

Properties at Infinity of Diffusion Semigroups and Stochastic Flows via Weak Uniform Covers

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Abstract. A unified treatment is given of some results of H. Donnelly, P. Li and L. Schwartz concerning the behaviour of heat semigroups on open manifolds with given compactifications, on one hand, and the relationship with the behaviour at infinity of solutions of related stochastic differential equations on the other. A principal tool is the use of certain covers of the manifold: which also gives a non-explosion test. As a corollary we obtain known results about the behaviour of Brownian motions on a complete Riemannian manifold with Ricci curvature decaying at most quadratically in the distance function.

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1. Introduction

Let M be a manifold, $\{\Omega, \mathcal{F}, \mathcal{F}_t, P\}$ a filtered probability space. A diffusion process on M is a stochastic process which is path continuous and strongly Markov. These diffusions are often given by solutions to stochastic differential equations of the following type:

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt \quad (1)$$

Here B_t is a R^m valued Brownian motion, X is C^2 from $M \times R^m$ to the tangent bundle TM with $X(x): R^m \rightarrow T_x M$ a linear map for each x in M , A is a C^1 vector field on M . Let u be an M -valued random variable independent of \mathcal{F}_0 . Denote by $F_t(u)$ the solution starting from u , $\xi(u)$ the explosion time. Then $\{F_t(x)\}$ is a diffusion for each $x \in M$. Associated with F_t there are also the probabilistic semigroup P_t and the corresponding infinitesimal generator \mathcal{A} .

1. MAIN RESULTS

The main aim [17] of this article is to give unified treatment to some of the results from H. Donnelly, P. Li and L. Schwartz. It gives first a probabilistic interpretation

and extends part of the latter. We first introduce weak uniform covers in an analogous way to uniform covers, which gives a nonexplosion test by using estimates on exit times of the diffusion considered. As a corollary this gives the known result on nonexplosion of a Brownian motion on a complete Riemannian manifold with Ricci curvature decaying at most quadratically in the distance function [14].

One interesting example is that a solution to a stochastic differential equation on R^n whose coefficients have linear growth has no explosion and has the C_0 property. Notice under this condition, its associated generator has quadratic growth. On the other hand let $M = R^n$, and let L be an elliptic differential operator:

$$L = \sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i}$$

where a_{ij} and b_i are C^2 . Let (s_{ij}) be the positive square root of the matrix (a_{ij}) . Let $X^i = \sum_j s_{ij} \partial/\partial x_j$, $A = \sum_j b_j \partial/\partial x_j$. Then the s.d.e. defined by:

$$(It\delta) \quad dx_t = \sum_i X^i(x_t) dB_t^i + A(x_t) dt$$

has generator L . Furthermore if $|(a_{ij})|$ has quadratic growth and b_i has linear growth, then both X and A in the s.d.e. above have linear growth. In this case any solution u_t to the following partial differential equation:

$$\frac{\partial u_t}{\partial t} = Lu_t$$

satisfies: $u_t \in C_0(M)$, if $u_0 \in C_0(M)$ (see next part).

II. PRELIMINARIES

A diffusion process is said to be a C_0 diffusion if its semigroup leaves invariant $C_0(M)$, the space of continuous functions vanishing at infinity, in which case the semigroup is said to have the C_0 property. A Riemannian manifold is said to be stochastically complete if the Brownian motion on it is complete, it is also said to have the C_0 property if the Brownian motion on it does. A Brownian motion is, by definition, a path continuous strong Markov process with generator $\frac{1}{2}\Delta$, where Δ denotes the Laplacian Beltrami operator. Equivalently a manifold is stochastically complete if the minimal heat semigroup satisfies: $P_t 1 \equiv 1$.

One example of a Riemannian manifold which is stochastically complete is a complete manifold with finite volume. See Gaffney [11]. More generally a complete Riemannian manifold with Ricci curvature bounded from below is stochastically complete and has the C_0 property as proved by Yau [20]. See also Ichihara [14], Dodziuk [6], L. Karp and P. Li [16], Bakry [2], Grigoryán [12], Hsu [13], Davies

[5], and Takeda [19] for further discussions in terms of volume growth and bounds on Ricci curvature. For discussions on the behaviour at infinity of diffusion processes, and the C_0 property, we refer the reader to Azencott [1] and Elworthy [10].

Those papers above are on a Riemannian manifold except for the last reference, for a manifold without a Riemannian structure, Elworthy [9] following Itô [15] showed that the diffusion solution to (1) does not explode if there is a uniform cover for the coefficients of the equation. See also Clarke [4]. In particular this shows that the s.d.e. (1) does not explode on a compact manifold if the coefficients are reasonably smooth. See [3], [8]. To apply this method to check whether a Riemannian manifold is stochastically complete, we usually construct a stochastic differential equation whose solution is a Brownian motion.

III. HEAT EQUATIONS, SEMIGROUPS, AND FLOWS

Let \bar{M} be a compactification of M , i.e., a compact Hausdorff space. We assume M is first countable. Let h be a continuous function on \bar{M} . Consider the following heat equation with initial boundary conditions:

$$\frac{\partial f}{\partial t} = \frac{1}{2} \Delta f, \quad x \in M \tag{2}$$

$$f(x, 0) = h(x), \quad x \in \bar{M} \tag{3}$$

$$f(x, t) = h(x), \quad x \in \partial M \tag{4}$$

It is known that there is a unique minimal solution satisfying the first two equations on a stochastically complete manifold, the solution is in fact given by the semigroup associated with Brownian motion on the manifold applied to h . So the above equation is not solvable in general. However, with a condition imposed on the boundary of the compactification, Donnelly and Li [7] showed that the heat semigroup satisfies (4). Here is the condition and the theorem:

THE BALL CONVERGENCE CRITERION. Let $\{x_n\}$ be a sequence in M converging to a point \bar{x} on the boundary, then the geodesic balls $\mathcal{B}_r(x_n)$, centered at x_n of radius r , converge to \bar{x} as n goes to infinity for each fixed r .

