On extensions of Myers' theorem

Xue-Mei Li*

Abstract

Let M be a compact Riemannian manifold and h a smooth function on M. Let $\rho^h(x) = \inf_{|v|=1} (Ric_x(v,v) - 2Hess(h)_x(v,v))$. Here Ric_x denotes the Ricci curvature at x and $Hess(h)$ is the Hessian of h. Then M has finite fundamental group if $\Delta^h - \rho^h < 0$. Here $\Delta^h =: \Delta + 2L_{\nabla h}$ is the Bismut-Witten Laplacian. This leads to a quick proof of recent results on extension of Myers' theorem to manifolds with mostly positive curvature. There is also a similar result for noncompact manifolds.

An early result of Myers says a complete Riemannian manifold with Ricci curvature bounded below by a positive number is compact and has finite fundamental group. See e.g. [9]. Since then efforts have been made to get the same type of result but to allow a little bit of negativity of the curvature (see Bérard and Besson[2]). Wu [12] showed that Myers' theorem holds if the manifold is allowed to have negative curvature on a set of small diameter, while Elworthy and Rosenberg [8] considered manifolds with some negative curvature on a set of small volume, followed by recent work of Rosenberg and Yang [10]. We use a method of Bakry [1] to obtain a result given in terms of the potential kernel related to $\rho(x) = \inf_{|v|=1} Ric_x(v,v)$, which gives a quick probabilistic proof of recent results on extensions of Myers' theorem. Here Ric_x denotes the Ricci curvature at x.

Let M be a complete Riemannian manifold, and h a smooth real-valued function on it. Assume Ric$-$2Hess(h) is bounded from below, where Hess(h)

*Research supported in part by NATO Collaborative Research Grants Programme 0232/87 and by SERC grant GR/H67263. 1991 Mathematical subject classification 60H30,53C21 is the hessian of h. Denote by Δ^h the Bismut-Witten Laplacian (with probabilistic sign convention) defined by: $\Delta h = \Delta + 2L_{\nabla h}$ on C_K^∞ the space of smooth differential forms with compact support. Here $L_{\nabla h}$ is the Lie derivative in direction of ∇h. Then the closure of Δ^h is a negative-definite self-adjoint differential operator on L^2 functions (or L^2 differential forms) with respect to $e^{2h}dx$ for dx the standard Lebesgue measure on M. We shall use the same notation for Δ^h and its closure. By the spectral theorem there is a heat semigroup P_t^h satisfying the following heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{2}\Delta^h u(x,t).$$

We shall denote by $P_t^h\phi$ the solution with initial value ϕ. For clarity, we also use $P_t^{h,1}$ for the corresponding heat semigroup for one forms. Then for a function f in C_K^∞,

$$dP_t^h f = P_t^{h,1}(df). \tag{1}$$

On M there is a h-Brownian motion $\{F_t(x) : t \geq 0\}$, i.e. a path continuous strong Markov process with generator $\frac{1}{2}\Delta^h$ for each starting point x. For a fixed point $x_0 \in M$, we shall write $x_t = F_t(x_0)$. Then $P_t^h f(x) = Ef(F_t(x))$ for all bounded L^2 functions.

Let $\{W_t^h(-), t \geq 0\}$ be the solution flow to the following covariant equation along h-Brownian paths $\{x_t\}$:

$$\begin{cases} \frac{D}{\partial t}W_t^h(v_0) &= -\frac{1}{2}Ric_{x_t}\left(W_t^h(v_0), -\right)^\# + Hess(h)_{x_t}\left(W_t^h(v_0), -\right)^\#, \\ W_0^h(v_0) &= v_0, \qquad v_0 \in T_{x_0}M. \end{cases} \tag{2}$$

Here # stands for the adjoint. The solution flow W_t^h is called the Hessian flow.

