{t\wedge S_j^K}}\chi_{S_j^K<t}\right) \\ &= |v_0|E\left(\mathscr{E}(M_t)e^{a^1_{t\wedge S_j^K}}\chi_{S_j^K<t}\right) \end{aligned}$$

by the optional stopping theorem. But by the Girsanov–Cameron–Martin formula ([10], [21]),

$$\begin{aligned} E\left(\mathscr{E}(M_t)e^{a^1_{t\wedge S_j^K}}\chi_{S_j^K<t}\right) &= Ee^{\frac{1}{2}\int_0^{t\wedge S_j^K} H_1(\tilde{x}_s)\left(\frac{\tilde{v}_s}{|\tilde{v}_s|}, \frac{\tilde{v}_s}{|\tilde{v}_s|}\right)ds}\chi_{S_j^K<t}\,. \\ &\leqq E\left(e^{\frac{1}{2}\int_0^{t\wedge S_j^K} f(\tilde{x}_s)ds}\chi_{S_j^K<t}\right) \\ &= E\left(e^{\frac{1}{2}\int_0^{S_j^K} f(x_s)ds}\chi_{S_j^K<t}\right) < \infty\,. \end{aligned}$$

Thus $\underline{\lim}_{j\to\infty} \sup_{x\in K} E|T_xF_{t\wedge S_j^K}|\chi_{S_j^K<t} < \infty$, and the strong 1-completeness follows. ∎

9 Application to differentiation of semigroups

Assume the derivative of the solution flow of equation (1) has first moment: $E|T_xF_s\chi_{s<\xi(x)}| < \infty$. We may define a semigroup (formally) of linear operators δP_t on bounded measurable 1-forms as follows: for $v \in T_xM$ and ϕ a 1-form

$$(\delta P_t)\phi(v) = E\phi(T_xF_t(v))\chi_{t<\xi(x)} \,. \tag{33}$$

It is in fact an L^p semigroup under suitable conditions on the derivative flow TF_t. On the other hand, $\delta P_t(df)$ is clearly the formal derivative of P_tf, which can be checked to be true if the SDS concerned is strongly 1-complete and if TF_t satisfy an integrability condition (see below). By virtue of the introduction of strong 1-completeness we can improve a theorem of Elworthy [10]. The assumption that the SDS is strongly 1-complete is, on the other hand, a natural assumption: first $dP_tf = (\delta P_t)(df)$ for $f \in BC^1$ implies completeness (take $f \equiv 1$), and in fact $dP_tf = (\delta P_t)(df)$ for $f \in C_K^\infty$ and $E|T_xF_t|_{\chi_{t<\xi(x)}} < \infty$ implies completeness [17]. Here BC^1 is the space of bounded functions with bounded continuous first derivatives. And also strong 1-completeness follows from completeness if for a complete Riemannian metric $\sup_{x\in K} E\sup_{s\leqq t}|T_xF_s| < \infty$ for all compact sets K (Theorem 3.1). For applications of results in this section, see [17], and [12].

Theorem 9.1 *Assume strong 1-completeness. Suppose the map $r \to E|T_{\sigma(r)}F_t|$ is continuous for r small, for all smooth curves $\sigma : [0,\ell] \to M$. If f is BC^1, then P_tf is C^1 and*

$$d(P_tf)(x) = \delta P_t(df)(x) \,.$$

Proof. Let $x \in M, v \in T_xM$. Take a geodesic curve $\sigma : [0,\ell] \to M$ starting from x with velocity v such that the image set is contained in a compact neighbourhood K of x. By the strong 1-completeness, $F_t(\sigma(s))$ is a.s. differentiable with respect to s. So for almostly all ω:

$$\frac{f(F_t(\sigma(s),\omega)) - f(F_t(x,\omega))}{s} = \frac{1}{s}\int_0^s df(T_{\sigma(r)}F_t(\dot\sigma(r),\omega))dr \,.$$

By the strong 1-completeness we know $T_{\sigma(r)}F_t(\dot\sigma(r))$ is continuous in r for almost all ω. Thus:

$$\begin{aligned} E\lim_{s\to 0}\frac{1}{s}\int_0^s df(T_{\sigma(r)}F_s(\dot\sigma(r),\omega))dr. &= E\lim_{s\to 0}\frac{1}{s}\int_0^s df(T_{\sigma(r)}F_t(\dot\sigma(r),\omega))dr \\ &= Edf(TF_t(v)) \,. \end{aligned}$$

On the other hand, $\lim_{s\to 0}\frac{1}{s}\int_0^s E|T_{\sigma(r)}F_t|dr = E|T_xF_t|$ if the map $r \longrightarrow E|T_{\sigma(r)}F_t|$ is continuous. But $|df(T_{\sigma(r)}F_t(\dot\sigma(r)))| \leqq |df|_\infty|T_{\sigma(r)}F_t|$, so $\lim_{s\to 0} EI_s = E\lim_{s\to 0} I_s$ giving $Edf(T_xF_t(v)) = d(P_tf)(v)$. ■

Let $\sigma(0) = x_0$, the required continuity of the map $r1 \longrightarrow E|T_{\sigma(r)}F_t|$ can be assured by one of the following conditions: (1) There is a constant $\delta > 0$ such that:

$$\sup_{x\in K} E|T_xF_t|^{1+\delta} < \infty ,$$

for a compact neighbourhood K of x_0. (2) $E\sup_{x\in K}|T_xF_t| < \infty$ for a compact set K containing x_0.

Corollary 9.2 *Let M be a complete Riemannian manifold. Suppose SDS (1) is complete and satisfies:*

$$H_{1+\delta}(v,v) \leqq k|v|^2 .$$

Then $dP_tf = \delta P_t(df)$ if both f and df are bounded. Here H is as defined in section 5.

Proof. First the system is strongly 1-complete by the boundedness of H_1. On the other hand, formula (18) in section 5 gives: $E|T_xF_t|^{1+\delta} \leqq e^{\frac{c(1+\delta)}{2}t}$.

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