An example of a manifold which satisfies the ball convergence criterion is R^n with sphere at infinity, but not the compactification of a cylinder with a circle at infinity added at each end. The one point compactification also satisfies the ball convergence criterion.

THEOREM 1.1 (H. Donnelly and P. Li). *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. The over determined system (2)–(4) is solvable*

for any given continuous function h on \bar{M} , if and only if the ball convergence criterion holds for \bar{M} .

Notice if the Brownian motion starting from x converges to some point on the boundary to which x converges, (2)–(4) is clearly solvable. See Section 5 for details. We would also like to consider the opposite question: Do we get any information on the diffusion processes if we know the behaviour at infinity of the associated semigroups? This is true for many cases. In particular for the one point compactification, Schwartz has the following theorem [18], which provides a partial converse to [10]:

THEOREM 1.2 (L. Schwartz). *Let \bar{F}_t be the standard extension of F_t to $\bar{M} = M \cup \{\infty\}$, the one point compactification. Then the map $(t, x) \mapsto \bar{F}_t(x)$ is continuous from $R_+ \times \bar{M}$ to $L^0(\Omega, P, \bar{M})$, the space of measurable maps with topology induced from convergence in probability, if and only if the semigroup P_t has the C_0 property and the map $t \mapsto P_t f$ is continuous from R_+ to $C_0(M)$.*

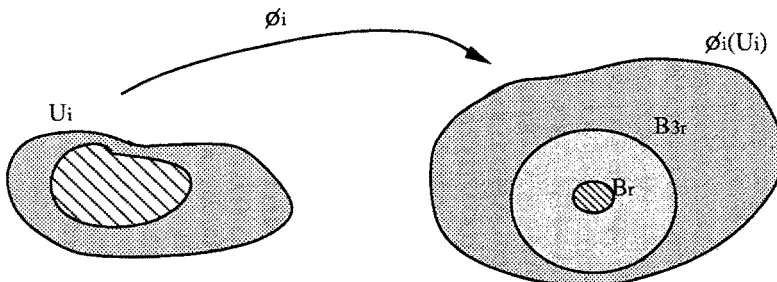
2. Weak Uniform Cover and Nonexplosion

DEFINITION 2.1 [9]. A stochastic dynamical system (1) is said to admit a uniform cover (radius $r > 0$, bound k), if there are charts $\{\phi_n, U_n\}$ of diffeomorphisms from open sets U_n of the manifold onto open sets $\phi_n(U_n)$ of R^n , such that:

1. $B_{3r} \subset \phi_i(U_i)$, each i . (B_α denotes the open ball about 0, radius α).
2. The open sets $\{\phi_i^{-1}(B_r)\}$ cover the manifold.
3. If $(\phi_i)_*(X)$ is given by:

$$(\phi_i)_*(X)(v)(e) = (D\phi_i)_{\phi_i^{-1}(v)} X(\phi_i^{-1}v)(e)$$

with $(\phi_i)_*(A)$ similarly defined, then both $(\phi_i)_*(X)$ and $\mathcal{A}(\phi_i)$ are bounded by k on B_{2r} , here \mathcal{A} is the generator for the dynamical system.



Let $\{e_j\}$ be an orthonormal basis for R^m , and define X^j to be $X(x)(e_j)$. Then for Stratonovich equation (1) on M ,

$$\mathcal{A}(\phi_i) = \frac{1}{2} \sum_{k,l=1}^m X^k X^l(\phi_i) + d\phi_i(A).$$

Let $M = R^n$. If we replace the Stratonovich differential in (1) by the Itô differential, then

$$\mathcal{A}(\phi_i)(x) = \frac{1}{2} \sum_{k,l=1}^m D^2\phi_i(x)(X^k(x), X^l(x)) + D\phi_i(x)(A(x)),$$

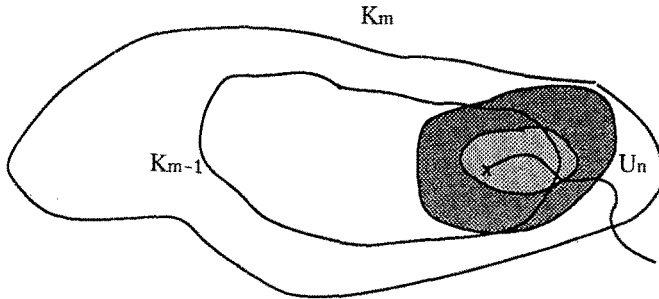
which does not involve any of the derivatives of X^j 's.

DEFINITION 2.2. A diffusion process $\{F_t, \xi\}$ is said to have a weak uniform cover if there are pairs of connected open sets $\{U_n^0, U_n\}$, and a non-increasing sequence $\{\delta_n\}$ with $\delta_n > 0$, such that:

1. $U_n^0 \subset U_n$, and the open sets $\{U_n^0\}$ cover the manifold. For $x \in U_n^0$ denote by $\tau^n(x)$ the first exit time of $F_t(x)$ from the open set U_n . Assume $\tau^n < \xi$ unless $\tau^n = \infty$.
2. There exists $\{K_n\}_{n=1}^\infty$, a family of increasing open subsets of M with $\cup K_n = M$, such that each U_n is contained in one of these sets and intersects at most one boundary from $\{\partial K_m\}_{m=1}^\infty$.
3. Let $x \in U_n^0$ and $U_n \subset K_m$, then for $t < \delta_m$:

$$P\{\omega: \tau^n(x) < t\} \leq Ct^2 \tag{5}$$

4. $\sum_{n=1}^\infty \delta_n = \infty$.