Let ϕ be a bounded 1-form, then for $x_0 \in M$, and $v_0 \in T_{x_0}M$

$$E\phi(W_t^h(v_0)) = P_t^{h,1}\phi(v_0). \tag{3}$$

if Ricci-2 Hess(h) is bounded from below. See e.g. [4] and [5].

Formula (3) gives the following estimates on the heat semigroup:

$$|P_t^{h,1}\phi| \leq |\phi|_\infty E|W_t^h|. \tag{4}$$

Let $\rho^h(x) = \inf_{|v|=1} (Ric_x(v,v) - 2Hess(h)(x)(v,v))$ and write ρ for ρ^h if $h = 0$. Then covariant equation (2) gives:

$$E|W_t^h| \leq E\mathrm{e}^{-\frac{1}{2}\int_0^t \rho^h(x_s)ds} \tag{5}$$

as in [4].

Let $P_t^{\rho^h}$ be the L^2 semigroup generated by the Schrödinger operator $\frac{1}{2}\Delta^h - \frac{1}{2}\rho^h$. Then

$$P_t^{\rho^h} f(x_0) = E\left[f(x_t)\mathrm{e}^{-\frac{1}{2}\int_0^t \rho^h(x_s)ds}\right]$$

by the Feyman-Kac formula. So equation (5) is equivalent to

$$E|W_t^h| \leq P_t^{\rho^h} 1.$$

Let Uf be the corresponding potential kernel defined by:

$$Uf(x_0) = \int_0^\infty E\left[f(x_t)\mathrm{e}^{-\frac{1}{2}\int_0^t \rho^h(x_s)ds}\right] dt.$$

Following Bakry's paper [1], we have the following theorem:

Theorem 1 *Let M be a complete Riemannian manifold with $Ric - 2Hess(h)$ bounded from below. Suppose*

$$\sup_{x\in K} U1(x) < \infty \tag{6}$$

for each compact set K. Then M has finite h-volume(i.e. $\int_M \mathrm{e}^{2h(x)}dx < \infty$), and finite fundamental group.

Proof: We follow [1]. Let $f \in C_K^\infty$, then $Hf = \lim_{t\to\infty} P_t f$ is an L^2 harmonic function. Assume h-vol$(M) = \infty$, then $Hf = 0$. We shall prove this is impossible. Let $f,\, g \in C_K^\infty$, then:

$$\begin{aligned}
&\int_M (P_t^h f - f) g\mathrm{e}^{2h}dx \\
&= \int_M \int_0^t \left(\frac{\partial}{\partial s} P_s^h f\right) g\mathrm{e}^{2h}dx \\
&= \int_0^t \int_M < \nabla P_s^h f, \nabla g > \mathrm{e}^{2h}dxds
\end{aligned}$$

$$
\begin{aligned}
&\leq |\nabla f|_\infty \int_M |\nabla g| \left(\int_0^t E|W_s^h| ds \right) \mathrm{e}^{2h} dx, \\
&\leq |\nabla f|_\infty \left(\sup_{x \in sup(g)} \int_0^\infty E|W_s^h| ds \right) |\nabla g|_{L^1} \\
&\leq c|\nabla f|_\infty |\nabla g|_{L^1}.
\end{aligned}
$$

Here $c = \sup_{x\in sup(g)} [U1(x)] = \sup_{x\in sup(g)} \left(\int_0^\infty E \left(\mathrm{e}^{-\int_0^t \rho^h(F_s(x))ds} \right) dt \right)$, and $sup(g)$ denotes the support of g.

Next take $f = h_n$, for h_n an increasing sequence of smooth functions approximating 1 with $0 \leq h_n \leq 1$ and $|\nabla h_n| \leq \frac{1}{n}$, see e.g. [1].

Then

$$\int_M (P_t^h h_n - h_n) g \mathrm{e}^{2h} dx \leq c \frac{1}{n} |\nabla g|_{L^1}.$$

First let t go to infinity, then let $n \to \infty$ to obtain:

$$-\int_M g \mathrm{e}^{2h} dx \leq 0.$$

This gives a contradiction with a suitable choice of g. So we conclude h-vol$(M) < \infty$.