Notice the introduction of $\{K_n\}$ is only for giving an order to the open sets $\{U_n\}$. This is quite natural when looking at concrete manifolds. In a sense the condition says the geometry of the manifold under consideration changes slowly as far as the diffusion process is concerned. In particular if δ_n can be taken all equal, we take $K_n = M$; e.g., when the number of open sets in the cover is finite. On a Riemannian manifold the open sets are often taken as geodesic balls.

LEMMA 2.1. Assume there is a uniform cover for the stochastic dynamic system (1), then the solution has a weak uniform cover with $\delta_n = 1$, all n .

Proof. This comes directly from Lemma 5, p. 127 in [9].

REMARKS.

1. Let T be a stopping time, the inequality (5) gives the following from the strong Markov property of the process:

let $V \subset U_n^0$, and $V \subset K_m$, then when $t < \delta_m$:

$$P\{\tau^n(F_T(x)) < t | F_T(x) \in V\} \leq Ct^2, \tag{6}$$

since

$$P\{\tau^n(F_T(x)) < t, F_T(x) \in V\} = \int_V P(\tau^n(y) < t) P_T(x, dy) \leq Ct^2 P\{F_T(x) \in V\},$$

here $P_T(x, dy)$ denotes the distribution of $F_T(x)$.

2. Denote by $P_t^{U_n}$ the heat solution on U_n with Dirichlet boundary condition, then (6) is equivalent to the following: when $x \in U_n^0$,

$$1 - P_t^{U_n}(1)(x) \leq Ct^2 \tag{7}$$

3. The methods in this article work in infinite dimensions to give analogous results.

EXIT TIMES. Given such a cover, let $x \in U_n^0$. We define stopping times $\{T_k(x)\}$ as follows: Let $T_0 = 0$. Let $T_1(x) = \inf\{t > 0: F_t(x, \omega) \notin \overline{U_n}\}$ be the first exit time of $F_t(x)$ from the set U_n . Then $F_{T_1}(x, \omega)$ must be in one of the open sets $\{U_k^0\}$. Let

$$\begin{aligned} \Omega_1^1 &= \{\omega: F_{T_1}(x)(\omega) \in U_1^0, T_1(x, \omega) < \infty\} \\ \Omega_k^1 &= \left\{ \omega: F_{T_1}(x) \in U_k^0 - \bigcup_{j=1}^{k-1} U_j^0, T_1 < \infty \right\} \end{aligned}$$

Then $\{\Omega_k^1\}$ are disjoint sets such that $\cup \Omega_k^1 = \{T_1 < \infty\}$. In general we only need to consider the nonempty sets of such. Define further the following: Let $T_2 = \infty$, if $T_1 = \infty$. Otherwise if $\omega \in \Omega_k^1$, let:

$$T_2(x, \omega) = T_1(x, \omega) + \tau^k(F_{T_1}(x, \omega)) \tag{8}$$

In a similar way, the whole sequence of stopping times $\{T_j(x)\}$ and sets $\{\Omega_k^j\}_{k=1}^\infty$ are defined for $j = 3, 4, \dots$. Clearly Ω_k^j so defined is measurable with respect to the sub-algebra \mathcal{F}_{T_j} .

LEMMA 2.2. *Given a weak uniform cover as above. Let $x \in U_n^0$ and $U_n \subset K_m$. Let $t < \delta_{m+k}$. Then*

$$P\{\omega: T_k(x, \omega) - T_{k-1}(x, \omega) < t, T_{k-1} < \infty\} \leq Ct^2. \tag{9}$$

Proof. Notice for such x , $F_{T_k}(x) \in K_{m+k-1}$. Therefore for $t < \delta_{m+k}$ we have

$$\begin{aligned} & P\{\omega: T_k(x, \omega) - T_{k-1}(x, \omega) < t, T_{k-1} < \infty\} \\ &= \sum_{j=1}^{\infty} P(\{\omega: T_k(x, \omega) - T_{k-1}(x, \omega) < t\} \cap \Omega_j^{k-1}) \\ &= \sum_{j=1}^{\infty} P\{\tau^j(F_{T_{k-1}}(x)) < t, \Omega_j^{k-1}\} \\ &\leq Ct^2 \sum_{j=1}^{\infty} P(\Omega_j^{k-1}) \leq Ct^2, \end{aligned}$$

as in Remark 1. Here χ_A is the characteristic function for a measurable set A , and E denotes taking expectation. ■

LEMMA 2.3. *If $\sum_n t_n = \infty$, $t_n > 0$ non-increasing. Then there is a non-increasing sequence $\{s_n\}$, such that $0 < s_n \leq t_n$:*

- (i) $\sum s_n = \infty$
- (ii) $\sum s_n^2 < \infty$.

Proof. Assume $t_n \leq 1$, all n . Group the sequence $\{t_n\}$ in the following way:

$$t_1; t_2, \dots, t_{k_2}; t_{k_2+1}, \dots, t_{k_3}; t_{k_3+1} \dots$$

such that $1 \leq t_2 + \dots + t_{k_2} \leq 2$, $1 \leq t_{k_i+1} + \dots + t_{k_{i+1}} \leq 2$, $i \geq 2$. Let $s_1 = t_1$, $s_2 = t_2/2, \dots, s_{k_2} = t_{k_2}/2$, $s_{k_2+1} = t_{k_2+1}/3, \dots, s_{k_3} = t_{k_3}/3$, $s_{k_3+1} = t_{k_3+1}/4, \dots$. Clearly the s_n 's so defined satisfy the requirements. ■

So without losing generality, we may assume from now on that the constants $\{\delta_n\}$ for a weak uniform cover fulfill the two conditions in the above lemma. With these established, we can now state the nonexplosion result. The proof is analogous to the argument of Theorem 6 on p. 129 in [9].

THEOREM 2.4. *If the solution $F_t(x)$ of the equation (1) has a weak uniform cover, then it is complete (nonexplosion).*

Proof. Let $x \in K_n$, $t > 0$, $0 < \varepsilon < 1$. Pick a number p (may be depending on ε and n), such that $\sum_{i=n+1}^{n+p} \varepsilon \delta_i > t$, which is possible from: $\sum_{i=1}^{\infty} \delta_i = \infty$. So

$$\begin{aligned}
 P\{\xi(x) < t\} &\leq P\{T_p(x) < t, T_{p-1} < \infty\} \\
 &= P\left\{\sum_{k=1}^p (T_k(x) - T_{k-1}(x)) < t, T_{p-1} < \infty\right\} \\
 &\leq \sum_{k=1}^p P\{T_k(x) - T_{k-1}(x) < \varepsilon\delta_{n+k}, T_{k-1} < \infty\} \\
 &\leq C\varepsilon^2 \sum_{k=n}^{n+p} \delta_k^2 \leq C\varepsilon^2 \sum_{k=1}^{\infty} \delta_k^2.
 \end{aligned}$$

Let $\varepsilon \rightarrow 0$, we get: $P\{\xi(x) < t\} = 0$. ■

REMARK. The argument in the above proof is valid if the definition of a weak uniform cover is changed slightly, i.e. replacing the constant C by C_n (with some slow growth condition) but keep all δ_n equal.

To understand more of the weak uniform cover, we look into an example:

EXAMPLE 1. Let $\{U_n\}$ be a family of relatively compact open (proper) subsets of M such that $U_n \subset U_{n+1}$ and $\cup_{n=1}^{\infty} U_n = M$. Assume there is a sequence of numbers $\{\delta_n\}$ with $\sum_n \delta_n = \infty$, such that the following inequality holds when $t < \delta_{n-1}$ and $x \in U_{n-1}$: $P\{\tau^{U_n}(x) < t\} \leq ct^2$. Then the diffusion concerned does not explode by taking $\{U_{n+1} - \overline{U_{n-1}}, U_n - \overline{U_{n-1}}\}$ to be a weak uniform cover and $K_n = \overline{U}_n$.

3. Boundary Behaviour of Diffusion Processes

To consider the boundary behaviour of diffusion processes, we introduce the following concept:

DEFINITION 3.1. A weak uniform cover $\{U_n^0, U_n\}$ is said to be regular (at infinity for M), if the following holds: let $\{x_j\}$ be a sequence in M converging to $\bar{x} \in \partial M$, and $x_j \in U_{n_j}^0 \in \{U_n^0\}_{n=1}^{\infty}$, then the corresponding open sets $\{U_{n_j}\}_{j=1}^{\infty} \subset \{U_n\}_1^{\infty}$ converges to \bar{x} as well. A regular uniform cover can be defined in a similar way.

For a point x in M , there are a succession of related open sets $\{W_x^p\}_{p=1}^{\infty}$, which are defined as follows: Let W_x^1 be the union of all open sets from $\{U_n\}$ such that U_n^0 contains x , and W_x^2 be the union of all open sets from $\{U_n\}$ such that U_n^0 intersects one of the open sets $U_{n_j}^0$ defining W_x^1 . The $\{W_x^p\}$ are defined similarly. These sets are well defined and in fact form an increasing sequence.

LEMMA 3.1. Assume F_t has a regular weak uniform cover. Let $\{x_n\}$ be a sequence in M which converges to a point $\bar{x} \in \partial M$. Then $W_{x_n}^p$ converges to \bar{x} as well for each fixed p .

Proof. We only need to prove the following: let $\{z_k\}$ be a sequence $z_k \in W_{x_k}^p$, then $z_k \rightarrow \bar{x}$, as $n \rightarrow \infty$. First, let $p = 2$.

By definition, for each x_k, z_k , there are open sets $U_{n_k}^0$ and $U_{m_k}^0$ such that $x_k \in U_{n_k}^0, z_k \in U_{m_k}^0$ and $U_{n_k}^0 \cap U_{m_k}^0 \neq \emptyset$. Furthermore $U_{n_k} \rightarrow \bar{x}$ as $k \rightarrow \infty$. Let $\{y_k\}$ be a sequence of points with $y_k \in U_{n_k}^0 \cap U_{m_k}^0$. But $y_k \rightarrow \bar{x}$ since U_{n_k} does. So $U_{m_k} \rightarrow \bar{x}$ again from the definition of a regular weak uniform cover. Therefore z_k converges to \bar{x} as $k \rightarrow \infty$, which is what we want. The rest can be proved by induction. ■

THEOREM 3.2. If the diffusion F_t admits a regular weak uniform cover for \bar{M} , with $\delta_n = \delta$, all n , then the map $F_t(-): M \rightarrow M$ can be extended to the compactification \bar{M} continuously in probability with the restriction to the boundary to be the identity map, uniformly in t in finite intervals. (We will say F_t extends.)

Proof. Take $\bar{x} \in \partial M$ and a sequence $\{x_n\}$ in M converging to \bar{x} . Let U be a neighbourhood of \bar{x} in \bar{M} . We want to prove for each t :

$$\lim_{n \rightarrow \infty} P\{\omega: F_s(x_n, \omega) \notin U, \text{ for some } s < t\} = 0.$$

Since x_n converges to \bar{x} , there is a number $N(p)$ for each p , such that if $n > N = N(p)$, $W_{x_n}^p \subset U$. Let $t > 0$, choose p such that $2t/p < \delta$. For a number $n > N(p)$ fixed, we have:

$$\begin{aligned} P\{\omega: F_s(x_n, \omega) \notin U, \text{ for some } s < t\} &\subset P\{\omega: F_s(x_n, \omega) \notin W_{x_n}^p, \text{ for some } s < t\} \\ &\leq P\{\omega: T_p(x_n)(\omega) < t, T_{p-1}(x_n) < \infty\} \\ &\leq \sum_{k=1}^p P\left\{T_k(x_n) - T_{k-1}(x_n) < \frac{t}{p}, T_{k-1}(x_n) < \infty\right\} \\ &\leq \frac{Ct^2}{p}. \end{aligned}$$

Here C is the constant in the definition of the weak uniform cover. Let p go to infinity to complete the proof. ■

REMARK. If δ_n can be taken all equal, Theorem 2.4, Theorem 3.2 hold if (5) is relaxed to:

$$P\{\tau^n(x) < t\} \leq f(t),$$

for some nonnegative function f satisfying $\lim_{t \rightarrow 0} f(t)/t = 0$.