Let $p\colon \tilde{M} \to M$ be the universal covering space for M with induced Riemannian metric on $\tilde{M}$. For $p(\tilde{x}) = x$, let $\{\tilde{F}_t(\tilde{x}), t \geq 0\}$ be the horizontal lift of $\{F_t(x)\}$ to $\tilde{M}$. Denote by $\tilde{Ric}$ the Ricci curvature on $\tilde{M}$, $\tilde{h}$ the lift of h to $\tilde{M}$, and $\tilde{\rho^h}$ the corresponding lower bound for $\tilde{Ric} - 2Hess(\tilde{h})$. Then the induced $\{\tilde{F}_t(\tilde{x}), t \geq 0\}$ is a h-Brownian motion on $\tilde{M}$. See e.g. [4]. Note also $\tilde{\rho^h}$ satisfies

$$\sup_{\tilde{x}\in\tilde{K}} \int_0^\infty E \left(\mathrm{e}^{-\frac{1}{2}\int_0^t \tilde{\rho^h}(\tilde{F}_s(\tilde{x}))ds} \right) dt = \sup_{x\in p(\tilde{K})} \int_0^\infty E \left(\mathrm{e}^{-\frac{1}{2}\int_0^t \rho^h(F_s(x))ds} \right) dt$$

for any $\tilde{K} \subset \tilde{M}$ compact. The same calculation as above will show that $\tilde{M}$ has finite h-volume, therefore p is a finite covering and so M has finite fundamental group. ■

Let $r(x)$ be the Riemannian distance between x and a fixed point of M and take h to be identically zero:

Corollary 2 *Let M be a complete Riemannian manifold with*

$$Ric_x > -\frac{n}{n-1}\frac{1}{r^2(x)}, \quad when\ r > r_0 \tag{7}$$

for some $r_0 > 0$. Then the manifold is compact if $\sup_{x\in K} U1(x) < \infty$ for each compact set K.

Proof: This is a consequence of the result of [3]: A complete Riemannian manifold with (7) has infinite volume. See also [11]. ■

For another extension of Myers' compactness theorem, see [3] where a diameter estimate is also obtained.

In the following we shall assume M is compact and get the following corollary:

Corollary 3 *Let M be a compact Riemannian manifold and h a smooth function on it. Then M has finite fundamental group if $\Delta^h - \rho^h < 0$.*

Proof: Let λ_0 be the minimal eigenvalue of $\Delta^h - \rho^h$. Then

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \sup_M \log E\left(\mathrm{e}^{-\frac{1}{2}\int_0^t \rho^h(x_s)ds}\right) \le \lambda_0 < 0.$$

See e.g. [7]. Thus there is a number T_0 such that if $t \ge T_0$,

$$\sup_M E\left(\mathrm{e}^{-\frac{1}{2}\int_0^t \rho^h(x_s)ds}\right) \le \mathrm{e}^{\lambda_0 t}.$$

Therefore

$$\begin{aligned}
&\sup_M \int_0^\infty E\left(\mathrm{e}^{-\frac{1}{2}\int_0^t \rho^h(x_s)ds}\right) dt \\
&\le \int_0^{T_0} E\left(\mathrm{e}^{-\frac{1}{2}\inf_{x\in M}\left[\rho^h(x)\right]t}\right) dt + \int_{T_0}^\infty E\left(\mathrm{e}^{-\frac{1}{2}\int_0^t \rho^h(x_s)ds}\right) dt \\
&< \infty.
\end{aligned}$$