The weak uniform cover condition is not very easy to apply for a general diffusion on a manifold. For a Brownian motion in a Riemannian manifold it is relatively simple. We often start with a uniform cover, ref. Lemma 2.1.

EXAMPLE 2. Let $M = R^n$ with the one point compactification. Consider the following s.d.e.:

$$(It\delta) \quad dx_t = X(x_t)dB_t + A(x_t)dt$$

Then if both X and A have linear growth, the solution has the C_0 property.

Proof. There is a well known uniform cover for this system. See [4], or [9]. A slight change gives us the following regular uniform cover:

Take a countable set of points $\{p_n\}_{n \geq 0} \subset M$ such that the open sets $U_n^0 = \{z: |z - p_n| < |p_n|/3\}$, $n = 1, 2, \dots$ and $U_0^0 = \{z: |z - p_0| < 2\}$ cover R^n . Let $U_0 = \{z: |z - p_0| < 6\}$; and $U_n = \{z: |z - p_n| < |p_n|/2\}$, for $n \neq 0$. Let ϕ_n be the chart map on U_n :

$$\phi_n(z) = \frac{z - p_n}{|p_n|}.$$

This certainly defines a uniform cover (for details see Example 3 below). Furthermore if $z_n \rightarrow \infty$ and $z_n \in U_n^0$, then any $y \in U_n$ satisfies the following:

$$|y| > \frac{|p_n|}{2} > \frac{1}{3}|z_n| \rightarrow \infty,$$

since $|p_n| \geq 3|z_n|/4$. Thus we have a regular uniform cover which gives the required C_0 property.

EXAMPLE 3. Let $M = R^n$, compactified with the sphere at infinity: $\bar{M} = R^n \cup S^{n-1}$. Consider the same s.d.e. as in the example above. Suppose both X and A have sublinear growth of power $\alpha < 1$:

$$|X(x)| \leq K(|x|^\alpha + 1)$$

$$|A(x)| \leq K(|x|^\alpha + 1)$$

for a constant K . Then there is no explosion. Moreover the solution F_t extends.

Proof. The proof is as in Example 2, we only need to construct a regular uniform cover for the s.d.e.:

Take points p_0, p_1, p_2, \dots in R^n (with $|p_0| = 1, |p_n| > 1$), such that the open sets $\{U_n^0\}$ defined by: $U_0^0 = \{z: |z - p_0| < 2\}$, $U_i^0 = \{z: |z - p_i| < |p_i|^\alpha/6\}$ cover R^n .

Let $U_0 = \{z: |z - p_0| < 6\}$, $U_i = \{z: |z - p_i| < |p_i|^\alpha/2\}$, and let ϕ_i be the chart map from U_i to R^n :

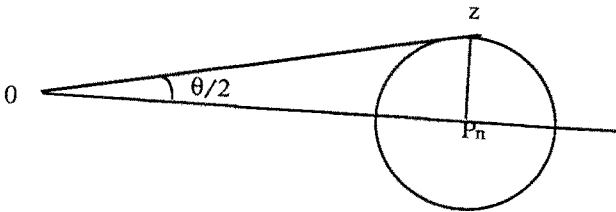
$$\phi_i(z) = \frac{6(z - p_i)}{|p_i|^\alpha}.$$

Then $\{\phi_i, U_i\}$ is a uniform cover for the stochastic dynamical system. In fact, for $i \neq 0$, and $y \in B_3 \subset R^n$:

$$|(\phi_i)_*(X)(y)| \leq \frac{K(1 + |\phi_i^{-1}(y)|^\alpha)}{|p_i|^\alpha} \leq \frac{K}{|p_i|^\alpha}(1 + 2|p_i|^\alpha) < 18K$$

Similarly $|(\phi_i)_*(A)(y)| \leq 18K$, and $D^2\phi_i = 0$.

Next we show this cover is regular. Take a sequence x_k converging to \bar{x} in ∂R^n . Assume $x_k \in U_k^0$. Let $z_k \in U_k$. We want to prove $\{z_k\}$ converge to \bar{x} . First the norm of z_k converges to infinity as $k \rightarrow \infty$, since $|p_k| > 2|x_k|/3$ and $|z_k| > |p_k| - \frac{1}{2}|p_k|^\alpha$.



Let θ be the biggest angle between points in U_k , then

$$\tan \frac{\theta}{2} \leq \sup_{z \in U_n} \frac{|z - P_n|}{|p_n|} \leq \frac{|p_n|^\alpha}{2|p_n|} \leq \frac{|p_n|^{\alpha-1}}{2} \rightarrow 0.$$

Thus $\{U_n, \phi_n\}$ is a uniform cover satisfying the convergence criterion for the sphere compactification. The required result holds from the theorem.

This result is sharp in the sense there is a s.d.e. with coefficients having linear growth but the solution to it does not extend to the sphere at infinity to be identity:

EXAMPLE 4. Let B be a one dimensional Brownian motion. Consider the following s.d.e. on the complex plane \mathcal{C} :

$$dx_t = ix_t dB_t$$

The solution starting from x is in fact $x e^{iB_t + t/2}$, which does not continuously extend to be the identity on the sphere at infinity.

4. Boundary Behaviour Continued

A diffusion process is a C_0 diffusion if its semigroup has the C_0 property. This is equivalent to the following [1]: let K be a compact set, and $T_K(x)$ the first entrance time to K of the diffusion starting from x , then $\lim_{x \rightarrow \infty} P\{T_K(x) < t\} = 0$ for each $t > 0$, and each compact set K .

The following theorem follows from Theorem 3.2 when δ_n in the definition of the weak uniform cover can be taken all equal:

THEOREM 4.1. *Let \bar{M} be the one point compactification. Then if the diffusion process $F_t(x)$ admits a regular weak uniform cover, it is a C_0 diffusion.*

Proof. Let K be a compact set with $K \subset K_j$; here $\{K_j\}$ is as in Definition 2.2. Let $\varepsilon > 0, t > 0$, then there is a number $N = N(\varepsilon, t)$ such that:

$$\delta_{j+2} + \delta_{j+4} + \dots + \delta_{j+2N-2} > \frac{t}{\varepsilon}.$$

Take $x \notin K_{j+2N}$. Assume $x \in K_m$, some $m > j + 2N$. Let T_0 be the first entrance time of $F_t(x)$ to K_{j+2N-1} , T_1 be the first entrance time of $F_t(x)$ to K_{j+2N-3} after T_0 , (if $T_0 < \infty$), and so on. But $P\{T_i < t, T_{i-1} < \infty\} \leq Ct^2$ for $t < \delta_{j+2N-2i}, i > 0$, since any open sets from the cover intersects at most one boundary of sets from $\{K_n\}$. Thus

$$\begin{aligned} P\{T_K(x) < t\} &\leq P\left\{\sum_1^{N-1} T_i(x) < t, T_{N-2} < \infty\right\} \\ &\leq \sum_{i=1}^{N-1} P\{T_i(x) < \varepsilon\delta_{j+2N-2i}, T_{i-1}(x) < \infty\} \\ &\leq C\varepsilon^2 \sum_1^{N-1} \delta_{j+2N-2i}^2 \leq C\varepsilon^2 \sum_1^\infty \delta_j^2. \end{aligned}$$

The proof is complete by letting $\varepsilon \rightarrow 0$. ■

EXAMPLE 5. Let M be a complete Riemannian manifold, p a fixed point in M . Denote by $\rho(x)$ the distance between x and p , $B_r(x)$ the geodesic ball centered at x of radius r , and $\text{Ricci}(x)$ the Ricci curvature at x .

ASSUMPTION A.

$$\int_1^\infty \frac{1}{\sqrt{K(r)}} dr = \infty. \tag{10}$$

Here K is defined as follows:

$$K(r) = -\left\{ \inf_{B_r(p)} \text{Ricci}(x) \wedge 0 \right\}.$$

Let $X_t(x)$ be a Brownian motion on M with $X_0(x) = x$. Consider the first exit time of $X_t(x)$ from $B_1(x)$:

$$T = \inf \{t \geq 0: \rho(x, X_t(x)) = 1\}.$$

Then we have the following estimate on T from [13]:

If $L(x) > \sqrt{K(\rho(x) + 1)}$, then

$$P \left\{ T(x) \leq \frac{c_1}{L(x)} \right\} \leq e^{-c_2 L(x)}$$

for all $x \in M$. Here c_1, c_2 are positive constants independent of L .

This can be rephrased into the form we are familiar with: when $0 \leq t < c_1/\sqrt{K(\rho(x) + 1)}$,

$$P\{T(x) \leq t\} \leq e^{-(c_1 c_2/t)}$$

But $\lim_{t \rightarrow 0} e^{-(c_1 c_2/t)/t^2} = 0$. So there is a $\delta_0 > 0$, such that: $e^{-(c_1 c_2/t)} \leq t^2$, when $t < \delta_0$. Thus:

ESTIMATION ON EXIT TIMES: when $t < c_1/\sqrt{K(\rho(x) + 1)} \wedge \delta_0$,

$$P\{T(x) < t\} \leq t^2. \tag{11}$$

Let $\delta_n = 1/\sqrt{K(3n + 1)} \wedge \delta_0$, then we also have the following:

$$\sum_1^\infty \delta_n \geq \sum_1^\infty \frac{1}{\sqrt{K(3n + 1)}} \geq \int_1^\infty \frac{1}{\sqrt{K(3r)}} dr = \infty \tag{12}$$

with this we may proceed to prove the following from Hsu:

COROLLARY [13]. *A complete Riemannian manifold M with Ricci curvature satisfying Assumption A is stochastically complete and has the C_0 property.*

Proof. There is a regular weak uniform cover as follows:

First take any $p \in M$, and let $K_n = \overline{B_{3n}(p)}$. Take points p_i such that $U_i^0 = B_1(p_i)$ covers the manifold. Let $U_i = B_2(p_i)$. Then $\{U_i^0, U_i\}$ is a regular weak uniform cover for $M \cup \Delta$.

REMARK. Grigoryán has the following volume growth test on nonexplosion. The Brownian motion does not explode on a manifold if there is a function f on M satisfying:

$$\int^\infty \frac{r}{\text{Ln}(\text{Vol}(B_R))} dr = \infty.$$

Here $\text{Vol}(B_R)$ denotes the volume of a geodesic ball centered at a point p in M . This result is stronger than the corollary obtained above by the following comparison theorem on an n dimensional manifold: let ω_{n-1} denote the volume of the $n - 1$ sphere of radius 1,

$$\text{Vol}(B_R) \leq \omega_{n-1} \int_0^R \left\{ \sqrt{\frac{(n-1)}{K(R)}} \text{Sinh} \left(\sqrt{\frac{K(R)}{(n-1)}} r \right) \right\}^{(n-1)} dr.$$

Notice $K(R)$ is positive when R is sufficiently big provided the Ricci curvature is not nonnegative everywhere. So Grigoryán’s result is stronger than the one obtained above.