The result follows from theorem 1. ■

Let $d_h = \mathrm{e}^h d \mathrm{e}^{-h}$. It has adjoint $\delta_h = \mathrm{e}^{-h}\delta \mathrm{e}^h$ on $L^2(M, dx)$. Let $\Box_h$ be the Witten Laplacian defined by:

$$\Box_h = -(d_h + \delta_h)^2.$$

By conjugacy of $\Box_h$ on $L^2(M, dx)$ with Δ^h on $L^2(M, e^{2h}dx)$, the condition "$\Delta^h - \rho^h < 0$" becomes:

$$\Box_h - \rho^h < 0$$

on $L^2(M, dx)$. On the other hand,

$$\Box_h = \Delta - ||dh||^2 - \Delta h.$$

See e.g. [6]. This gives: a compact manifold has finite fundamental group if

$$\Delta - ||dh||^2 - \Delta h - \rho^h < 0$$

on $L^2(M, dx)$.

Corollary 3 leads to the following theorem from [10]: Let $\mathcal{N} = \mathcal{N}(K, D, V, n)$ be the collection of n-dimensional Riemannian manifolds with Ricci curvature bounded below by K, diameter bounded above by D, and volume bounded below by V.

Corollary 4 (Rosenberg& Yang) *Choose $R_0 > 0$. There exists $a = a(\mathcal{N}, R_0)$ such that a manifold $M \in \mathcal{N}$ with $\mathrm{vol}\{x : \rho(x) < R_0\} < a$ has finite fundamental group. Here "vol" denotes the volume of the relevant set.*

Proof: Let $h = 0$ in corollary 3. Then under the assumptions in the corollary, $\Delta - \rho < 0$ according to [8]. The conclusion follows from corollary 3. ■

Acknowledgement: The author is grateful to Professor D. Elworthy and Professor S. Rosenberg for helpful comments and encouragement.

References

[1] D. Bakry. Un critére de non-explosion pour certaines diffusions sur une varíeté Riemannienne compléte. *C. R. Acad. Sc. Paris*, t. 303, Série I(1):23–26, (1986).

[2] P. Bérard and G. Besson. On the number of bound states and estimates on some geometric invariants. *Springer Lect. Notes*, 1986.

[3] M. Cheeger, M. Gromov, and M. Taylor. Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. *J. of Diff. Geom.*, 17:15–53, (1982).

[4] K. D. Elworthy. Geometric aspects of diffusions on manifolds. In P. L. Hennequin, editor, *Ecole d'Eté de Probabilités de Saint-Flour XV-XVII, 1985, 1987. Lecture Notes in Mathematics*, volume 1362, pages 276–425. Springer-Verlag, 1988.

[5] K. D. Elworthy. Stochastic flows on Riemannian manifolds. In M. A. Pinsky and V. Wihstutz, editors, *Diffusion processes and related problems in analysis, volume II. Birkhauser Progress in Probability*, pages 37–72. Birkhauser, Boston, 1992.

[6] K.D. Elworthy and S. Rosenberg. The Witten Laplacian on negatively curved simply connected manifolds. To appear in Tokyo J. Math.

[7] K.D. Elworthy and S. Rosenberg. Generalized Bochner theorems and the spectrum of complete manifolds. *Acta Appl. Math.*, 12:1–33, 1988.

[8] K.D. Elworthy and S. Rosenberg. Manifolds with wells of negative curvature. *Invent. Math.*, 103:471–495, 1991.

[9] S. Gallot, D Hulin, and J. Lafontaine. *Riemannian geometry, second edition*. Springer-Verlag, (1990).

[10] S. Rosenberg and D. Yang. Bounds on the fundamental group of a manifold with almost non-negative Ricci curvature. To appear in J. Math. Soc. Japan.

[11] H. WU. Subharmonic functions and the volume of a noncompact manifold. In *Differential geometry, a symposium in honour of Manfred do Carmo*. Longman scientific & technical, 1991.

[12] J-Y Wu. Complete manifolds with a little negative curvature. *Amer. J. Math.*, 113:387–395, (1991).

Address:

Mathematics Institute, University of Warwick,

Coventry CV4,7AL

U.K.

e-mail: xl@maths.warwick.ac.uk.bitnet