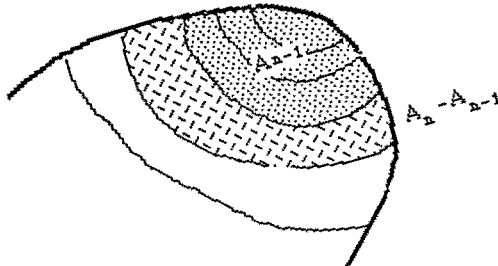
The definition of weak uniform cover is especially suitable for the one point compactification. For general compactifications the following definition explores more of the geometry of the manifolds and gives a better result:

DEFINITION 4.1. Let \bar{M} be a compactification of M , $\bar{x} \in \partial M$. A diffusion process F_t is said to have a uniform cover at the point \bar{x} , if there is a sequence A_n of open neighbourhoods of \bar{x} in \bar{M} and positive numbers δ_n and a constant $c > 0$, such that:

1. The sequence of A_n is strictly decreasing, with $\cap A_n = \bar{x}$, and $A_n \supset \partial A_{n+1}$.
2. The sequence of numbers δ_n is non-increasing with $\sum \delta_n = \infty$ and $\sum \delta_n^2 < \infty$.
3. When $t < \delta_n$, and $x \in A_n - A_{n+1}$,

$$P\{\tau^{A_{n-1}}(x) < t\} \leq ct^2.$$

Here $\tau^{A_n}(x)$ denotes the first exit time of $F_t(x)$ from the set A_n .



PROPOSITION 4.2. *If there is a uniform cover for $\bar{x} \in \partial M$, then $F_t(x)$ converges to \bar{x} continuously in probability, uniformly in t in finite interval, as $x \rightarrow \bar{x}$.*

Proof. The existence of $\{A_n\}$ will ensure $F_{\tau^{A_n}}(x) \subset A_{n-1}$, which allows us to apply a similar argument as in the case of the one point compactification. Here we denote by τ^A the first exit time of the process $F_t(x)$ from a set A .

Let U be a neighbourhood of \bar{x} . For this U , by compactness of \bar{M} , there is a number m such that $A_m \subset U$, since $\cap_{k=1}^\infty A_k = \bar{x}$. Let $0 < \varepsilon < 1$, $\bar{\varepsilon} = (\varepsilon/c \sum_k \delta_k^2)^{1/2}$. We may assume $\bar{\varepsilon} < 1$. Choose $p = p(\varepsilon) > 0$ such that:

$$\delta_m + \delta_{m+1} + \dots + \delta_{m+p-1} > \frac{t}{\bar{\varepsilon}}.$$

Let $x \in A_{m+p+2}$. Denote by $T_0(x)$ the first exit time of $F_t(x)$ from A_{m+p+1} , $T_1(x)$

the first exit time of $F_{T_0}(x)$ from A_{m+p} where defined. Similarly $T_i, i > 1$ are defined.

Notice if $T_i(x) < \infty$, then $F_{T_i(x)} \in A_{m+p-i} - A_{m+p+1-i}$, for $i = 0, 1, 2, \dots$. Thus for $i > 0$ there is the following inequality from the definition and the Markov property:

$$P\{T_i(x) < \bar{\varepsilon}\delta_{m+p-i}\} < c\bar{\varepsilon}^2\delta_{m+p-i}^2.$$

Therefore we have:

$$\begin{aligned} P\{\tau^U(x) < t\} &\leq P\{\tau^{A_m}(x) < t\} \\ &\leq P\{T_p + \dots + T_1 < t, T_{p-1} < \infty\} \\ &\leq \sum_{i=1}^p P\{T_i < \bar{\varepsilon}\delta_{m+p-i}, T_{i-1} < \infty\} \\ &\leq c\bar{\varepsilon}^2 \sum_{i=1}^p \delta_{m+p-i}^2 < \varepsilon. \end{aligned}$$

This finishes the proof. ■

5. Property at Infinity of Semigroups

Recall a semigroup is said to have the C_0 property, if it sends $C_0(M)$, the space of continuous functions on M vanishing at infinity, to itself. Let \bar{M} be a compactification of M . Denote by Δ the point at infinity for the one point compactification. Corresponding to the C_0 property of semigroups we consider the following C_* property for \bar{M} :

DEFINITION 5.1. A semigroup P_t is said to have the C_* property for \bar{M} , if for each continuous function f on \bar{M} , the following holds: let $\{x_n\}$ be a sequence converging to \bar{x} in ∂M , then

$$\lim_{n \rightarrow \infty} P_t f(x_n) = f(\bar{x}). \tag{13}$$

To justify the definition, we notice if \bar{M} is the one point compactification, condition C_* will imply the C_0 property of the semigroup. On the other hand if P_t has the C_0 property, it has the C_* property for $M \cup \Delta$ assuming nonexplosion. This is observed by subtracting a constant function from a continuous function f on $M \cup \Delta$: let $g(x) = f(x) - f(\Delta)$, then $g \in C_0(M)$. So $P_t g(x) = P_t f(x) - f(\Delta)$. Thus

$$\lim_{n \rightarrow \infty} P_t f(x_n) = \lim_{n \rightarrow \infty} P_t g(x_n) + f(\Delta) = f(\Delta),$$

if $\lim_{n \rightarrow \infty} x_n = \Delta$.

In fact the C_* property holds for the one point compactification if and only if there is no explosion and the C_0 property holds. These properties are often possessed by

processes, e.g. a Brownian motion on a Riemannian manifold with Ricci curvature which satisfies (10) has this property.

Before proving this claim, we observe first that:

LEMMA 5.1. *If P_t has the C_* property for any compactification \bar{M} , it must have the C_* property for the one point compactification.*

Proof. Let $f \in C(M \cup \Delta)$. Define a map β from \bar{M} to $M \cup \Delta$: $\beta(x) = x$ on the interior of M , and $\beta(x) = \Delta$, if x belongs to the boundary. Then β is a continuous map from \bar{M} to $M \cup \Delta$, since for any compact set K , the inverse set $\bar{M} - K = \beta^{-1}(M \cup \Delta - K)$ is open in \bar{M} .

Let g be the composition map of f with β : $g = f \circ \beta: \bar{M} \rightarrow R$. Thus $g(x)|_M = f(x)|_M$, and $g(x)|_{\partial M} = f(\Delta)$. So for a sequence $\{x_n\}$ converging to $\bar{x} \in \partial M$, $\lim_n P_t f(x_n) = \lim_n P_t g(x_n) = g(\bar{x}) = f(\Delta)$. ■

We are ready to prove the following theorem.

THEOREM 5.2. *If a semigroup P_t has the C_* property, the associated diffusion process F_t is complete.*

Proof. We may assume the compactification under consideration is the one point compactification from the lemma above. Take $f \equiv 1$, $P_t f(x) = P\{t < \zeta(x)\}$. But $P\{t < \zeta(x)\} \rightarrow 1$ as $x \rightarrow \Delta$ from the assumption. More precisely for any $\varepsilon > 0$, there is a compact set K , such that if $x \notin K_\varepsilon$, $P\{t < \zeta(x)\} > 1 - \varepsilon$.

Let K be a compact set containing K_ε . Denote by τ the first exit time of $F_t(x)$ from K . So $F_\tau(x) \notin K$, on the set $\{\tau < \infty\}$. Thus:

$$\begin{aligned} P\{t < \zeta(x)\} &\geq P\{\tau < \infty, t < \zeta(F_\tau(x))\} + P\{\tau = \infty\} \\ &= E\{\chi_{\tau < \infty} E\{\chi_{t < \zeta(F_\tau(x))} | \mathcal{F}_\tau\}\} + P\{\tau = \infty\}. \end{aligned}$$

Here χ_A denotes the characteristic function of set A . Applying the strong Markov property of the diffusion we have:

$$E\{\chi_{\tau < \infty} E\{\chi_{t < \zeta(F_\tau(x))} | \mathcal{F}_\tau\}\} = E\{\chi_{\tau < \infty} E\{t < \zeta(y) | F_\tau = y\}\}.$$

However

$$E\{\chi_{t < \zeta(y)} | F_\tau = y\} > 1 - \varepsilon,$$

so

$$\begin{aligned} P\{t < \zeta(x)\} &\geq P\{\tau = \infty\} + E\{\chi_{\tau < \infty}(1 - \varepsilon)\} \\ &= 1 - \varepsilon P\{\tau < \infty\}. \end{aligned}$$

Therefore $P\{t < \zeta(x)\} = 1$, since ε is arbitrary. ■

In the following we examine the relation between the behaviour at ∞ of diffusion processes and the diffusion semigroups.

THEOREM 5.3. *The semigroup P_t has the C_* property if and only if the diffusion process F_t is complete and can be extended to \bar{M} continuously in probability with $F_t(x)|_{\partial M} = x$.*

Proof. Assume F_t is complete and extends. Take a point $\bar{x} \in \partial M$, and sequence $\{x_n\}$ converging to \bar{x} . Thus

$$\lim_{n \rightarrow \infty} P_t f(x_n) = \lim_{n \rightarrow \infty} Ef(F_t(x_n)) = Ef(x) = f(x)$$

for any continuous function on \bar{M} , by the dominated convergence theorem.

On the other hand P_t does not have the C_* property if the assumption above is not true. In fact, let x_n be a sequence converging to \bar{x} , such that for some neighbourhood U of \bar{x} , and a number $\delta > 0$:

$$\lim_{n \rightarrow \infty} P\{F_t(x_n) \notin U\} = \delta.$$

There is therefore a subsequence $\{x_{n_i}\}$ such that:

$$\lim_{i \rightarrow \infty} P\{F_t(x_{n_i}) \notin U\} = \delta.$$

Thus there exists $N > 0$, such that if $i > N$:

$$P\{F_t(x_{n_i}) \in \bar{M} - U\} > \frac{\delta}{2}.$$

But since \bar{M} is a compact Hausdorff space, there is a continuous function f from M to $[0, 1]$ such that $f|_{\bar{M}-U} = 1$, and $f(x)|_G = 0$, for any open set G in U . Therefore

$$\begin{aligned} P_t f(x_n) &= Ef(F_t(x_n)) \\ &\geq \int_{\{\omega: F_t(x_n) \in \bar{M} - U\}} f(F_t(x_n)) P(d\omega) \\ &= P\{F_t(x_n) \in \bar{M} - U\} > \frac{\delta}{2}. \end{aligned}$$

So $\lim P_t f(x_n) \neq f(x) = 0$. ■

COROLLARY 5.4. *Assume the diffusion process F_t admits a weak uniform cover regular for $M \cup \Delta$, then its diffusion semigroup P_t has the C_* property for $M \cup \Delta$. The same is true for a general compactification if all δ_n in the weak uniform cover can be taken equal.*

EXAMPLE 6 [7]. Let M be a complete connected Riemannian manifold with Ricci curvature bounded from below. Let \bar{M} be a compactification such that the ball convergence criterion holds (ref. Section 1). In particular the over determined equation (2)–(4) is solvable for any continuous function f on \bar{M} if the ball convergence criterion holds.

Proof. We keep the notation of Example 5 here. Let $K = -\{\inf_x \text{Ricci}(x) \wedge 0\}$, $\delta = c_1/k$, where c_1 is the constant in Example 5. Let $p \in M$ be a fixed point, and $K_n = \overline{B_{3n}(p)}$ be compact sets in M . Take points $\{p_i\}$ in M such that $\{B_1(p_i)\}$ cover the manifold. Then $\{B_1(p_i), B_2(p_i)\}$ is a weak uniform cover from (11) and (12). Moreover this is a regular cover if the ball convergence criterion holds for the compactification.

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